Resolvement of the St. Petersburg Paradox and Improvement of Pricing Theory

Dahang Li

Abstract

The St. Petersburg Paradox was proposed before two centuries. In the paper we proposed a new pricing theory with several rules to solve the paradox and state that the fair pricing should be judged by buyer and seller independently. The pricing theory we proposed can be applied to financial market to solve the confusion with fat tails.

Index Terms

St. Petersburg Paradox, Pricing theory

I. Introduction

The St. Petersburg Paradox is still an open issue. The infinite expected value of the St. Petersburg Paradox has been a source of contention within probability theory since its inception in the early 18th century [1]. The St. Petersburg Game is that: Peter tosses a fair coin repeatedly until it shows heads. He agrees to pay Paul two ducats if it shows heads on the first toss, four ducats if the first head appears on the second toss, eight ducats if the first head appears on the third toss, sixteen if on the fourth toss, etc. How much should Peter charge Paul as an entrance fee to this game so that the game will be fair? To determine the amount Peter should charge Paul as an entrance fee so that the St. Petersburg Game will be fair, we calculate Paul's expected income. Let k be any positive integer, then the probability that the game ends at the kth toss is $p(x_k) = 2^{-k}$, at which time Peter will pay Paul $x_k = 2^k$ ducats. Let E denote the expected value of Peter's payout, therefore

$$E = \sum_{k=1}^{\infty} x_k p(x_k) = \sum_{k=1}^{\infty} 2^k 2^{-k} = \sum_{k=1}^{\infty} 1 = \infty \tag{1}$$

proves that no finite amount of money can be a fair entrance fee. In short, Paul should be willing to pay an infinite price to enter this game. However, almost no rational person would agree to do that. People will only pay what they think is a moderate fee, and the fee is very limited, in fact, it is generally no more than 20 ducats. That is the St. Petersburg Paradox.

II. THE ST. PETERSBURG GAME CAN'T BE PRICED AS A LIMITED VALUE

Several approaches have been proposed for solving the paradox, such as Expected utility theory, Probability weighting, Rejection of mathematical expectation, Finite St. Petersburg lotteries. These views are very enlightening, but they all are not so good. And they are basically non quantitative descriptive explanations, so it is difficult to apply them directly to the actual decision-making.

At present, it is generally believed that William Feller offers a mathematical correct solution involving sampling [2]. It can be understood intuitively to "perform this game with a large number of people and calculate the expected value from the sample extraction". In this method, when the games of infinite number of times are possible, the expected value would be infinity, and in the case of finite, the expected value will be a much smaller value. Accurately, when the games times is N, the fair price should be $\log_2 N$. William Feller did a good job in quantifying the fair price of the St. Petersburg Game. We come to a different conclusion from other perspectives. First, take a counter example to prove its irrationality: according to the statement, if Peter (banker) lets Paul (player) play 1024 games, it's fair for Peter to charge $\log_2(1024) = 10$ ducats each time, and it's also fair for Paul to pay 10 ducats each time. The problem is, if Paul plays 1024 games at the banker Peter, and then he goes to another banker Tom to play another 1024 games. Well, Paul should make a lot of money (Because the game is worth 11 ducats each time for playing 2048 times). But neither Peter nor Tom will lose. How does this work? Therefore, it is not appropriate that the so-called fair pricing is related to the number of times player plays. Next, I will strictly prove that the reasonable quotation of the St Petersburg game's banker should indeed be infinite (that is, the banker can't quote). Here, we use 1 to represent the heads side of the coin that was tossed, and 0 to represent the other side. The process of playing the St Petersburg Game once can be represented by a binary number string with infinite bits in which each digit is independent and random. If the nth digit from left to right is the first digit 1, it means that the first head appears on the nth toss, Paul can get 2^n ducats. The numbers after the nth digit do not affect the results of the game. Now, define a very simple Lottery Game K, where K is a positive integer . The lottery number is a binary number sequence with infinite digits , and each digit is independent, identical distribution random variable and selected from $\{0, 1\}$ with equal probability. If the first K digits of the lottery are $\overbrace{0 \cdots 0}^{K-1} 1$, the player wins 2^K ducats. Otherwise, the player gets nothing. For $E(x) = 1/2^K \times 2^K = 1$, the fair price of this lottery game is 1 ducat. Now, take the same binary number string as the process of the St Petersburg Game, and as the lottery number for Lottery Game 1, Lottery Game 2, Lottery Game 3, and so on as well. Every lottery game that player plays is worth 1 ducat. Therefore, you cannot play this series of unlimited lottery games at any limited price. In addition, it is important that for the same string of numbers, the income from playing a St. Petersburg Game once is equal to the total income from playing this series of lottery games. Therefore, you can't play the St Petersburg Game with a limited price.

The result is proven.

III. The New Pricing Theory

The St. Petersburg Paradox seems to be back to its original point. Now let's put the St. Petersburg Paradox aside and think about whether we can improve the pricing theory generally. For a discrete event, the buyer's return is a discrete random variable $X = [x_1, x_2, x_3, \cdots]$, the probability mass function $p(x) = P(X = x), x \in X$. The expected value $E(X) = \sum_{i=1}^{\infty} x_i p(x_i)$ According to the conventional pricing theory, it will be considered fair for the seller to price the event as $\mu = E(X)$. For the continuous random variable $X = f(x)$, it is similar. The current pricing theory will suffer from the inability to explain the St. Petersburg Paradox, and the inability to price options with certain fat tails distributions. In our paper, a new pricing theory and several pricing rules are proposed, which can solve above problems.

The gist of our new pricing theory is that a fair offer by the seller is not necessarily an offer that the buyer is willing to accept. Buyers and sellers have their own decision-making mechanisms.

Rule 1: For the quotation μ offered by the seller, the buyer shall judge whether the quotation is acceptable according to the following process. At first, we define two parameters:

1) Hopeless Probability $\epsilon \in [0, 1]$: let ϵ denote the probability that the buyer will ignore. More formally, the buyer ignores the possibility that the probability of some opportunities is not greater than ϵ, and does not pay for such opportunities.
2) Cost-effectiveness Factor k : let k denote the buyer's investment preference. k=1 means that the buyer seeks fair dealing, and $k < 1$ means that the buyer seeks stable profit opportunities (such as the wrong pricing of the seller), and $k > 1$ means that the buyer is speculating (such as gaming).

Specifically, every buyer can choose his own ϵ and k values. Suppose that N_ϵ is the minimum positive integer meeting:

$$\sum_{i=N_\epsilon+1}^{\infty} p(x_i) \leq \epsilon \tag{2}$$

since

$$\sum_{i=1}^{N_\epsilon} p(x_i) + \sum_{i=N_\epsilon+1}^{\infty} p(x_i) = \sum_{i=1}^{\infty} p(x_i) = 1. \tag{3}$$

we have

$$\sum_{i=1}^{N_\epsilon} p(x_i) = 1 - \sum_{i=N_\epsilon+1}^{\infty} p(x_i) \geq 1 - \epsilon \tag{4}$$

and let

$$E_\epsilon = \sum_{i=1}^{N_\epsilon} x_i p(x_i) \tag{5}$$

then $\mu$is an acceptable price for the buyer if $\mu \leq kE_\epsilon$.

Rule 2: In the case that the contract must be executed at the end, the seller determines the quotation μ with the expected value just like traditional pricing theory. That is

$$\mu \geq E(X) = \sum_{i=1}^{\infty} x_i p(x_i) \tag{6}$$

Rule 3: In the case that the seller can close the contract during the process (such as stock option), the seller uses a similar method as the buyer in Rule 1 to determines the quotation μ. thus, we have

$$\epsilon \in [0, 1] \xrightarrow{(2)} N_\epsilon \xrightarrow{(5)} E_\epsilon \tag{7}$$

and let $\mu = kE_\epsilon, k \geq 1$ for gaining profit. When $\epsilon = 0$ and $k = 1$, we have $\mu = E$, this is the conventional fair price. When $\epsilon > 0$, the seller should choose $k > 1$, because he bears the additional risk of ϵ without calculating the cost, so he is also gambling. The premise that sellers can use Rule 3 is that the exchange has a margin system for risk control.

IV. Resolve The St. Petersburg Paradox

Based on the pricing rules proposed in section III, the seller's pricing should be infinite for the St. Petersburg game. That is, the seller cannot provide appropriate price for the game to the buyer. This conclusion has been proved in section II, but it does not explain most people's doubts.

Many researchers, such as Hayden and Camerer, had discussed how much the St. Petersburg Game is worth paying for [1] [3]. Usually, the values are small, and almost do not exceed 20 ducats.

Let's first look at the American Powerball, with a bet of two dollars, and the odds of winning the jackpot are one in 292.2 million. And in 2019, Jackpot Amount averaged about 171 million.

Back to the St. Petersburg Game, it is appropriate to calculate how much buyers pay rationally based on Rule 1.

Let $\epsilon = \frac{1}{2^{28}}$, we have

$$\frac{1}{2^{28}} = \epsilon \geq \sum_{i=N_\epsilon+1}^{\infty} p(x_i) = \sum_{i=N_\epsilon+1}^{\infty} 2^{-i} = 2^{-N_\epsilon} \tag{8}$$

Hence,$N_\epsilon = 28$,we will omit those bonuses of 2^{29},2^{30},2^{31},$\cdots$

$$\begin{aligned} p(X_{N_\epsilon}) = 2^{-28} = \frac{1}{268435456} > \frac{1}{292200000} \\ (X_{N_\epsilon}) = 2^{28} = 268435456 > 1.71 \times 10^8 \end{aligned} \tag{9}$$

It can be seen from the Powerball lottery that people are willing to bet for 1.71×10^8 jackpots for $1/292200000$ chances. However, in fact, people are willing to buy Powerball lottery but not to play The St. Petersburg Game. An important difference between the Powerball lottery and the St. Petersburg Game is that Powerball lottery only costs 2 (dollars), while St. Petersburg Game is 28 (ducats). With close odds and amount of jackpot, the St. Petersburg Game costs a lot more. In this regard, it can only be said that the St. Petersburg Game is poorly designed, with a large bet amount, everyone wins, and has not concentrated the cost on the players' attention. What sellers price based on cost is not necessarily what buyers like. Moreover, for gambling, it is not very important whether the cost-effectiveness factor k is less than 1. The St. Petersburg Game's $k = 1$, the Powerball's $k > 1$, but people prefer to buy Powerball.

V. Fat tails and option pricing

The correct description of asset return distribution is directly related to the correctness of portfolio selection, the effectiveness of risk management and the rationality of option pricing. In the classical Efficient Market Hypothesis, stock returns are usually assumed to follow the normal distribution, the ends of the curve distribution are thin tails. But in reality, the markets don't behave this way. The possibility of extreme events occurring similar to the financial crisis appears much more frequently than imagined, and the empirical distribution of returns has obvious fat tails. There are two wonderful passages in Taleb's book [4]: ”There is an example academic literature trying to maintain us that options are not rational to own because some options are overpriced, and they are embedded overpriced according to business school methods of computing risks that do not take into account the possibility of rare events” and ”Further, casino bets and lottery tickets also have a known maximum upside – in real life, the sky is often the limit, and the difference between the two cases can be significant. Risk taking ain't gambling, and optionality ain't lottery tickets”. Coincidentally, this paper also considers lottery and options together. The conventional financial theories have been challenged for their inability to realistically explain risk. The financial asset returns do not simply follow the Geometric Brownian Motion. Officer, Benoit Mandelbrot [5], Eugene F. Fama [6], and Stanley J.Kon [7] all report evidence that stock returns are not consistent with the Random Walk Theory. It is of great significance to describe the statistical characteristics of the distribution of financial asset returns accurately, which is the premise of the correct pricing of options. Finance engineers often choose the Lévy Distribution to model price changes in markets.The 'fat tail' or slow fall off that this distribution models is a good match for what happens after prices change. The normal (Gaussian) distribution is a special case of a Lévy distribution, for which $\alpha = 2$. And one case of a non-Gaussian Levy distribution is the Cauchy distribution, for which $\alpha = 1$. The Lévy Stable Distribution ($1 \leq \alpha < 2$) implies infinite variance. Having an infinite variance does not prevent a distribution from becoming quite proper, but it does make it quite peculiar. In standard financial theory, volatility is the most important parameter. Volatility is used in the financial calculation of risk and option pricing. Volatility σ is usually measured by the standard deviation of the return of a security or market index. The variance of Lévy Stable Distribution $\sigma^2 = \infty$ will make the option pricing formula give meaningless answers. For example, for Cauchy distribution, its density function is

$$p(x) = \frac{1}{\pi(x^2+1)} \tag{10}$$

Calculating the European call option price C, where $S(T)$ is the price of the stock at time T, K is the exercise price, and the risk-free rate of return is r, then

$$\begin{aligned}
C &= e^{-rT}E[\max(S(T)-K,0)] \\
&= e^{-rT}\int_K^{\infty}(x-K)p(x)dx = e^{-rT}\int_K^{\infty}(x-K)\frac{1}{\pi(x^2+1)}dx \\
&= e^{-rT}\left(\int_K^{\infty}\frac{x}{\pi(x^2+1)}dx - K\int_K^{\infty}\frac{1}{\pi(x^2+1)}dx\right) > e^{-rT}\left(\int_K^{\infty}\frac{x}{\pi(x^2+1)}dx - K\right) \\
&\geq e^{-rT}\left(\int_{\max(K,2)}^{\infty}\frac{x}{\pi(x^2+1)}dx - K\right) \geq e^{-rT}\left(\int_{\max(K,2)}^{\infty}\frac{x}{\pi(x^2+x)}dx - K\right) \\
&= e^{-rT}\left(\int_{\max(K,2)}^{\infty}\frac{1}{\pi(x+1)}dx - K\right) = e^{-rT}\left(\frac{\ln(x+1)}{\pi}|_{\max(K,2)}^{\infty} - K\right) = \infty
\end{aligned} \tag{11}$$

That's the problem.

Now, according to the new pricing theory (rules) proposed in section III, it is not important for $E[\max(S(T)-K,0)]$ and $E[\max(K-S(T),0)]$ to be finite.

When calculating the option price, select a sufficiently small $\epsilon[0,1]$ as the hopeless probability. And assume U satisfies:

$$\int_U^{\infty} p(x)dX \leq \epsilon \tag{12}$$

then

$$C = e^{-rT}\int_{-\infty}^{U}\max(x-K,0)p(x)dx = e^{-rT}\int_K^U (x-K)p(x)dx \tag{13}$$

Considering the actual situation of the real securities market, we can consider that ST<100*S is absolutely true, or that $U = 100S$ meets (12). So

$$C = e^{-rT}\int_K^{100S}(x-K)p(x)dx \tag{14}$$

Similarly, we have

$$P = e^{-rT}\int_{0.01S}^{K}(K-x)p(x)dx \tag{15}$$

Finally, the conclusion is that you don't need to worry about the probability density distribution causing the option pricing formula to fail. You can truncate the upper and lower limits during the integration operation.

VI. Conclusions

In this paper, we proposed a new Pricing Theory. The fair offer for the seller is not necessarily the offer that the buyer is willing to accept. The buyer and the seller have their own decision-making mechanisms. The seller is concerned about costs, and the buyer is concerned about the realistic prospect of returns. We also proposed three specific quantitative pricing rules. Moreover, our paper solves the St. Peter's paradox perfectly, and gives a method for pricing options of financial assets with fat tails distribution. In the next step, we think that there will be good prospects for studying the distribution model (with fat tails) of actual stock/index and the method of determining the Hopeless Probability ϵ.

References

[1] B. Hayden and M. Platt, "The mean, the median, and the st. petersburg paradox," Judgment and Decision Making, vol. 4, no. 4, pp. 256–272, Aug 2009.

[2] W. Feller, An Introduction to Probability Theory and Its Applications. John Wiley and Sons, Inc., Oct 1968, vol. 1, no. 3.

[3] C. Camerer, "psychological, theoretical, empirical — for loss aversion," Journal of Marketing Research, vol. 42, pp. 129–133, July 2005.

[4] Taleb and N. Nicholas, Antifragile: Things That Gain from Disorder, May 2012, no. 12.

[5] B. Mandelbrot, "The variation of certain speculative prices," Journal of Business, vol. 36, pp. 394–419, Sep. 1963.

[6] E. Fama, "The behavior of stock market prices," Journal of Business, vol. 33, pp. 34–105, Sep. 1965.

[7] S. J. Kon, "Models of stock returns-a comparison," Journal of Finance, vol. 39, no. 1, pp. 147–165, Dec 1984.