The Hamiltonian of the f(R) gravity

Faisal. A. Y. Abdelmohssin[1], Osman. M. H. El Mekki[2]

Abstract

We derived the Hamiltonian of the source free $f(R)$ gravity from its Euler-Poisson equation. Interpreting it as energy, we have shown that it vanishes for linear Lagrangians in Ricci scalar curvature without source, which is the same result, obtained using the stress-energy tensor equation.

Keywords: Calculus of variation, Einstein field equations, f(R) gravity

Contents

(1). Introduction

H. A. Buchdahl, [1] proposed his generalization to the Einstein field equations by considering a generalization of the gravitational Lagrangian $\phi(R)$ to be a general function of the Ricci scalar rather than just a linear function proportional to the Ricci scalar curvature. Nowadays it is called $f(R)$ gravity. He suggested a Lagrangian functional of the form

$$L = \phi(R) \quad (1.0)$$

Where $\phi(R)$ is unspecified. He has given the generalization to the Einstein field equations as a tensor equation, which contains derivatives of $\phi(R)$ with respect to the Ricci scalar as well as derivatives of the Ricci scalar with respect to space-time coordinates.

[1] Department Of Physics, Faculty of Science and Technology, AL-Neelain University, P.O. Box, 12702, Khartoum, Sudan. email address: f.a.y.abdelmohssin@gmail.com
[2] Faculty of Mathematical Sciences, University of Khartoum, P.O. Box, 11115, Khartoum, Sudan. email address: oelmekki@uofk.edu

(2). Euler-Poisson equation

The Euler-Poisson equation of a general Lagrangian functional L is derived from the calculus of variation by varying the Lagrangian functional with respect to its independent variables and set the variation equals to zero which results in following equation [2]

$$\frac{\partial L}{\partial q}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)+\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right)=0 \quad (2.1)$$

With the boundary term given by

$$\left[\underbrace{\left(\frac{\partial L}{\partial \dot{q}}-\frac{d}{dt}\frac{\partial L}{\partial \ddot{q}}\right)}_{p_{(q)}}\delta q+\underbrace{\left(\frac{\partial L}{\partial \ddot{q}}\right)}_{p_{(\dot{q})}}\delta \dot{q}\right]_{q_1}^{q_2} \quad (2.2)$$

Which vanishes if δq and $\delta \dot{q}$ vanish at the two end points q_1 and q_2.
To derive field equations for any functional of the fundamental metric tensor $g_{mp}(x^e)$ we make the following change of variables into the Euler-Poisson equation

$$\begin{aligned}
&t \Rightarrow x^a\\
&q(t) \Rightarrow g_{mp}(x^e)\\
&\dot{q}(t)=\frac{dq(t)}{dt} \Rightarrow g_{mp,s}(x^e)=\frac{\partial g_{mp}(x^e)}{\partial x^s}\\
&\ddot{q}(t)=\frac{d^2q(t)}{dt^2} \Rightarrow g_{mp,sr}(x^e)=\frac{\partial}{\partial x^s}\left(\frac{\partial g_{mp}(x^e)}{\partial x^r}\right)=\frac{\partial^2 g_{mp}(x^e)}{\partial x^s\,\partial x^r}\\
&L\left(t,\ q(t),\ \dot{q}(t),\ \ddot{q}(t)\right) \Rightarrow L\left(x^e,\ g_{mp}(x^e),\ g_{mp,s}(x^e),\ g_{mp,sr}(x^e)\right)
\end{aligned} \quad (2.3)$$

The Lagrangian of the $f(R)$ gravity is

$$L(g_{ab},\ g_{ab,c},\ g_{ab,cd})=\sqrt{-g}\ L_{Gravity} \quad (2.4)$$

Where

$$L_{Gravity}(g_{ab},\ g_{ab,c},\ g_{ab,cd})=f(R) \quad (2.5)$$

Substituting Eqs. (2.3 - 2.5) in Eq. (2.1), the Euler-Poisson equation of the Lagrangian of $f(R)$ gravity may be written explicitly as

$$\frac{\partial[\sqrt{-g}\ f(R)]}{\partial g_{mp}} - \frac{\partial}{\partial x^{s}}\left(\frac{\partial[\sqrt{-g}\ f(R)]}{\partial g_{mp,s}}\right) + \frac{\partial^{2}}{\partial x^{r}\,\partial x^{s}}\left(\frac{\partial[\sqrt{-g}\ f(R)]}{\partial g_{mp,sr}}\right) = 0 \quad (2.6)$$

We derived the equation of motion of the $f(R)$ gravity using the Euler-Poisson equation in Eq. (2.6) [9].

(3). Derivation of the Hamiltonian of the $f(R)$ gravity

From the Euler-Poisson equation for a single particle given in Eq. (2.1), we can construct the Hamiltonian using the Legendre transformation

$$\begin{aligned} H &= p\,\dot{q} + \left(\frac{\partial L}{\partial \ddot{q}}\right)\ddot{q} - L \\ &= p_{(q)}\,\dot{q} + p_{(\dot{q})}\,\ddot{q} - L \end{aligned} \quad (3.1)$$

Where we have used the notation in the boundary term in Eq. (2.2), in which we defined the conjugate momenta $p_{(q)}$ and $p_{(\dot{q})}$ corresponding to the conjugate coordinate variable q and $\dot{q}$ respectively.

The changes in Eq. (2.3) result in changing the following variables:

$$p_{(q)} = \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}}\right) \Rightarrow\Rightarrow p^{mps}{}_{(g_{mp}(x^{e}))} = \frac{\partial L}{\partial g_{mp,s}(x^{e})} - \frac{\partial}{\partial x^{r}}\left(\frac{\partial L}{\partial g_{mp,sr}(x^{e})}\right),$$

$$\frac{\partial L}{\partial \ddot{q}} \Rightarrow\Rightarrow \frac{\partial L}{\partial g_{mp,sr}(x^{e})};$$

$$H = p\,\dot{q} + \left(\frac{\partial L}{\partial \ddot{q}}\right)\ddot{q} - L \Rightarrow\Rightarrow H = p^{mps}{}_{(g_{mp}(x^{e}))}\, g_{mp,s}(x^{e}) + \frac{\partial L}{\partial g_{mp,sr}(x^{e})}\, g_{mp,sr}(x^{e}) - L$$

and

$$L(t,\ q(t),\ \dot{q}(t),\ \ddot{q}(t)) \Rightarrow\Rightarrow L(x^{e},\ g_{mp}(x^{e}),\ g_{mp,s}(x^{e}),\ g_{mp,sr}(x^{e})).$$

By considering a local coordinate system in which $g_{mp,s}(x^{e}) = 0$ and for which $\frac{\partial L}{\partial g_{mp,s}(x^{e})} = 0$, the momentum tensor and the Hamiltonian of the any Lagrangian functional L are then given by

$$p^{mps}{}_{(g_{mp}(x^e))} = -\frac{\partial}{\partial x^r}\left(\frac{\partial L}{\partial g_{mp,sr}(x^e)}\right)$$
$$H = \frac{\partial L}{\partial g_{mp,sr}(x^e)}\ g_{mp,sr}(x^e) - L \qquad (3.2)$$

Substituting $L = \sqrt{-g}\ f(R)$, the momentum tensor becomes

$$\begin{aligned} p^{mps}{}_{(g_{mp}(x^e))} &= -\frac{\partial}{\partial x^r}\left(\frac{\partial L}{\partial g_{mp,sr}(x^e)}\right) = -\frac{\partial}{\partial x^r}\left(\frac{\partial[\sqrt{-g}\ f(R)]}{\partial g_{mp,sr}(x^e)}\right) \\ &= -\frac{\partial}{\partial x^r}\left\{\left(\sqrt{-g}\ \frac{\partial f(R)}{\partial g_{mp,sr}(x^e)}\right) + \left(f(R)\ \frac{\partial[\sqrt{-g}\,]}{\partial g_{mp,sr}(x^e)}\right)\right\} \\ &= -\frac{\partial}{\partial x^r}\left\{\left(\sqrt{-g}\ \frac{df(R)}{dR}\ \frac{\partial R}{\partial g_{mp,sr}(x^e)}\right) + \left(f(R)\ (0)\right)\right\} \qquad (3.3) \\ &= -\frac{\partial}{\partial x^r}\left(\sqrt{-g}\ \frac{df(R)}{dR}\ \frac{\partial R}{\partial g_{mp,sr}(x^e)}\right) \\ &= -\frac{\partial}{\partial x^r}\left(\sqrt{-g}\ f_R(R)\ \frac{\partial R}{\partial g_{mp,sr}(x^e)}\right) \end{aligned}$$

Recall the Ricci scalar R may written in terms of the metric tensor and its partial derivatives by

$$\begin{aligned} R = {} & \frac{1}{2}\ g^{ab}\ g^{ch}\ [(g_{ch,ab} - g_{ac,hb}) - (g_{bh,ac} - g_{ab,hc})] \\ & - (1/4)\ g^{ab}\ g^{ce}\ g^{hq}\ [(g_{ah,c} + g_{ch,a} - g_{ac,h})\ (g_{bq,e} + g_{eq,b} - g_{be,q}) \qquad (3.4) \\ & - (g_{ah,b} + g_{bh,a} - g_{ab,h})\ (g_{cq,e} + g_{eq,c} - g_{ce,q})] \end{aligned}$$

The Ricci scalar in local coordinates like a geodesic coordinate system [4], a local inertial frame [5], or a Riemann Normal Coordinate system [7], which all are characterized by

$$\frac{\partial g_{ab}}{\partial x^c} = 0, \qquad \Gamma_{abc} = 0, \qquad \Gamma_{ab}{}^{c} = 0 \quad (3.5)$$

And

$$\frac{\partial}{\partial x^{c}}\left(\frac{\partial g_{ab}}{\partial x^{d}}\right)\neq 0, \qquad \frac{\partial \Gamma_{abc}}{\partial x^{d}}\neq 0, \qquad \frac{\partial \Gamma_{ab}{}^{c}}{\partial x^{d}}\neq 0 \quad (3.6)$$

Where Γ_{abc} is the Christoffel symbol of the first kind so, the Ricci Scalar in Eq. (3.4) may be rewritten as

$$R=\frac{1}{2}\ g^{ab}\ g^{ch}\ [(g_{ch,ab}-g_{ac,hb})-(g_{bh,ac}-g_{ab,hc})] \quad (3.7)$$

In which the first partial derivate of the metric tensor vanishes.
Since $(a,\ b,\ c,\ h)$ are dummy indices ($\equiv$summed over), the Ricci Scalar R in Eq. (3.7) may as well be rewritten as

$$\begin{aligned}
R &=\frac{1}{2}\ g^{ab}\ g^{ch}\ [(g_{ch,ab}-g_{ac,hb})-(g_{bh,ac}-g_{ab,hc})] \\
&=\frac{1}{2}\ g^{ab}\ g^{ch}\ (g_{ch,ab}-g_{ac,hb}-g_{bh,ac}+g_{ab,hc}) \\
&=\frac{1}{2}\ (g^{ab}\ g^{ch}\ g_{ch,ab}-g^{ab}\ g^{ch}\ g_{ac,hb}-g^{ab}\ g^{ch}\ g_{bh,ac}+g^{ab}\ g^{ch}\ g_{ab,hc}) \\
&=\frac{1}{2}\ (g^{ab}\ g^{ch}\ g_{ch,ab}-g^{ab}\ g^{ch}\ g_{ac,hb}-g^{ba}\ g^{hc}\ g_{ac,bh}+g^{ch}\ g^{ab}\ g_{ch,ab}) \quad (3.8) \\
&=\frac{1}{2}\ (2\ g^{ab}\ g^{ch}\ g_{ab,ch}-2\ g^{ab}\ g^{ch}\ g_{ac,bh}) \\
&=\ (g^{ab}\ g^{ch}\ g_{ab,ch}-g^{ab}\ g^{ch}\ g_{ac,bh}) \\
&=\ g^{ab}\ g^{ch}\ (g_{ab,ch}-g_{ac,bh})
\end{aligned}$$

This is resulting from making the indices changes $(a\rightarrow c,\ b\rightarrow h)$ in the first term and $(a\rightarrow b,\ c\rightarrow h)$ in the second term, respectively.
To determine the various derivatives in the three terms in the brackets in Eq. (2.6) in the Euler-Lagrange equation, we make use of the derivation of Einstein field equation from Einstein-Hilbert Lagrangian [4-7] in which

$$\frac{\delta\ R}{\delta\ g^{ab}}=R_{ab} \quad (3.9)$$

Which when written as

$$\delta\ R=\frac{\partial R}{\partial g^{ab}}\ \delta g^{ab} \quad (3.10)$$

Implies

$$\frac{\partial R}{\partial g^{ab}} = R_{ab} \quad (3.11)$$

Since we are using the covariant metric tensor $g_{mp}(x^e)$, we may transform the Eq. (3.11) to be rewritten in terms of the covariant metric tensor as

$$\frac{\partial R}{\partial g_{mp}} = \frac{\partial R}{\partial g^{ab}} \frac{\partial g^{ab}}{\partial g_{mp}} = R_{ab} \frac{\partial g^{ab}}{\partial g_{mp}} \quad (3.12)$$

Using the identity

$$\frac{\partial g^{ab}}{\partial g_{mp}} = - g^{am} g^{bp} \quad (3.13)$$

This is resulting from differentiating $g^{ab} g_{bc} = \delta^a{}_c$ with respect to g_{mp}.
We get the result of the derivative of the Ricci scalar with respect to the covariant metric tensor as

$$\frac{\partial R}{\partial g_{mp}} = \frac{\partial R}{\partial g^{ab}} (- g^{am} g^{bp}) = R_{ab} (- g^{am} g^{bp}) = (- g^{am} g^{bp} R_{ab}) = - R^{mp} \quad (3.14)$$

Since we are considering local coordinates in which the Christoffel symbols of both kinds and the first derivative of the metric tensor vanish and do not appear in R expression in Eq. (3.8), this gives

$$\frac{\partial R}{\partial g_{mp,s}} = 0 \quad (3.15)$$

The derivative of Ricci scalar in Eq. (3.8) with respect to second derivative of the metric tensor with respect to the coordinates is given by

$$
\begin{aligned}
\frac{\partial R}{\partial g_{mp,sr}} &= \frac{\partial}{\partial g_{mp,sr}} \left[g^{ab}\ g^{ch}\ (g_{ab,ch} - g_{ac,bh}) \right] = g^{ab}\ g^{ch}\ \frac{\partial}{\partial g_{mp,sr}} \left(g_{ab,ch} - g_{ac,bh} \right) \\
&= g^{ab}\ g^{ch} \left(\frac{\partial}{\partial g_{mp,sr}} (g_{ab,ch}) + \frac{\partial}{\partial g_{mp,sr}} (-\ g_{ac,bh}) \right) \\
&= g^{ab}\ g^{ch} \left(\frac{\partial}{\partial g_{mp,sr}} (g_{ab,ch}) - \frac{\partial}{\partial g_{mp,sr}} (g_{ac,bh}) \right) \\
&= g^{ab}\ g^{ch}\ [(\delta^{a}{}_{m}\ \delta^{b}{}_{p}\ \delta^{c}{}_{s}\ \delta^{h}{}_{r}) - \delta^{b}{}_{m}\ \delta^{h}{}_{p}\ \delta^{a}{}_{s}\ \delta^{c}{}_{r})] \\
&= g^{ab}\ g^{ch}\ (\delta^{c}{}_{m}\ \delta^{h}{}_{p}\ \delta^{a}{}_{s}\ \delta^{b}{}_{r} - \delta^{a}{}_{m}\ \delta^{c}{}_{p}\ \delta^{h}{}_{s}\ \delta^{b}{}_{r}) \\
&= (g^{ab}\ g^{ch}\ \delta^{c}{}_{m}\ \delta^{h}{}_{p}\ \delta^{a}{}_{s}\ \delta^{b}{}_{r}) - (g^{ab}\ g^{ch}\ \delta^{a}{}_{m}\ \delta^{c}{}_{p}\ \delta^{h}{}_{s}\ \delta^{b}{}_{r}) \\
&= (g^{mp}\ g^{sr}) - (g^{ms}\ g^{pr}) = (g^{mp}\ g^{sr} - g^{mr}\ g^{ps})
\end{aligned}
$$

(3.16)

The results of derivatives in Eqs. (3.14 - 3.16) are needed to derive the momentum tensor and the Hamiltonian in a local coordinate system. To obtain the momentum tensor we substitute Eqs. (3.14 - 3.16), in Eq. (3.3), which yields

$$
\begin{aligned}
p^{mps}{}_{(g_{mp}(x^{e}))} =& -\sqrt{-g}\ f_{R}(R)\ \frac{\partial}{\partial x^{r}} \left(\frac{\partial R}{\partial g_{mp,sr}(x^{e})} \right) - \sqrt{-g}\ \frac{\partial R}{\partial g_{mp,sr}(x^{e})}\ \frac{\partial}{\partial x^{r}} \left(f_{R}(R) \right) \\
& - f_{R}(R)\ \frac{\partial R}{\partial g_{mp,sr}(x^{e})}\ \frac{\partial}{\partial x^{r}} \left(\sqrt{-g} \right) \\
=& -\sqrt{-g}\ f_{R}(R)\ \frac{\partial}{\partial x^{r}} \left(g^{mp}\ g^{sr} - g^{mr}\ g^{ps} \right) \\
& - \sqrt{-g}\ \frac{\partial R}{\partial g_{mp,sr}(x^{e})}\ [\frac{df_{R}(R)}{dR} \frac{\partial}{\partial x^{r}} R] \\
& - f_{R}(R)\ \frac{\partial R}{\partial g_{mp,sr}(x^{e})}\ \frac{\partial}{\partial g_{mp}} \left(\sqrt{-g} \right)\ \frac{\partial}{\partial x^{r}} \left(g_{mp} \right) \\
=& -\sqrt{-g}\ f_{R}(R)\ (0) - \sqrt{-g}\ \left(g^{mp}\ g^{sr} - g^{mr}\ g^{ps} \right) f_{RR}(R)\ R_{,r} \\
& - f_{R}(R) \left(g^{mp}\ g^{sr} - g^{mr}\ g^{ps} \right) \frac{\partial}{\partial g_{mp}} \left(\sqrt{-g} \right)\ (0) \\
=& -\sqrt{-g}\ (g^{mp}\ g^{sr} - g^{mr}\ g^{ps})\ f_{RR}(R)\ R_{,r} \\
=& -\sqrt{-g}\ (g^{mp}\ g^{sr}\ R_{,r} - g^{mr}\ g^{ps}\ R_{,r})\ f_{RR}(R) \\
=& -\sqrt{-g}\ (g^{mp}\ R^{,s} - g^{ps}\ R^{,m})\ f_{RR}(R)
\end{aligned}
$$

(3.17)

Contracting Eq. (3.17) - by multiplying both sides with g_{mp}, we get the momentum vector

$$g_{mp}\ p^{mps}{}_{(g_{mp}(x^e))} = p^{m}{}_{m}{}^{s}{}_{(g_{mp}(x^e))} = p^{s}{}_{(g_{mp}(x^e))}$$
$$= g_{mp}\left(-\sqrt{-g}\ (g^{mp}\ R^{,s} - g^{ps}\ R^{,m})\ f_{RR}(R)\right)$$
$$= -\sqrt{-g}\ (g_{mp}\ g^{mp}\ R^{,s} - g_{mp}\ g^{ps}\ R^{,m})\ f_{RR}(R)$$
$$= -\sqrt{-g}\ (4\ R^{,s} - \delta^{s}{}_{m}\ R^{,m})\ f_{RR}(R)$$
$$= -\sqrt{-g}\ (4\ R^{,s} - R^{,s})\ f_{RR}(R)$$
$$= -\sqrt{-g}\ (3\ R^{,s})\ f_{RR}(R)$$
$$= -3\ \sqrt{-g}\ R^{,s}\ f_{RR}(R)$$

(3.18)

The momentum vector $p^{s}{}_{(g_{mp}(x^e))}$ vanishes for linear $f(R)$ gravity for example the Einstein-Hilbert Lagrangian $f(R) = \frac{1}{2\chi} R$, and also vanishes for constant Ricci scalar. Moreover, we derive the Hamiltonian as follows

$$H = \frac{\partial L}{\partial g_{mp,sr}(x^e)}\ g_{mp,sr}(x^e) - L$$
$$= \frac{\partial[\sqrt{-g}\ f(R)]}{\partial g_{mp,sr}(x^e)}\ g_{mp,sr}(x^e) - \sqrt{-g}\ f(R)$$
$$= \left(\sqrt{-g}\ \frac{\partial f(R)}{\partial g_{mp,sr}(x^e)} + f(R)\ \frac{\partial[\sqrt{-g}\,]}{\partial g_{mp,sr}(x^e)}\right) g_{mp,sr}(x^e) - \sqrt{-g}\ f(R)$$
$$= \left(\sqrt{-g}\ \frac{df(R)}{dR}\ \frac{\partial R}{\partial g_{mp,sr}(x^e)} + f(R)\ (0)\right) g_{mp,sr}(x^e) - \sqrt{-g}\ f(R)$$
$$= \sqrt{-g}\ f_R(R)\ \frac{\partial R}{\partial g_{mp,sr}(x^e)}\ g_{mp,sr}(x^e) - \sqrt{-g}\ f(R)$$

(3.19)

Then substituting Eq. (3.16) in Eq. (3.19), we get

$$H = \sqrt{-g}\; f_R(R)\; (g^{mp}\; g^{sr} - g^{mr}\; g^{ps})\; g_{mp,sr}(x^e) - \sqrt{-g}\; f(R) \quad (3.20)$$

The expression $(g^{mp}\; g^{sr} - g^{mr}\; g^{ps})\; g_{mp,sr}(x^e)$ may be written as

$$\begin{aligned}(g^{mp}\; g^{sr} - g^{mr}\; g^{ps})\; g_{mp,sr}(x^e) \\ &= g^{mp}\; g^{sr}\; g_{mp,sr}(x^e) - g^{mr}\; g^{ps}\; g_{mp,sr}(x^e) \\ &= g^{mp}\; g^{sr}\; g_{mp,sr}(x^e) - g^{mp}\; g^{sr}\; g_{mr,ps}(x^e) \\ &= g^{mp}\; g^{sr}\; [g_{mp,sr}(x^e) - g_{mr,ps}(x^e)]\end{aligned} \quad (3.21)$$

However, from Eq. (3.8), $g^{mp}\; g^{sr}\; [g_{mp,sr}(x^e) - g_{mr,ps}(x^e)] = R$ - is the Ricci scalar curvature, - so H becomes

$$\begin{aligned}H &= \sqrt{-g}\; f_R(R)\; R - \sqrt{-g}\; f(R) \\ &= \sqrt{-g}\; [f_R(R)\; R - f(R)]\end{aligned} \quad (3.22)$$

This is a local Hamiltonian of the $f(R)$ gravity, since it is a scalar; it has the same value in any system of coordinates.
In the dynamics of a closed system of single particle or scalar field the Hamiltonian is interpret as the "energy" of the system, we assume that the Hamiltonian given in Eq. (3.22) is to the energy of the $f(R)$ gravity.
Now, we prove the Hamiltonian in Eq. (3.22) vanishes for linear Lagrangians. Substituting the Einstein-Hilbert Lagrangian $f(R) = \frac{1}{2\chi}\; R$ and it its first derivative with respect to R (i.e.$f_R(R) = \frac{1}{2\chi}$) in the Hamiltonian in Eq. (3.22), we get

$$\begin{aligned}H &= \sqrt{-g}\; [f_R(R)\; R - f(R)] \\ &= \sqrt{-g}\; [(\frac{1}{2\chi})\; R - \frac{1}{2\chi}\; R] = 0\end{aligned} \quad (3.23)$$

(4). Stress-Energy tensor of the Hilbert-Einstein Lagrangian

The stress-energy momentum tensor of a source field - not including gravitational field energy- is defined as [8]

$$T_{ab} = -2 \frac{\delta L_M}{\delta g^{ab}} + g_{ab} L_M \quad (4.1)$$

Where L_M is the matter and energy Lagrangian.

The Hilbert-Einstein Lagrangian $f(R) = \frac{1}{2\chi} R$ does not contain any matter and energy source, so the left hand side of Eq. (4.1) should vanish when the Hilbert-Einstein Lagrangian is substituted in Eq. (4.1)

$$\begin{aligned} T_{ab} &= -2 \frac{\delta (\frac{1}{2\chi} R)}{\delta g^{ab}} + g_{ab} (\frac{1}{2\chi} R) \\ &= -2 \frac{1}{2\chi} \frac{\delta R}{\delta g^{ab}} + \frac{1}{2\chi} g_{ab} R \qquad (4.2) \\ &= \frac{1}{2\chi} \left(-2 \frac{\delta R}{\delta g^{ab}} + g_{ab} R \right) \end{aligned}$$

Inserting the identity in Eq. (3.11) in Eq. (4.2), we get

$$\begin{aligned} T_{ab} &= \frac{1}{2\chi} \left(-2 R_{ab} + g_{ab} R \right) \\ &= \frac{1}{2\chi} \left[-2 \left(R_{ab} - \frac{1}{2} g_{ab} R \right) \right] \end{aligned} \quad (4.3)$$

However, the expression in the bracket in the right hand side in Eq. (4.3) is the Einstein field equations in absence of a source of matter and energy derived from the variation of the Einstein-Hilbert action. The expression vanishes so does the stress-energy tensor in the left hand side of Eq. (4.3)

5. Conclusion

When the Hamiltonian of source free $f(R)$ gravity is interpreted as energy it vanishes for linear Lagrangians in Ricci scalar curvature without source, which is the same result, obtained using the stress-energy tensor equation.

6. References

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