Upper and Lower Rarely α-continuous Multifunctions *

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Abstract

Recently the notion of rarely α-continuous functions has been introduced and investigated by Jafari [1]. This paper is devoted to the study of upper (and lower) rarely α-continuous multifunctions.

1 Introduction

Let A be a subset of a topological space (X, τ). We will denote the interior and the closure of A by int A and cl A, respectively. A is called *α-open* [2] if $A \subseteq \text{int}(\text{cl}(\text{int } A))$. The complement of an α-open set will be called an *α-closed* set. The *α-interior* of A is defined as the union of all α-open sets contained in A and is denoted by α-int A. It is well known that $\alpha\text{-int } A = A \cap \text{int}(\text{cl}(\text{int } A))$. The collection of all α-open sets is denoted by α(X) and we set $\alpha(X, x) = \{ U : x \in U \text{ and } U \in \alpha(X) \}$. A *rare* set is a codense set, i.e. its interior is empty. Finally, A is called *regular open* if $A = \text{int}(\text{cl } A)$.

Definition 1 A function $f : X \to Y$ is said to be *rarely α-continuous* [1] (briefly r.α.c.) if for each $x \in X$ and each open set $V \subseteq Y$ containing $f(x)$, there exist a rare set R_V and a $U \in \alpha(X, x)$ such that $f(U) \subseteq V \cup R_V$. (Clearly we may assume that $V \cap R_V$ is empty.)

Definition 2 A function $f : X \to Y$ is said to be *weakly α-continuous* [3] (briefly w.α.c.) if for each $x \in X$ and each open set $V \subseteq Y$ containing $f(x)$, there exists a $U \in \alpha(X, x)$ such that $f(U) \subseteq cl\ V$.

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2 Rarely α-continuous multifunctions

We will follow the notations in [6] . As usual, if $F : X \to Y$ is a multifunction, the upper and lower inverses of a set $V \subseteq Y$ will be denoted by $F^+(V)$ and $F^-(V)$, respectively. We have $F^+(V) = \{x \in X : F(x) \subseteq V\}$ and $F^-(V) = \{x \in X : F(x) \cap V \neq \emptyset\}$.

Definition 3 (1) The multifunction $F : X \to Y$ is called *upper rarely α-continuous* (briefly u.r.α.c.) if for each $x \in X$ and each open set $V \subseteq Y$ with $F(x) \subseteq V$, there exist a rare set R_V disjoint from V and $U \in \alpha(X, x)$ such that $F(U) \subseteq V \cup R_V$.

(2) The multifunction $F : X \to Y$ is called *lower rarely α-continuous* (briefly l.r.α.c.) if for each $x \in X$ and each open set $V \subseteq Y$ with $F(x) \cap V \neq \emptyset$, there exist a rare set R_V disjoint from V and $U \in \alpha(X, x)$ such that $F(y) \cap (V \cup R_V) \neq \emptyset$ for each $y \in U$.

For the definitions of the following abbreviations we refer the reader to [4], [5], [6] and [7] . We clearly have the implications :

1) u.w.c. $\Rightarrow$ u.r.c. $\Rightarrow$ u.r.α.c.

2) u.w.c. $\Rightarrow$ u.w.α.c. $\Rightarrow$ u.r.α.c.

3) l.w.c. $\Rightarrow$ l.r.c. $\Rightarrow$ l.r.α.c.

4) l.w.c. $\Rightarrow$ l.w.α.c. $\Rightarrow$ l.r.α.c.

3 Some properties of rarely α-continuous multifunctions

Theorem 3.1 For a multifunction $F : X \to Y$ the following are equivalent :

(1) F is u.r.α.c. at $x \in X$,

(2) For each open set $V \subseteq Y$ with $F(x) \subseteq V$ there exists a rare set R_V disjoint from V such that $x \in \alpha\text{-int}(F^+(V \cup R_V))$,

(3) For each open set $V \subseteq Y$ with $F(x) \subseteq V$ there exists a rare set R_V with cl $V \cap R_V = \emptyset$ such that $x \in \alpha\text{-int}(F^+(cl\ V \cup R_V))$,

(4) For each regular open set $V \subseteq Y$ with $F(x) \subseteq V$ there exists a rare set R_V disjoint from V such that $x \in \alpha\text{-int}(F^+(V \cup R_V))$,

(5) For each open set $V \subseteq Y$ with $F(x) \subseteq V$ there exists $U \in \alpha(X,x)$ such that $F(U) \cap (Y \setminus V)$ has empty interior ,

(6) For each open set $V \subseteq Y$ with $F(x) \subseteq V$ there exists $U \in \alpha(X,x)$ such that int $F(U) \subseteq$ cl (V) .

Proof. (1) $\Rightarrow$ (2) : Let V be an open subset of Y such that $F(x) \subseteq V$. By hypothesis there exist a rare set R_V disjoint from V and $U \in \alpha(X,x)$ such that $F(U) \subseteq V \cup R_V$. Hence $x \in U \subseteq (F^+(V \cup R_V))$. It follows that $x \in \alpha\text{-int}(F^+(V \cup R_V))$.

(2) $\Rightarrow$ (3) : Let V be an open subset of Y such that $F(x) \subseteq V$. Then there exists a rare set R_V disjoint from V such that $x \in \alpha\text{-int}(F^+(V \cup R_V))$. Let $S_V = R_V \cap (Y \backslash \text{cl } V)$. Then S_V is disjoint from cl V. Since $V \cup R_V \subseteq \text{cl } V \cup S_V$, we have $x \in \alpha\text{-int}(F^+(cl\ V \cup S_V))$.

(3) $\Rightarrow$ (4) : Let V be a regular open subset of Y such that $F(x) \subseteq V$, and let R_V be a rare set with cl $V \cap R_V = \emptyset$ and $x \in \alpha\text{-int}(F^+(\text{cl } V \cup R_V))$. If $S_V = R_V \cup (\text{cl } V \setminus V)$ then S_V is a rare set disjoint from V satisfying $x \in \alpha\text{-int}(F^+(V \cup S_V))$.

(4) $\Rightarrow$ (5) : Let V be an open subset of Y with $F(x) \subseteq V$ and let $W = \text{int}(\text{cl } V)$. Then W is regular open and $V \subseteq W$. By assumption there exists a rare set R_V disjoint from W such that $x \in \alpha\text{-int}(F^+(W \cup R_V))$. If $U = \alpha\text{-int}(F^+(W \cup R_V))$ then $U \in \alpha(X,x)$ and $F(U) \subseteq W \cup R_V$. We now have $\text{int}(F(U) \cap (Y \backslash V)) = \text{int}F(U) \cap \text{int}(Y \setminus V) \subseteq \text{int}(\text{cl } V \cup R_V) \cap (Y \backslash \text{cl } V) = \emptyset$.

(5) $\Rightarrow$ (6) : This is obvious.

(6) $\Rightarrow$ (1) : Let V be an open subset of Y with $F(x) \subseteq V$. There exists $U \in \alpha(X,x)$ such that int $F(U) \subseteq$ cl V . Now, $N = (\text{cl } V) \setminus V$ is nowhere dense and $M = (\text{cl } F(U) \setminus \text{int } F(U)) \cap (Y \setminus V)$ is a rare set, so $R_V = M \cup N$ is also a rare set disjoint from V and we have $F(U) \subseteq V \cup R_V$. Hence F is u.r.α.c. at $x \in X$. □

Our next result provides a characterization of l.r.α.c. multifunctions. Its proof is very similar to the proof of Theorem 3.1 (1) - (4), so we will omit it.

Theorem 3.2 For a multifunction $F : X \to Y$ the following are equivalent :

(1) F is l.r.α.c. at $x \in X$,

(2) For each open set $V \subseteq Y$ with $F(x) \cap V \neq \emptyset$ there exists a rare set R_V disjoint from V such that $x \in \alpha\text{-int}(F^{-}(V \cup R_V))$,

(3) For each open set $V \subseteq Y$ with $F(x) \cap V \neq \emptyset$ there exists a rare set R_V with cl $V \cap R_V = \emptyset$ such that $x \in \alpha\text{-int}(F^{-}(cl\ V \cup R_V))$,

(4) For each regular open set $V \subseteq Y$ with $F(x) \cap V \neq \emptyset$ there exists a rare set R_V disjoint from V such that $x \in \alpha\text{-int}(F^{-}(V \cup R_V))$.

Corollary 3.3 ([1], Theorem 3.1)

For a function $f : X \to Y$ the following are equivalent :

(1) f is r.α.c. at $x \in X$,

(2) For each open set $V \subseteq Y$ containing $f(x)$ there exists a rare set R_V disjoint from V such that $x \in \alpha\text{-int}(f^{-1}(V \cup R_V))$,

(3) For each open set $V \subseteq Y$ containing $f(x)$ there exists a rare set R_V with cl $V \cap R_V = \emptyset$ such that $x \in \alpha\text{-int}(f^{-1}(cl\ V \cup R_V))$,

(4) For each regular open set $V \subseteq Y$ containing $f(x)$ there exists a rare set R_V disjoint from V such that $x \in \alpha\text{-int}(f^{-1}(V \cup R_V))$,

(5) For each open set $V \subseteq Y$ containing $f(x)$ there exists $U \in \alpha(X, x)$ such that $f(U) \cap (Y \setminus V)$ has empty interior ,

(6) For each open set $V \subseteq Y$ containing $f(x)$ there exists $U \in \alpha(X, x)$ such that $\text{int}\ f(U) \subseteq \text{cl}\ (V)$.

Theorem 3.4 If $F : X \to Y$ is an u.r.α.c. multifunction then for any open set $U \subseteq X$ containing x and any open set $V \subseteq Y$ with $F(x) \subseteq V$ there exists a rare set R_V disjoint from V and a nonempty open set $W \subseteq U$ such that $F(W) \subseteq V \cup R_V$.

Proof. Let $V \subseteq Y$ be open with $F(x) \subseteq V$, and let $U \subseteq X$ be an open set containing x . By Theorem 3.1, there exists $G \in \alpha(X, x)$ such that $F(G) \cap (Y \setminus V)$ has empty interior and is therefore a rare set, say R_V . In addition, R_V is disjoint from V. Then $U \cap G$ is an α-open set containing x. If $W = \text{int}(U \cap G)$ then W is a nonempty open set contained in U. Consequently, $F(W) \subseteq F(G) \subseteq V \cup (F(G) \cap (Y \setminus V)) \subseteq V \cup R_V$. □

Corollary 3.5 If $f : X \to Y$ is a r.α.c. function then for any open set $U \subseteq X$ containing x and any open set $V \subseteq Y$ containing $f(x)$ there exist a rare set R_V disjoint from V and a nonempty open set $W \subseteq U$ such that $f(W) \subseteq V \cup R_V$.

Recall that a subset A of a topological space X is called *semi-open* if $A \subseteq$ cl (int A). We will call a multifunction $F : X \to Y$ *always semi-open* if the image of each α-open set is semi-open.

Theorem 3.6 If $F : X \to Y$ is an always semi-open, u.r.α.c. multifunction then F is also u.w.α.c.

Proof. Let $x \in X$ and $V \subseteq Y$ be an open set with $F(x) \subseteq V$. Since F is u.r.α.c. there exists a rare set R_V disjoint from V and $U \in \alpha(X, x)$ such that $F(U) \subseteq V \cup R_V$. We have $F(U) \cap (Y \setminus \text{cl } V) \subseteq R_V$ and so cl (int $F(U)) \cap (Y \setminus \text{cl } V) = \emptyset$. Since F is always semi-open, $F(U)$ is semi-open and hence $F(U) \subseteq \text{cl } V$, i.e. F is u.w.α.c. □

Recall that a function $f : X \to Y$ *r.α-open* [1] if the image of each α-open set is open.

Corollary 3.7 ([1], Theorem 3.8) If $f : X \to Y$ is r.α-open and r.α.c., then f is w.α.c.

In conclusion, we shall present two more results whose proofs are easy and left to the reader.

Definition 4 For a multifunction $F : X \to Y$, the *graph multifunction* $G_F : X \to X \times Y$ is defined as follows : $G_F(x) = \{(x, y) : y \in F(x)\}$ for each $x \in X$.

Theorem 3.8 If $F : X \to Y$ is an u.r.α.c. multifunction such that $F(x)$ is compact for each $x \in X$ then G_F is u.r.α.c.

Theorem 3.9 Let $\{U_i : i \in I\}$ be an open cover of X . A multifunction $F : X \to Y$ is u.r.α.c. if and only if the multifunctions $F|_{U_i} : U_i \to Y$ are u.r.α.c. for each $i \in I$.

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