Maxwell's Equations: A Brief Exploration

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Abstract

We perform a brief analysis of Maxwell's equations in this writing to bring out a bewildering aspect that an infinitude of boundary conditions are possible for any given global source distribution of charges and currents. This concept is distinct from an infinitude of boundary conditions resulting from different source distributions. For a given source distribution inside a finite [or semi infinite] region an infinite number of boundary conditions might be possible for several distributions outside the enclosure. But for any global source distribution we do not expect an infinitude of boundary conditions. The analysis to follow lead us to certain contradictory features.

Introduction

For a specified source distribution inside a finite [or a semi-infinite] region an infinite number of boundary conditions[1] might be possible for several distributions outside the enclosure. But for any global source distribution we do not expect an infinitude of boundary conditions. Nevertheless in our analysis we derive the abnormal possibility of having an infinite number of boundary conditions for any global distribution of charges and currents. The analysis to follow lead us to certain contradictory features.

Maxwell's Equations and some Mathematical Consequences

We first write the traditional Maxwell's equations[2] in the SI system using conventional notations:

$$\nabla\vec{E}=\frac{\rho}{\varepsilon_0} \qquad (1.1)$$

$$\nabla\vec{B}=0 \qquad (1.2)$$

$$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \qquad (1.3)$$

$$\nabla\times\vec{B}=\mu_0\vec{J}+\varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} \qquad (1.4)$$

We may replace $\vec{E}$ by $\vec{E}$+∇λand $\vec{B}$ by $\vec{B} + \nabla\chi$ with the condition that λ and χ *are time independent* scalars and that they satisfy Laplace's

equation:

1) $\nabla^2\lambda = 0$ (2.1)
2) $\nabla^2\chi = 0$ (2.2)

[Prime below does not denote differentiation but transformation]
Using (1.1) and (2.1) we have:
$\nabla\overrightarrow{\mathrm{E}'} = \nabla(\vec{\mathrm{E}} + \nabla\lambda) = \nabla\,\vec{\mathrm{E}} + \nabla^2\lambda = \nabla\vec{E} = \frac{\rho}{\varepsilon_0}$since$\nabla^2\lambda = 0$
Using (1.2) and (2.2) we have
$\nabla\overrightarrow{\mathrm{B}'} = \nabla(\vec{\mathrm{B}} + \nabla\chi) = \nabla\vec{B} + \nabla^2\chi = 0$since$\nabla^2\chi = 0$
Using $\nabla\times\nabla\lambda = 0$ and $\nabla\chi$ independent of time we have
$\nabla\times\overrightarrow{\mathrm{E}'} = \nabla\times(\vec{E} + \nabla\lambda) = -\frac{\partial(\vec{B}+\nabla\chi)}{\partial t} = -\frac{\partial\vec{B}'}{\partial t}$since$\lambda$ is independent of time

$$\nabla\times\vec{\mathrm{B}}' = \nabla\times(\vec{\mathrm{B}} + \nabla\chi) = \nabla\times\vec{B} + \nabla^2\chi = \nabla\times\vec{B}$$

[Since $\nabla^2\chi = 0$]
Or,
Using $\nabla\times\nabla\chi = 0$ and $\nabla\lambda$ independent of time we have

$$\nabla\times\vec{\mathrm{B}}' = \nabla\times(\vec{B} + \nabla\chi) = \mu_0\vec{J} + \varepsilon_0\mu_0\frac{\partial(\vec{E} + \nabla\lambda)}{\partial t} = \mu_0\vec{J} + \varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}$$

[since$\lambda$is independent of time]
We have :
If $\vec{E}$,$\vec{B}$ are solutions to Maxwell's equations for configuration inside a region,
$\vec{E}' = \vec{E} + \nabla\lambda$and$\vec{B}' = \vec{B} + \nabla\chi$ will also be solutions for the same source configuration

Provided

1) $\nabla^2\lambda = 0$
2) $\nabla^2\chi = 0$
3) And λ and χ are time independent quantities.

It is an inherent fact in our transformations that charge density and current densities remain unaltered by these transformations that is by $\vec{E}' = \vec{E} + \nabla\lambda$and$\vec{B}' = \vec{B} + \nabla\chi$, λ and χ being time independent.

Indeed

$$\nabla\vec{E} = \frac{\rho}{\varepsilon_0} \Rightarrow \rho = \varepsilon_0 \nabla\vec{E} \ \ (3)$$

On transformation

$$\rho' = \varepsilon_0 \nabla\vec{E}' = \varepsilon_0 \vec{\nabla}\left(\mathrm{E} + \vec{\nabla}\lambda\right) = \varepsilon_0 \vec{\nabla}\vec{\mathrm{E}} + \nabla^2\lambda$$

Since $\nabla^2\lambda = 0$ by our choice

$$\rho' = \varepsilon_0 \vec{\nabla}\vec{\mathrm{E}} = \rho \Rightarrow \rho' = \rho \ \ (4)$$

Again

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \ \ (5)$$

$$\vec{J} = \frac{1}{\mu_0}\left[\nabla \times \vec{B} - \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}\right] \ \ (6)$$

On transformation we have

$$\vec{J}' = \frac{1}{\mu_0}\left[\nabla \times \left(\vec{B} + \vec{\nabla}\chi\right) + \varepsilon_0 \mu_0 \frac{\partial\left(\vec{E} + \vec{\nabla}\lambda\right)}{\partial t}\right]$$

$$\vec{J}' = \frac{1}{\mu_0}\left[\nabla \times \vec{B} + \vec{\nabla} \times \vec{\nabla}\chi\left(+\vec{\nabla}\chi\right) + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} + \varepsilon_0 \mu_0 \frac{\partial\left(\vec{\nabla}\lambda\right)}{\partial t}\right]$$

Now $\vec{\nabla} \times \vec{\nabla}\chi = 0$ and $\frac{\partial(\vec{\nabla}\lambda)}{\partial t} = 0$ since λ is time independent

$$\vec{J}' = \frac{1}{\mu_0}\left[\nabla \times \vec{B} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}\right] \ \ (7)$$

From (6) and (7) we have,

$$\vec{J}' = \vec{J} \ \ (21)$$

Our transformations do not change the distribution of the sources asides maintaining Maxwell's equations[preserving their form].The functions representing charge and current densities do not change. We must keep in mind that our transformations are not the Lorentz transformations [the Lorentz transformations , incidentally, treat $(\rho, \vec{J})$ as a four vector]

With that in mind we calculate

$$\frac{\mathrm{Q_{extra}}}{\epsilon_0} = \oiint \left[\nabla\lambda - \frac{\partial \vec{\mathrm{A}}}{\partial \mathrm{t}}\right] . d\vec{S}$$

But

$$\mathrm{Q_{extra}} = 0$$

Therefore for any time independent λ satisfying $\nabla^2\lambda = 0$

$$\oiint \left[\vec{\nabla}\lambda - \frac{\partial\vec{\mathrm{A}}}{\partial \mathrm{t}}\right].d\vec{S} = 0$$

$\vec{\mathrm{A}}$ is the extra vector potential introduced by the transformation. Since both λ and χ are time independent the extra $\vec{\mathrm{A}}$ should be time independent: $\frac{\partial\vec{\mathrm{A}}}{\partial \mathrm{t}} = 0$ .Therefore for any time independent $\lambda(\mathrm{x}\,,\mathrm{y},\mathrm{z})$ with $\nabla^2\lambda = 0$

$$\oiint \vec{\nabla}\lambda.\,d\vec{S} = 0$$

irrespective of boundary conditions.If $\lambda = 0$ or a non zero constant on the boundary then $\lambda = 0$ or the same non zero constant everywhere inside by the effect of the uniqueness theorem. We maintain $\lambda \neq 0$ [or not non zero constant]on the boundary which could be at a finite distance or at an infinite distance. Therefore λ is a variable inside the boundary. It should not be zero or a non zero constant everywhere inside. Therefore, in general, $\vec{\nabla}\lambda \neq 0$ inside the boundary. We have sources inside or outside (for the finite case) the boundary that create the non zero extra field $\vec{\nabla}\lambda$. We are not supposed to have any extra source even globally, as discussed earlier, by our transformation [$\vec{E}_{extra} = \vec{\nabla}\lambda - \frac{\partial\vec{\mathrm{A}}}{\partial \mathrm{t}}$;but $\frac{\partial\vec{\mathrm{A}}}{\partial \mathrm{t}} = 0$since both λ and χ are time independent⇒ $\vec{E}_{extra} = \vec{\nabla}\lambda$]

Integral Laws and Sources on Transformation

With our substitutions[transformations]the macroscopic charge values and the currents remain unaltered. We may come to this conclusion in an equivalent manner[for the macroscopic case] by considering the Integral form of the laws: Gauss law and Ampere's circuital law.

$$\oiint \vec{E}\,.\,d\vec{S} = \frac{q}{\epsilon_0} \quad (8)$$

Since Maxwell's equations are preserved for our transformations, we have,

$$\oiint \vec{E}^{'}\,.\,d\vec{S} = \frac{q'}{\epsilon_0} \; (9)$$

$$\frac{q'}{\epsilon_0} = \oiint \vec{E}'\,.\,d\vec{S} = \oiint \left(\vec{E} + \vec{\nabla}\lambda\right).\,d\vec{S} = \oiiint \vec{\nabla}\left(\vec{E} + \vec{\nabla}\lambda\right).\,d\vec{S} = \oiiint \left(\overrightarrow{\nabla.E} + \nabla^2\lambda\right).\,d\vec{S}$$

$$\oiint \vec{E}'\,.\,d\vec{S} = \oiiint \overrightarrow{\nabla.E}.\,d\vec{S} = \oiint \vec{E}\,.\,d\vec{S} = \frac{q}{\epsilon_0}\ (10)$$

For our transformations, for any arbitrary volume,

$$q = q'\ (11)$$

We come to Ampere's circuital law

$$\oint \vec{B}.\,d\vec{l} = \mu_0 i \quad (12)$$

Since Maxwell's equations are preserved on transformations we have

$$\oint \vec{B}'.\,d\vec{l} = \mu_0 i' (13)$$

$$\mu_0 i' = \oint \overrightarrow{B'}.\,d\vec{l} = \oint(\vec{B} + \vec{\nabla}\lambda).\,d\vec{l} = \oint \vec{B}\,.\,d\vec{l} + \oint \vec{\nabla}\lambda\,.\,d\vec{l} = \iint \vec{\nabla} \times \vec{B}.\,d\vec{S} + \iint \vec{\nabla}\vec{\nabla}\lambda \times .\,d\vec{S}$$
$$= \iint \vec{\nabla} \times \vec{B}.\,d\vec{S}$$

$$\mu_0 i' = \oint \overrightarrow{B'}.\,d\vec{l} = \iint \vec{\nabla} \times \vec{B}.\,d\vec{S} = \oint \vec{B}.\,d\vec{l} = \mu_0 i$$

For any arbitrary surface if we consider currents passing through it, we have,

$$i' = i \quad (14)$$

Our transformations change the values of $\vec{E}$ and $\vec{B}$ without disturbing the sources.

For a given source distribution we have an infinitude of $\left(\vec{E} + \vec{\nabla}\lambda, \vec{B} + \vec{\nabla}\chi\right)$ where $\vec{\nabla}\lambda$ and $\vec{\nabla}\chi$are time independent and also $\nabla^2\lambda = 0$ and $\nabla^2\chi = 0$. The bewildering aspect is that an infinitude of boundary conditions are possible for a given source distribution [distribution of charges and currents. We may consider as a particular instance the sources to be confined primarily to a finite region of space so that the fields $\vec{E}$ and $\vec{B}$ tending to zero at infinity but not becoming zero. In this situation an infinitude of values of $\vec{E}$ and $\vec{B}$ will exist at each point. If a part of the stated source distribution is enclosed by a surface we shall have an infinitude of boundary conditions on this surface the fields falling off to zero at an infinite distance.

Helmholtz Theorem

With reference to Helmholtz theorem[3] let us consider following the surface integral of the reference

$$\oiint \frac{C(\vec{r}')}{|\vec{r}-\vec{r}'|}dS$$

Sources at an infinite distance are not zero. For such points we do not have the approximation $|\vec{r}-\vec{r}'| \approx |\vec{r}|$. But these points are material to the context when we consider points on a surface of infinite radius. No matter how far the boundary is from the origin we may consider non zero sources[very small but non zero] sufficiently close to the boundary so that the approximation $|\vec{r}-\vec{r}'| \approx |\vec{r}|$ breaks down in a serious manner. Right on the surface $|\vec{r}-\vec{r}'| = 0$ for any non zero source density present. No matter how small this source density is, so long as it is non zero, we have an infinitely large values of the integrand at points where sources are present. There is a possibility of the integrand blowing up. The integral will either blowup or it will become a finite quantity. The 'strong' charges in the excluded regions might save the situation. We will not be able to write the equations we wrote to derive Helmholtz's theorem. There would be drastic modifications .We will not have the delta function in the integrands .

From reference [3]

$$\nabla\vec{F} = D(\vec{r})$$

$$\nabla \times \vec{F} = \vec{C}(\vec{r})$$

$$\vec{F} = -\nabla \mathrm{U} + \nabla \times \overrightarrow{\mathrm{W}}$$

If $\vec{F} \equiv \vec{E}$

$$\nabla\vec{E} = \frac{\rho}{\epsilon_0} = D(\vec{r})$$

$$\nabla \times \vec{E} = -\frac{\partial\vec{B}}{\partial t} = \vec{C}(\vec{r})$$

$$\vec{E} = -\nabla\phi - \frac{\partial\overrightarrow{\mathrm{A}}}{\partial \mathrm{t}} = -\nabla \mathrm{U} + \nabla \times \overrightarrow{\mathrm{W}}$$

$$\nabla(\mathrm{U}-\phi) = \frac{\partial\overrightarrow{\mathrm{A}}}{\partial \mathrm{t}} + \nabla \times \overrightarrow{\mathrm{W}}$$

$$\nabla^2(\mathrm{U}-\phi) = \frac{\partial\nabla\overrightarrow{\mathrm{A}}}{\partial \mathrm{t}} + \nabla.\nabla \times \overrightarrow{\mathrm{W}}$$

$\nabla\overrightarrow{\mathrm{A}} = 0$[Coulomb gauge: $\nabla\vec{F} = -\nabla^2 U$ in the reference assumes Coulomb gauge if we consider$\vec{F} \equiv \vec{E}$ ']

$$\nabla^2(\mathrm{U}-\phi) = 0$$

If both U and ϕ tend to zero st a large distance then $\mathrm{U} - \phi = 0 \Rightarrow \mathrm{U} = \phi$

We recall

$$\vec{E} = -\nabla\phi - \frac{\partial\vec{\mathrm{A}}}{\partial\mathrm{t}} = -\nabla\mathrm{U} + \nabla\times\vec{\mathrm{W}}$$

$$\nabla\times\vec{\mathrm{W}} = -\frac{\partial\vec{\mathrm{A}}}{\partial\mathrm{t}}$$

$$\frac{1}{4\pi}\nabla\times\int\frac{\mathrm{C}(\vec{\mathrm{r}}')}{|\vec{\mathrm{r}}-\vec{\mathrm{r}}'|}\mathrm{d}\vec{\mathrm{r}}' = -\frac{\partial\vec{\mathrm{A}}}{\partial\mathrm{t}}$$

$$\frac{1}{4\pi}\nabla\times\int\frac{1}{|\vec{\mathrm{r}}-\vec{\mathrm{r}}'|}\frac{\partial\vec{B}}{\partial t}\mathrm{d}\vec{\mathrm{r}}' = -\frac{\partial\vec{\mathrm{A}}}{\partial\mathrm{t}}$$

$$\nabla\times\frac{1}{|\vec{\mathrm{r}}-\vec{\mathrm{r}}'|}\frac{\partial\vec{B}}{\partial t} = \nabla\left(\frac{1}{|\vec{\mathrm{r}}-\vec{\mathrm{r}}'|}\right)\times\frac{\partial\vec{B}}{\partial t} + \frac{1}{|\vec{\mathrm{r}}-\vec{\mathrm{r}}'|}\nabla\times\frac{\partial\vec{B}}{\partial t}$$

$$= \frac{\vec{\mathrm{r}}-\vec{\mathrm{r}}'}{|\vec{\mathrm{r}}-\vec{\mathrm{r}}'|^3}\times\frac{\partial\vec{B}}{\partial t} + \frac{1}{|\vec{\mathrm{r}}-\vec{\mathrm{r}}'|}\frac{\partial\left(\nabla\times\vec{B}\right)}{\partial t}$$

$$\frac{1}{4\pi}\int\left[\frac{\vec{\mathrm{r}}-\vec{\mathrm{r}}'}{|\vec{\mathrm{r}}-\vec{\mathrm{r}}'|^3}\times\frac{\partial\vec{B}}{\partial t} + \frac{1}{|\vec{\mathrm{r}}-\vec{\mathrm{r}}'|}\frac{\partial\left(\nabla\times\vec{B}\right)}{\partial t}\right]\mathrm{d}\vec{\mathrm{r}}' = -\frac{\partial\vec{\mathrm{A}}}{\partial\mathrm{t}}$$

Subject to Coulomb gauge$\nabla\vec{\mathrm{A}} = 0$ we may have various functions representing $\vec{\mathrm{A}}$: $\frac{\partial\vec{\mathrm{A}}}{\partial\mathrm{t}}$ is not unique. But the left side is unique, $\vec{B}$ being a physical quantity which can be measured.

Having said all that one must appreciate the fact that it is possible to have charges and currents spreading up to infinity, the fields being arbitrarily finite at any arbitrary distance from the reckoned origin.

More on Vanishing Divergences

Let us consider the PDE

$$\vec{\nabla}.\vec{V} = 0$$

For an arbitrary volume which does not include a source

$$\iiint\vec{\nabla}.\vec{V} = 0$$

$$\oiint\vec{V}\,.d\vec{S} = \iiint\vec{\nabla}.\vec{V} = 0$$

For any arbitrary surface inside the mentioned volume

$$\oiint \vec{V} \,.\, d\vec{S} = 0$$

That does not necessarily mean $\vec{V} = 0$. One may consider asource free region in the vicinity of a point charge

We may consider

$$\vec{\nabla}.\vec{V} = 0$$

in the vicinity of a point charge. We do have a finite continuous region where

$$\oiint \vec{V} \,.\, d\vec{S} = 0$$

for any arbitrary closed surface lying inside it. But we cannot claim that $\vec{V} = 0$ for all points in this region.

Conclusion

An infinitude of boundary conditions are possible for any global distribution of currents and charges. As claimed we arrive at contradictions.

References

1. Wikipedia, Boundary Value Problem, https://en.wikipedia.org/wiki/Boundary_value_problem

2. Griffiths D J, Introduction to Electrodynamics, Pearson India Education Services Pvt Ltd Copyright © 2015,Appendix: Basic Equations of Electrodynamics

3. Griffiths D J, Introduction to Electrodynamics, Pearson India Education Services Pvt Ltd Copyright © 2015,Appendix: Basic Equations of Electrodynamics, Equation (8)