an electron in a precessing magnetic field: a tutorial

Anindya Kumar Biswas, Bhabani Prasad Mandal*
Department of Physics;
North-Eastern Hill University,
Mawkynroh-Umshing, Shillong-793022.
Department of Physics; Banaras Hindu University, Banaras, UP.

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In this pedagogical article, we elucidate on direct derivation of wave function of an electron in a precessing magnetic field of constant magnitude.

* anindya@nehu.ac.in, bhabani.mandal@gmail.com

I. INTRODUCTION

Apparently, this problem was first considered by Julian Schwinger,[1] and was solved by rotaing coordinated method, [1],[2].Eighty years have passed since then. But what is missing uniformly in all quantum mechanics books is a straightforward derivation of the wavefunction. David Griffiths poses it as a problem for two special cases in his quantum mechanics book,[3]. We could not locate a paper in which it is done. Quite likely it has been done. Still to assist students, we have worked it out and in this pedagogical article, are presenting one straightforward derivation. This problem has played important role in the development of Berry phase, [4],[5], research.

II. EIGENVALUES OF PAULI HAMILTONIAN AND EIGENFUNCTIONS

Pauli equation, [6], for an electron with charge -e, mass m_0, is

$$i\hbar\frac{\partial\chi}{\partial t} = \frac{e\hbar}{2m_0}\vec{\sigma}.\vec{B}\chi.$$

If the magnetic field is along z-direction: $\vec{B} = B_0\hat{k}$, then hamiltonian $H = \frac{e\hbar}{2m_0}\vec{\sigma}.\vec{B} = \frac{\hbar\omega_0}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

Eigenvalues of the hamiltonian are $E_\pm = \pm\frac{\hbar\omega_0}{2}$, where, ω_0 is $\frac{eB_0}{2m_0}$.

Corresponding eigenfunctions of the Hamiltonian are $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$.

III. TIME-DEPENDENT WAVEFUNCTION OF PAULI EQUATION/SCHR$\ddot{o}$DINGER EQUATION

If the magnetic field is precessing about z-direction, making a constant angle α, with angular frequency ω, then $\vec{B} = B_0(\hat{i}\sin\alpha\cos\omega t, \hat{i}\sin\alpha\sin\omega t, \cos\alpha)$. Moreover, hamiltonian $H = \frac{e\hbar}{2m_0}\vec{\sigma}.\vec{B} = \frac{\hbar\omega_1}{2}\begin{pmatrix}\cos\alpha & \sin\alpha e^{-i\omega t}\\ \sin\alpha e^{i\omega t} & -\cos\alpha\end{pmatrix}$. Eigenvalues of the hamiltonian are $E_\pm = \pm\frac{\hbar\omega_1}{2}$ where, ω_1 is $\frac{eB_0}{m_0}$ i.e. ω_0 is equal to $\frac{\omega_1}{2}$. (Here, we differ from Griffiths,[3], in notation. our ω_1 is -Ω_1 in Griffiths.) Corresponding eigenfunctions of the Hamiltonian are $\begin{pmatrix}\cos\frac{\alpha}{2}\\ \sin\frac{\alpha}{2}e^{i\omega t}\end{pmatrix}$ and $\begin{pmatrix}\sin\frac{\alpha}{2}\\ -\cos\frac{\alpha}{2}e^{i\omega t}\end{pmatrix}$.

Eigenfunctions do not satisfy the schr$\ddot{o}$dinger equation: $i\hbar\frac{\partial}{\partial t}\chi = H\chi$. Wavefunction $\chi(t)$ which satisfies the schr$\ddot{o}$dinger equation: $i\hbar\frac{\partial}{\partial t}\chi = H\chi$ is

$$\chi(t) = A(t)e^{-i\frac{\omega_1}{2}t}\begin{pmatrix}\cos\frac{\alpha}{2}\\ \sin\frac{\alpha}{2}e^{i\omega t}\end{pmatrix} + B(t)e^{i\frac{\omega_1}{2}t}\begin{pmatrix}\sin\frac{\alpha}{2}\\ -\cos\frac{\alpha}{2}e^{i\omega t}\end{pmatrix} \tag{1}$$

which is the same as

$$\chi(t) = \begin{pmatrix}\cos\frac{\alpha}{2}e^{-i\frac{\omega_1}{2}t} & \sin\frac{\alpha}{2}e^{i\frac{\omega_1}{2}t}\\ \sin\frac{\alpha}{2}e^{i\omega t}e^{-i\frac{\omega_1}{2}t} & -\cos\frac{\alpha}{2}e^{i\omega t}e^{i\frac{\omega_1}{2}t}\end{pmatrix}\begin{pmatrix}A(t)\\ B(t)\end{pmatrix}$$

where, $A(t)$ and $B(t)$ are determined as a consequence.

IV. EQUATIONS OF A(T) AND B(T)

$A(t)$ and $B(t)$ are required to satisfy the following equations

$$i\hbar\frac{dA}{dt} - \hbar\omega A(t)\sin^2\frac{\alpha}{2} + \frac{\hbar\omega}{2}B(t)\sin\alpha e^{i\omega_1 t} = 0$$
$$i\hbar\frac{dB}{dt} - \hbar\omega B(t)\cos^2\frac{\alpha}{2} + \frac{\hbar\omega}{2}A(t)\sin\alpha e^{-i\omega_1 t} = 0$$

The two equations cannot be decoupled by differentiation.
Moreover,

$$|A|^2 + |B|^2 = 1$$

as $\chi^\dagger \chi = 1$.

A. putting in matrix form

We put in the matrix form the two differential equations as

$$i\hbar \begin{pmatrix} \frac{dA}{dt} \\ \frac{dB}{dt} \end{pmatrix} = \begin{pmatrix} \hbar\omega \sin^2 \frac{\alpha}{2} & -\frac{\hbar\omega}{2} \sin\alpha e^{i\omega_1 t} \\ -\frac{\hbar\omega}{2} \sin\alpha e^{-i\omega_1 t} & \hbar\omega \cos^2 \frac{\alpha}{2} \end{pmatrix} \begin{pmatrix} A(t) \\ B(t) \end{pmatrix} \tag{2}$$

B. eigenvalue of rhs matrix

Solving

$$0 = \begin{pmatrix} \sin^2 \frac{\alpha}{2} - \lambda & -\frac{1}{2} \sin\alpha e^{i\omega_1 t} \\ -\frac{1}{2} \sin\alpha e^{-i\omega_1 t} & \cos^2 \frac{\alpha}{2} - \lambda \end{pmatrix}$$

we obtain the eigenvalues i.e. values of λ as zero and one.
Normalised eigenvector corresponding to eigenvalue zero is

$$\begin{pmatrix} \cos \frac{\alpha}{2} e^{i\omega_1 t/2} \\ \sin \frac{\alpha}{2} e^{-i\omega_1 t/2} \end{pmatrix}$$

Normalised eigenvector corresponding to eigenvalue one is

$$\begin{pmatrix} -\sin \frac{\alpha}{2} e^{i\omega_1 t/2} \\ \cos \frac{\alpha}{2} e^{-i\omega_1 t/2} \end{pmatrix}$$

C. spectral value decomposition

We verify that the spectral value decomposition is correct i.e.

$$0 \begin{pmatrix} \cos \frac{\alpha}{2} e^{i\omega_1 t/2} \\ \sin \frac{\alpha}{2} e^{-i\omega_1 t/2} \end{pmatrix} \times \begin{pmatrix} \cos \frac{\alpha}{2} e^{-i\omega_1 t/2} & \sin \frac{\alpha}{2} e^{i\omega_1 t/2} \end{pmatrix} + 1 \begin{pmatrix} -\sin \frac{\alpha}{2} e^{i\omega_1 t/2} \\ \cos \frac{\alpha}{2} e^{-i\omega_1 t/2} \end{pmatrix} \times \begin{pmatrix} -\sin \frac{\alpha}{2} e^{-i\omega_1 t/2} & \cos \frac{\alpha}{2} e^{i\omega_1 t/2} \end{pmatrix} = \begin{pmatrix} \sin^2 \frac{\alpha}{2} & -\frac{1}{2} \sin\alpha e^{i\omega_1 t} \\ -\frac{1}{2} \sin\alpha e^{-i\omega_1 t} & \cos^2 \frac{\alpha}{2} \end{pmatrix}$$

D. time-dependent orthogonal transformation

Hence the time-dependent orthogonal transformation diagonalising the R.H.S matrix is

$$O = \begin{pmatrix} \cos \frac{\alpha}{2} e^{i\omega_1 t/2} & -\sin \frac{\alpha}{2} e^{i\omega_1 t/2} \\ \sin \frac{\alpha}{2} e^{-i\omega_1 t/2} & \cos \frac{\alpha}{2} e^{-i\omega_1 t/2} \end{pmatrix}$$

or,

$$O^\dagger \times \begin{pmatrix} \sin^2 \frac{\alpha}{2} & -\frac{1}{2} \sin\alpha e^{i\omega_1 t} \\ -\frac{1}{2} \sin\alpha e^{-i\omega_1 t} & \cos^2 \frac{\alpha}{2} \end{pmatrix} \times O = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

One can easily show that

$$O^\dagger \times O = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

as required for O to be an orthogonal matrix.

V. EQUATIONS OF $a(t)$ AND $b(t)$

Multiplying the eq.(2) from the left by $O^\dagger$, we obtain

$$i\hbar O^\dagger \times \begin{pmatrix} \frac{dA}{dt} \\ \frac{dB}{dt} \end{pmatrix} = O^\dagger \begin{pmatrix} \hbar\omega \sin^2\frac{\alpha}{2} & -\frac{\hbar\omega}{2}\sin\alpha e^{i\omega_1 t} \\ -\frac{\hbar\omega}{2}\sin\alpha e^{-i\omega_1 t} & \hbar\omega\cos^2\frac{\alpha}{2} \end{pmatrix} \begin{pmatrix} A(t) \\ B(t) \end{pmatrix}$$

which is the same as

$$i\hbar O^\dagger \times \frac{d}{dt} O O^\dagger \begin{pmatrix} A(t) \\ B(t) \end{pmatrix} = O^\dagger \begin{pmatrix} \hbar\omega \sin^2\frac{\alpha}{2} & -\frac{\hbar\omega}{2}\sin\alpha e^{i\omega_1 t} \\ -\frac{\hbar\omega}{2}\sin\alpha e^{-i\omega_1 t} & \hbar\omega\cos^2\frac{\alpha}{2} \end{pmatrix} O O^\dagger \begin{pmatrix} A(t) \\ B(t) \end{pmatrix}$$

which is the same as

$$i\hbar O^\dagger \times \frac{d}{dt} O O^\dagger \begin{pmatrix} A(t) \\ B(t) \end{pmatrix} = \hbar\omega \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} O^\dagger \begin{pmatrix} A(t) \\ B(t) \end{pmatrix}$$

It is easily verifiable,

$$i\hbar O^\dagger \times \frac{d}{dt} O = -\frac{\hbar\omega_1}{2} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & -\cos\alpha \end{pmatrix}$$

We define,

$$O^\dagger \begin{pmatrix} A(t) \\ B(t) \end{pmatrix} = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}. \tag{3}$$

A. putting in matrix form

We obtain,

$$i\hbar \frac{d}{dt} \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \frac{\hbar\omega_1}{2} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & -\cos\alpha + \frac{2\omega}{\omega_1} \end{pmatrix} \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} \tag{4}$$

The square matrix in the R.H.S is time-independent. Hence, we can diagonalise it using time-independent orthogonal transformation.

B. eigenvalue of rhs matrix

Solving

$$0 = \begin{pmatrix} \cos\alpha - \lambda & -\sin\alpha \\ -\sin\alpha & -\cos\alpha + \frac{2\omega}{\omega_1} - \lambda \end{pmatrix}$$

one obtains eigenvalues $\lambda_\pm = \frac{\omega}{\omega_1} \pm \sqrt{(\frac{\omega}{\omega_1})^2 + 1 - 2\frac{\omega}{\omega_1}\cos\alpha}$. Normalised eigenvector corresponding to the eigenvalue, λ_+ is

$$\frac{1}{\sqrt{1 + (\lambda_+)^2 - 2\lambda_+\cos\alpha}} \begin{pmatrix} \sin\alpha \\ \cos\alpha - \lambda_+ \end{pmatrix}.$$

Normalised eigenvector corresponding to the eigenvalue, λ_- is

$$\frac{1}{\sqrt{1 + (\lambda_+)^2 - 2\lambda_+\cos\alpha}} \begin{pmatrix} \lambda_+ - \cos\alpha \\ \sin\alpha \end{pmatrix}$$

.

C. spectral value decomposition

One can verify the spectral value decomposition i.e. show that

$$\lambda_+ \begin{pmatrix} \sin\alpha \\ \cos\alpha - \lambda_+ \end{pmatrix} \times \begin{pmatrix} \sin\alpha & \cos\alpha - \lambda_+ \end{pmatrix} + \lambda_- \begin{pmatrix} \lambda_+ - \cos\alpha \\ \sin\alpha \end{pmatrix} \times \begin{pmatrix} \lambda_+ - \cos\alpha & \sin\alpha \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & -\cos\alpha + \frac{2\omega}{\omega_1} \end{pmatrix}$$

by using $\lambda_+ - \lambda_- = 2\sqrt{(\frac{\omega}{\omega_1})^2 + 1 - 2\frac{\omega}{\omega_1}\cos\alpha}$ and $\lambda_+\lambda_- + 1 = 2\frac{\omega}{\omega_1}\cos\alpha$.

D. time-independent orthogonal transformation

Hence the time-independent orthogonal transformation, O_1 diagonalising the square matrix in the R.H.S of the eq.(4) is

$$O_1 = \frac{1}{\sqrt{1 + (\lambda_+)^2 - 2\lambda_+\cos\alpha}} \begin{pmatrix} \sin\alpha & \lambda_+ - \cos\alpha \\ \cos\alpha - \lambda_+ & \sin\alpha \end{pmatrix}$$

E. equations of linear combinations of a(t) and b(t)in diagonal form

Multiplying the eq.(4) from the left by $O_1^\dagger$, we obtain

$$i\hbar O_1^\dagger \frac{d}{dt} \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \frac{\hbar\omega_1}{2} O_1^\dagger \begin{pmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & -\cos\alpha + \frac{2\omega}{\omega_1} \end{pmatrix} \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$$

which is the same as

$$i\hbar O_1^\dagger \frac{d}{dt} O_1 O_1^\dagger \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \frac{\hbar\omega_1}{2} O_1^\dagger \begin{pmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & -\cos\alpha + \frac{2\omega}{\omega_1} \end{pmatrix} O_1 O_1^\dagger \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$$

which is the same as

$$i\hbar \frac{d}{dt} O_1^\dagger \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \frac{\hbar\omega_1}{2} \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} O_1^\dagger \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$$

which is the same as

$$i\hbar \frac{d}{dt} \begin{pmatrix} \sin\alpha & -\lambda_+ + \cos\alpha \\ -\cos\alpha + \lambda_+ & \sin\alpha \end{pmatrix} \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \frac{\hbar\omega_1}{2} \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} \begin{pmatrix} \sin\alpha & -\lambda_+ + \cos\alpha \\ -\cos\alpha + \lambda_+ & \sin\alpha \end{pmatrix} \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$$

which is the same as

$$\frac{d}{dt} \begin{pmatrix} a(t)\sin\alpha + (-\lambda_+ + \cos\alpha)b(t) \\ (-\cos\alpha + \lambda_+)a(t) + b(t)\sin\alpha \end{pmatrix} = \begin{pmatrix} \frac{-i\omega_1\lambda_+}{2}(a(t)\sin\alpha + (-\lambda_+ + \cos\alpha)b(t)) \\ \frac{-i\omega_1\lambda_-}{2}((-\cos\alpha + \lambda_+)a(t) + b(t)\sin\alpha) \end{pmatrix}.$$

F. linear combinations of a(t) and b(t) as a function of time

As a result,

$$\begin{pmatrix} a(t)\sin\alpha + (-\lambda_+ + \cos\alpha)b(t) \\ (-\cos\alpha + \lambda_+)a(t) + b(t)\sin\alpha \end{pmatrix} = \begin{pmatrix} e^{\frac{-i\omega_1\lambda_+ t}{2}}(a(0)\sin\alpha + (-\lambda_+ + \cos\alpha)b(0)) \\ e^{\frac{-i\omega_1\lambda_- t}{2}}((-\cos\alpha + \lambda_+)a(0) + b(0)\sin\alpha) \end{pmatrix}.$$

which is the same as

$$\begin{pmatrix} \sin\alpha & -\lambda_+ + \cos\alpha \\ -\cos\alpha + \lambda_+ & \sin\alpha \end{pmatrix} \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \begin{pmatrix} e^{\frac{-i\omega_1\lambda_+ t}{2}} & 0 \\ 0 & e^{\frac{-i\omega_1\lambda_- t}{2}} \end{pmatrix} \begin{pmatrix} \sin\alpha & -\lambda_+ + \cos\alpha \\ -\cos\alpha + \lambda_+ & \sin\alpha \end{pmatrix} \cdot \begin{pmatrix} a(0) \\ b(0) \end{pmatrix}$$

which is the same as

$$\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \frac{1}{1+(\lambda_+)^2-2\lambda_+\cos\alpha} \begin{pmatrix} \sin\alpha & \lambda_+ - \cos\alpha \\ \cos\alpha - \lambda_+ & \sin\alpha \end{pmatrix} \begin{pmatrix} e^{\frac{-i\omega_1\lambda_+ t}{2}} & 0 \\ 0 & e^{\frac{-i\omega_1\lambda_- t}{2}} \end{pmatrix} \begin{pmatrix} \sin\alpha & -\lambda_+ + \cos\alpha \\ -\cos\alpha + \lambda_+ & \sin\alpha \end{pmatrix} \begin{pmatrix} a(0) \\ b(0) \end{pmatrix} \tag{5}$$

From the eq.(3), we get

$$O^\dagger \begin{pmatrix} A(t) \\ B(t) \end{pmatrix} = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}.$$

which is the same as

$$\begin{pmatrix} A(t) \\ B(t) \end{pmatrix} = O \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}.$$

Multiplying eq.(5) by O(t), from left we get

$$O(t)\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \frac{1}{1+(\lambda_+)^2-2\lambda_+\cos\alpha} O(t) \begin{pmatrix} \sin\alpha & \lambda_+ - \cos\alpha \\ \cos\alpha - \lambda_+ & \sin\alpha \end{pmatrix} \begin{pmatrix} e^{\frac{-i\omega_1\lambda_+ t}{2}} & 0 \\ 0 & e^{\frac{-i\omega_1\lambda_- t}{2}} \end{pmatrix} \begin{pmatrix} \sin\alpha & -\lambda_+ + \cos\alpha \\ -\cos\alpha + \lambda_+ & \sin\alpha \end{pmatrix} O^\dagger(0) O(0) \begin{pmatrix} a(0) \\ b(0) \end{pmatrix}$$

which is the same as

$$\begin{pmatrix} A(t) \\ B(t) \end{pmatrix} = \frac{1}{1+(\lambda_+)^2-2\lambda_+\cos\alpha} O(t) \begin{pmatrix} \sin\alpha & \lambda_+ - \cos\alpha \\ \cos\alpha - \lambda_+ & \sin\alpha \end{pmatrix} \begin{pmatrix} e^{\frac{-i\omega_1\lambda_+ t}{2}} & 0 \\ 0 & e^{\frac{-i\omega_1\lambda_- t}{2}} \end{pmatrix} \begin{pmatrix} \sin\alpha & -\lambda_+ + \cos\alpha \\ -\cos\alpha + \lambda_+ & \sin\alpha \end{pmatrix} O^\dagger(0) \begin{pmatrix} A(0) \\ B(0) \end{pmatrix}$$

which is the same as

$$\begin{pmatrix} A(t) \\ B(t) \end{pmatrix} = \frac{1}{1+(\lambda_+)^2-2\lambda_+\cos\alpha} \begin{pmatrix} \cos\frac{\alpha}{2} e^{i\omega_1 t/2} & -\sin\frac{\alpha}{2} e^{i\omega_1 t/2} \\ \sin\frac{\alpha}{2} e^{-i\omega_1 t/2} & \cos\frac{\alpha}{2} e^{-i\omega_1 t/2} \end{pmatrix} \begin{pmatrix} \sin\alpha & \lambda_+ - \cos\alpha \\ \cos\alpha - \lambda_+ & \sin\alpha \end{pmatrix} \begin{pmatrix} e^{\frac{-i\omega_1\lambda_+ t}{2}} & 0 \\ 0 & e^{\frac{-i\omega_1\lambda_- t}{2}} \end{pmatrix}$$
$$\begin{pmatrix} \sin\alpha & -\lambda_+ + \cos\alpha \\ -\cos\alpha + \lambda_+ & \sin\alpha \end{pmatrix} \begin{pmatrix} \cos\frac{\alpha}{2} & \sin\frac{\alpha}{2} \\ -\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix} \begin{pmatrix} A(0) \\ B(0) \end{pmatrix}. \tag{6}$$

VI. INITIAL CONDITION CORRESPONDING TO EXAMPLE 10.1.3: WAVEFUNCTION

Statement of the example 10.1.3 in Griffiths,[3]: Suppose the electron starts out wirth spin up, along $\vec{B}(0)$, the exact solution to the time-dependent Schrödinger equation is

$$\chi(t) = \begin{pmatrix} e^{-i\omega t/2}\cos\frac{\alpha}{2}[\cos\frac{\lambda t}{2} + i\frac{\omega+\omega_1}{\lambda}\sin\frac{\lambda t}{2}] \\ e^{i\omega t/2}\sin\frac{\alpha}{2}[\cos\frac{\lambda t}{2} + i\frac{-\omega+\omega_1}{\lambda}\sin\frac{\lambda t}{2}] \end{pmatrix},$$

where, $\lambda = \sqrt{\omega^2 + \omega_1^2 + 2\omega\omega_1\cos\alpha}$.

Solution: From eq.(1), we get A(0)=1, B(0)=0. Putting in the eq.(6), we get after several matrix multiplications,

$$\begin{pmatrix} A(t) \\ B(t) \end{pmatrix} = \frac{1}{1+(\lambda_+)^2-2\lambda_+\cos\alpha} \begin{pmatrix} I \\ II \end{pmatrix},$$

where,

$$I = \cos\frac{\alpha}{2} e^{i\omega_1 t/2}[e^{\frac{-i\omega_1\lambda_+ t}{2}}(1+\lambda_+)\sin\alpha\sin\frac{\alpha}{2} + e^{\frac{-i\omega_1\lambda_- t}{2}}(-1+\lambda_+)(\lambda_+ - \cos\alpha)\cos\frac{\alpha}{2}]$$
$$-\sin\frac{\alpha}{2} e^{i\omega_1 t/2}[e^{\frac{-i\omega_1\lambda_+ t}{2}}(1+\lambda_+)(-\lambda_+ + \cos\alpha)\sin\frac{\alpha}{2} + e^{\frac{-i\omega_1\lambda_- t}{2}}(-1+\lambda_+)\cos\frac{\alpha}{2}\sin\alpha]$$

and

$$II = \sin\frac{\alpha}{2}e^{-i\omega_1 t/2}[e^{\frac{-i\omega_1\lambda_+ t}{2}}(1+\lambda_+)\sin\alpha\sin\frac{\alpha}{2} + e^{\frac{-i\omega_1\lambda_- t}{2}}(-1+\lambda_+)(\lambda_+ - \cos\alpha)\cos\frac{\alpha}{2}]$$
$$+\cos\frac{\alpha}{2}e^{-i\omega_1 t/2}[e^{\frac{-i\omega_1\lambda_+ t}{2}}(1+\lambda_+)(-\lambda_+ + \cos\alpha)\sin\frac{\alpha}{2} + e^{\frac{-i\omega_1\lambda_- t}{2}}(-1+\lambda_+)\cos\frac{\alpha}{2}\sin\alpha].$$

As a result, we obtain

$$A(t) = \frac{1}{1+(\lambda_+)^2 - 2\lambda_+\cos\alpha}e^{i(\omega_1-\omega)t/2}[e^{\frac{-i\Gamma t}{2}}(1+\lambda_+)^2\sin^2\frac{\alpha}{2} + e^{\frac{i\Gamma t}{2}}(-1+\lambda_+)^2\cos^2\frac{\alpha}{2}],$$
$$B(t) = \frac{-i(1-(\lambda_+)^2)}{1+(\lambda_+)^2 - 2\lambda_+\cos\alpha}e^{-i(\omega_1+\omega)t/2}\sin\frac{\Gamma t}{2}\sin\alpha, \tag{7}$$

where, $\Gamma = \sqrt{\omega^2 + \omega_1^2 - 2\omega\omega_1\cos\alpha}$.
Combining eq.(7) with eq.(1) we obtain

$$\chi(t) = \frac{1}{1+(\lambda_+)^2 - 2\lambda_+\cos\alpha}\begin{pmatrix} U \\ L \end{pmatrix},$$

where, on simplification using

$$(\lambda_+)^2 - 1 = \frac{2\omega}{\omega_1}(\lambda_+ - \cos\alpha),$$
$$1+(\lambda_+)^2 - 2\lambda_+\cos\alpha = 2\Gamma(\lambda_+ - \cos\alpha) \tag{8}$$

one gets

$$\frac{U}{1+(\lambda_+)^2 - 2\lambda_+\cos\alpha} = e^{-i\omega t/2}\cos\frac{\alpha}{2}[\cos\frac{\Gamma t}{2} + i\frac{\omega-\omega_1}{\Gamma}\sin\frac{\Gamma t}{2}],$$
$$\frac{L}{1+(\lambda_+)^2 - 2\lambda_+\cos\alpha} = e^{i\omega t/2}\sin\frac{\alpha}{2}[\cos\frac{\Gamma t}{2} - i\frac{\omega+\omega_1}{\Gamma}\sin\frac{\Gamma t}{2}],$$

i.e.

$$\chi(t) = \begin{pmatrix} e^{-i\omega t/2}\cos\frac{\alpha}{2}[\cos\frac{\Gamma t}{2} + i\frac{\omega-\omega_1}{\Gamma}\sin\frac{\Gamma t}{2}] \\ e^{i\omega t/2}\sin\frac{\alpha}{2}[\cos\frac{\Gamma t}{2} - i\frac{\omega+\omega_1}{\Gamma}\sin\frac{\Gamma t}{2}] \end{pmatrix}.$$

By changing ω_1 to -ω_1 and Γ to λ, we obtain the asked solution.

VII. INITIAL CONDITION CORRESPONDING TO PROBLEM 9.19: WAVEFUNCTION

Statement of the problem 9.19 in Griffiths,[3]: Suppose the electron starts out wirth spin up, along the z-direction, the exact solution to the time-dependent Schrödinger equation is

$$\chi(t) = \begin{pmatrix} e^{-i\omega t/2}[\cos\frac{\lambda t}{2} + i\frac{\omega+\omega_1\cos\alpha}{\lambda}\sin\frac{\lambda t}{2}] \\ i\frac{\omega_1}{\lambda}e^{i\omega t/2}\sin\alpha\sin\frac{\lambda t}{2}] \end{pmatrix},$$

where, $\lambda = \sqrt{\omega^2 + \omega_1^2 + 2\omega\omega_1\cos\alpha}$.
Solution: From eq.(1), we get

$$\chi(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

which implies

$$\begin{pmatrix} A(0) \\ B(0) \end{pmatrix} = \begin{pmatrix} \cos\frac{\alpha}{2} \\ \sin\frac{\alpha}{2} \end{pmatrix}$$

which implies in association with eq.(6),

$$\begin{pmatrix}A(t)\\B(t)\end{pmatrix}=$$

$$\frac{1}{1+(\lambda_+)^2-2\lambda_+\cos\alpha}\begin{pmatrix}\cos\frac{\alpha}{2}e^{i\omega_1 t/2} & -\sin\frac{\alpha}{2}e^{i\omega_1 t/2}\\ \sin\frac{\alpha}{2}e^{-i\omega_1 t/2} & \cos\frac{\alpha}{2}e^{-i\omega_1 t/2}\end{pmatrix}\begin{pmatrix}\sin\alpha & \lambda_+-\cos\alpha\\ \cos\alpha-\lambda_+ & \sin\alpha\end{pmatrix}\begin{pmatrix}e^{\frac{-i\omega_1\lambda_+ t}{2}} & 0\\ 0 & e^{\frac{-i\omega_1\lambda_- t}{2}}\end{pmatrix}$$
$$\begin{pmatrix}\sin\alpha & -\lambda_++\cos\alpha\\ -\cos\alpha+\lambda_+ & \sin\alpha\end{pmatrix}\begin{pmatrix}\cos\frac{\alpha}{2} & \sin\frac{\alpha}{2}\\ -\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2}\end{pmatrix}\begin{pmatrix}\cos\frac{\alpha}{2}\\ \sin\frac{\alpha}{2}\end{pmatrix},$$

which further implies in attachment with eq.(1),

$$\chi(t)=\frac{1}{1+(\lambda_+)^2-2\lambda_+\cos\alpha}\begin{pmatrix}\cos\frac{\alpha}{2}e^{-i\frac{\omega_1}{2}t} & \sin\frac{\alpha}{2}e^{i\frac{\omega_1}{2}t}\\ \sin\frac{\alpha}{2}e^{i\omega t}e^{-i\frac{\omega_1}{2}t} & -\cos\frac{\alpha}{2}e^{i\omega t}e^{i\frac{\omega_1}{2}t}\end{pmatrix}\begin{pmatrix}\cos\frac{\alpha}{2}e^{i\omega_1 t/2} & -\sin\frac{\alpha}{2}e^{i\omega_1 t/2}\\ \sin\frac{\alpha}{2}e^{-i\omega_1 t/2} & \cos\frac{\alpha}{2}e^{-i\omega_1 t/2}\end{pmatrix}$$
$$\begin{pmatrix}\sin\alpha & \lambda_+-\cos\alpha\\ \cos\alpha-\lambda_+ & \sin\alpha\end{pmatrix}\begin{pmatrix}e^{\frac{-i\omega_1\lambda_+ t}{2}} & 0\\ 0 & e^{\frac{-i\omega_1\lambda_- t}{2}}\end{pmatrix}\begin{pmatrix}\sin\alpha & -\lambda_++\cos\alpha\\ -\cos\alpha+\lambda_+ & \sin\alpha\end{pmatrix}$$
$$\begin{pmatrix}\cos\frac{\alpha}{2} & \sin\frac{\alpha}{2}\\ -\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2}\end{pmatrix}\begin{pmatrix}\cos\frac{\alpha}{2}\\ \sin\frac{\alpha}{2}\end{pmatrix},$$

which after several matrix multiplications followed by simplification using eq.(8), yields to

$$\chi(t)=\begin{pmatrix}e^{-i\omega t/2}[\cos\frac{\Gamma t}{2}+i\frac{\omega-\omega_1\cos\alpha}{\Gamma}\sin\frac{\Gamma t}{2}]\\ -i\frac{\omega_1\sin\alpha}{\Gamma}e^{i\omega t/2}\sin\frac{\Gamma t}{2}\end{pmatrix}.$$

By changing ω_1 to -ω_1 and Γ to λ, we obtain the asked solution.

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