# A map of a research programme for subtlety theory

Chris Goddard
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**Abstract**

*The scope of this short note is to outline a research programme for the exploration of 'subtlety theory', which can be thought of as a framework for exploring various classes of structure associated to higher categories. It is hoped that this might form a logical springboard for researchers wishing to explore said ideas and potentially take them further.*

## 1 Foreword

First it might be useful for pedagogical purposes to provide a rough overview of the research programme sketched in earlier works.

The beginnings of this were seeded in a work in 1999 by Frieden titled "Physics from Fisher Information" [2]. In this book Roy Frieden considered the idea of deriving hamiltonians from information theoretic principles, focusing in particular on the subject of the title of his book.

From 2005 through to 2010, the author further developed these ideas and demonstrated how they could be used to derive general relativity, classical quantum mechanics, gain some understanding as to why Lorentzian space (3 spatial dimensions and one temporal dimension) is most natural as an emergent geometry, make inroads in describing turbulence and other exotic geometries, and touch on number theoretic consequences [11], [3].

In 2013, the author looked briefly into a bridging mechanism between these lower order exotic differential structures, and some slightly more higher order ones that were to follow, in a talk given at the mathematics of planet earth conference 2013 [7].

In 2015, the author indicated how these ideas can be turned on their head and used in statistics, through information functional duality; in short, that the Fisher information is part of a family of information functionals at a given level of complexity (being just two) [6].

In 2017, the author demonstrated deeper number theoretic results [8], and also sketched a rudimentary demonstration as to the full flowering of geometric information theory in terms of a theory space of information functionals *itself* admitting a functional, to describe soliton solutions to the dynamics of space filling curves [4].

In 2018, topological consequences were explored in relation to game theory, wherein it was demonstrated how optimal strategies in a game could be thought of as geodesics with respect to a suitable information functional [9], [10].

In 2019, algebraic consequences were mapped out, and the geometric, topological, and algebraic strands demonstrated to be different pieces of a single construct [5].

Now, looking beyond this, we are interested in the study of more abstract structures. We also expect progress here to be potentially harder as the problems are now much more subtle. Within the results outlined above, the key atomic units of structure, working behind the scenes, were primes in the standard sense - i.e., primes within the field $\mathcal{Z}$. But in order to go beyond this and deal with deeper questions, we need to go beyond this primitive notion of primality. In particular, we need to consider harmonics, such as the Bernoulli numbers. We need to consider irreducible polynomials over $\mathcal{Z}[x]$, $\mathcal{Z}[x][y]$, and respectively also *their* first harmonics over $\mathcal{Q}[x]$, $\mathcal{Q}[x][y]$, and potentially also quotient spaces. We need to consider subtler things still. In short, we need to expand the scope of what constitutes the atoms, or drivers of our structure.

One potential path towards looking into deeper structures might be to consider the *bulk*, i.e. consider models with two spatial coordinates, one being effectively Lorentzian space $M$ and the other an element of $M^4 := M \times M \times M \times M$. In particular, we would be interested in this case in functions

$$f(m, \eta | a, b) := \delta(\delta(\epsilon_{ij}(m, \eta) - a) - b)$$

with $m \in M$ and $\eta \in M^4$, with the idea of a "bulk metric" $\mathcal{B}$ defined as

$$\mathcal{B} := \begin{bmatrix} \delta\delta\epsilon_{00} & \dots & \delta\delta\epsilon_{03} \\ \vdots & \ddots & \vdots \\ \delta\delta\epsilon_{30} & \dots & \delta\delta\epsilon_{33} \end{bmatrix}$$

... which is really just another name for the statistical distribution $f$. Here

$$\epsilon_{ij}(m, \eta) : T_i^{(1)} M^4 \times T_j^{(1)} M^4 \to R$$

is a metric for each tuple $(i, j)$, where $T^{(1)}M^4$ is the first jet bundle on $M^4$, and $T_i^{(1)}M^4$ is its $i$th component.

*Remark.* To be clear about the terminology here, for an $n$-manifold $M$, we define the tangent bundle $TM$ (interchangeably $T^{(0)}M$) to be the set of tuples $(m, v)$ with $v \in T_mM$ an $n$-vector, where $T_mM$ is the tangent space to $M$ at $m \in M$. The first jet bundle $T^{(1)}M$ is the set of tuples $(m, V)$ with $V \in T_m^{(1)}M$ an $n \times n$ matrix, withe $T_m^{(1)}M$ the first jet bundle space to $M$ at $m \in M$.

We would also be interested in a secondary statistical distribution using said bulk metric in order to describe geometry:

$$g(m, \eta|a, b, c) := \delta(\mathcal{B}(m, \eta|a, b) - c)$$

In particular, I claim that such considerations might allow one to expand one's remit of atomic units of structure from $\mathcal{Z}$ to $\mathcal{Z}[x]$. Within number theory, there are also intriguing lines of potential investigation towards looking into questions concerning criticality properties of 3-tuples of integers, the criticality properties of rules acting on sequences of integers (eg, the two rules that formulate the Collatz conjecture), as well as potentially criticality properties of 1-tuples - i.e., pertaining to the distribution - of the set of irreducible integer polynomials.

Things become more intriguing still if we observe that $p(0) = 1$ (distribution of irreducible integer polynomials), $p \circ p(0) = 2$ (pairs of rules on sequences of integers), $p \circ p \circ p(0) = 3$ (3-tuples of integers) relate to different levels of structure that can be described in a critical way by such considerations, but the combinatorial nature of such a sequence itself may be mutable and might - nay should - be more abstract at deeper levels of subtlety. Indeed, one might wish to consider instead of $p : N \to N$ certain irreducible integer polynomials instead mapping between more complex topologies, or other constructions.

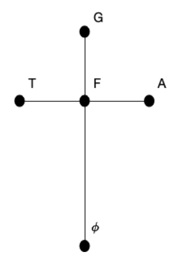
Apart from the digression above, further progression necessitates a deeper delve into higher category theory - in particular, the theory of higher topoi [17], [18], [14], [12]. We need to be aware as to how we should decorate standard notions of higher category with topological information as metadata - in particular, with Lens spaces, and more exotic topologies.

Finally, we need to consider combinatorics carefully in what we do, particularly in regards to Ramsay theory [21], [1], [23].

## 2 Consideration of structure of structure

The structures for geometry ([4]), algebra ([5]), topology ([9], [10]), as well as foundations ( [11] , [3] ) form a diagram for first subtlety as follows:

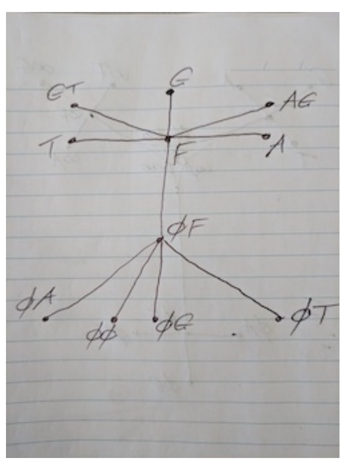
*Figure 1. Structure of structures at first subtlety*




This is interesting, and combined with observations about the distribution of elliptic curves documented here [15], indicates that there may be a connection. Regardless, adopting the convention that this describes first subtlety, what about 2nd subtlety?

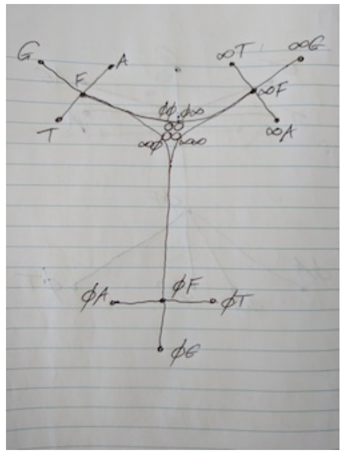
Moving to deeper abstraction, there are a number of different ways we can extend this diagram to deeper subtlety, and two of these are as follows:

*Figure 2. Structure of structure with blowup of null and extension of generality*



and:

*Figure 3. Structure of structures with blowup of null and infinity*




We now abuse notation and use $\phi$ for first infinitesimal infinity and $\infty$ for first uncountable infinity (the cardinality of the reals, say). A better way to think of this is in extending the reals to the surreal numbers. To see how that plays out in practice, consider the prototypical example of a signal function over a statistical manifold:

$$f(m, \alpha) := \delta(\sigma(m) - \alpha)$$

where $\sigma : M \to \{TM \times TM \to R\}$ is a metric on a differentiable manifold $M$.

If we lift this to the idea of considering maybe two copies of $M$:

$$f(a, b, \alpha) := \delta(\sigma(a, b) - \alpha)$$

with $\sigma : M \times M \to \{TM \times TM \to R\}$ this seems on the face of it to be a good way to wrap a higher infinity (two copies of $M$) into a smaller one (one copy of $M$) - consider say $M = \mathcal{R}$, the reals. But this is nonsense because the cardinality of $M^2$ is the same as the cardinality of $M$! Hence it is more logical to consider

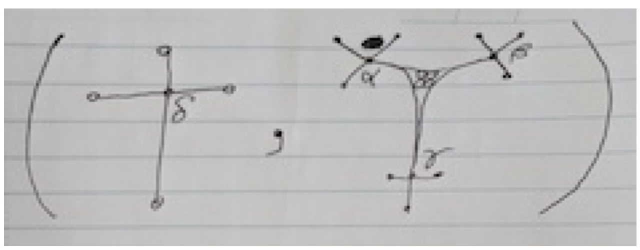
$$f(\phi, \alpha) := \delta(\sigma(\phi) - \alpha)$$

as our prototypical signal function, with $\sigma : \{f|f : M \to M\} \to \{TM \times TM \to R\}$. So $\phi$ is now no longer a point in $M$, but a function in the set of maps from $M$ to itself. The cardinality of $\{f|f : M \to M\}$ is greater than that of $M$ as it contains the power set of $M$, hence this is a logical way to 'compute surreal type quantities'.

Now, observe above that the choice of form of extension of a diagram at first subtlety to classification of structures of deeper subtlety was not unique. Therefore we might be interested in a classification of the structure of diagrams that describe structure, i.e. the structure of structure of structure.

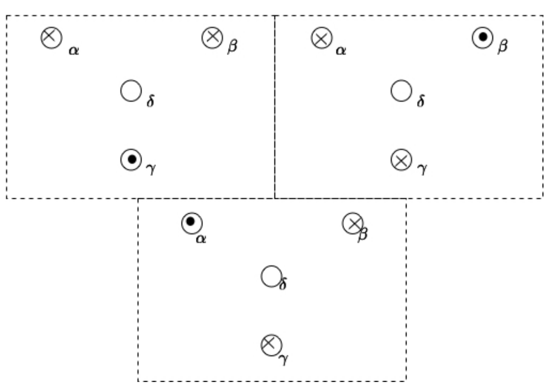
A logical way to represent diagrams that describe things like this is as with the above, but now a 2-tuple containing a diagram in each position. For instance:

*Figure 4. Structure of Structure of Structures*



It seems reasonable to posit some form of braid structure between these two entries, such as the winding of the second entry around the first with additional structural information associated to the braid group $S_3$.

*Figure 5. Structural tuple braid structure*



These above diagrams in particular are, according to the aesthetic sense and intuition of the author, the most symmetric and natural way to describe the structure of structure of structure, and therefore form a natural cornerstone on which to consider theories of subtlety 2. Note that in this diagram the two components wrap around each other with a braid group isomorphic to the symmetry group with 3 elements, $S_3$. Here three of the six elements of this group are shown. A circle with a cross denotes deeper into the page, a dot indicates 'on the page', and a circle with a dot indicates 'above the page'.

Certainly these notions could be made much more precise. Indeed, regardless of the veracity of the second figure, it seems clear that a better understanding of the concerns here could potentially be served well by reviewing some related concepts in the literature [19], [20], [16], [13], [24], [22].

Foundations for this sort of diagram are best described from the outset with a natural idea of understanding the concept of a 'pyramid' category, or a $\Delta^2$-category. Intuition for this is roughly as follows. Consider a 1-category. This is our first vertex, and in the theory of circle categories, contains information about things like geometry. Now consider the idea of 2-categories, 3-categories, $\cdots$, $\infty$-categories. Evidently there is no way to describe an $\infty$-category meaningfully in full generality with a finite amount of information, so we need a way to make them meaningful through construction of a compressed form. Suppose we have a way of creating a 1-projection or '1-submersion' of an $n$-category to first subtlety. So consider this 1-projection of an $\infty$-category. This will manifest in terms of stochasticity or the Ito calculus, and is within the remit of the practitioners of such curiosities as loop quantum gravity. This is our 2nd vertex.

There is a duality between the two vertices mentioned above (the 1-projection and the 1-category) - i.e., between the discrete and the continuous, between the stochastic and the deterministic. Examples of ways to get from one thing to the other exist - the prototypical are known as Fourier Transforms. So consider the space of such transforms to first subtlety. This forms our 3rd vertex.

Finally, consider the notion of stepping from an $n$-category to an $(n+1)$-category, or from an $n$-category to an $(n-1)$-category. In one way we extend, and in the other direction we compress. Description of how to do this (to take 1-steps between successive levels of abstraction) can be described by another object. This forms our 4th and final vertex.

Consequently, intuitively in aggregate these four vertices form a 2-simplex. To conduct the most natural analysis to truly lay the foundation $\mathcal{F}^{(2)}$ for the study of

2nd subtlety, of course, we need to find a way to treat the four as one, rather than as separate entities. i.e., we need a way to attach this meta-data concerning topology ($\Delta^2$) to an object, and then from this formalism define in a natural manner the idea of a $\Delta^2$-category. This is a challenge for a future paper.

(If one wanted to 'make everything topology', one would consider first subtlety as the study of $\bigcirc$-categories (circle categories) rather than 1-categories, and the study of second subtlety as the study of $\Delta^{\bigcirc \otimes \bigcirc}$-categories, or equivalently $\Delta^{\mathcal{T}}$-categories, where $\mathcal{T}$ signifies the metadata for a torus. Indeed, maybe $\Delta^2$-categories are not the correct objects to study, but rather $\mathcal{T}$-categories! That might make more sense, because with lift to third subtlety it seems natural to consider (Lens) $L_{p(0),p(1)}$-spaces, so that $L_{p(0),p(1)}$-categories become the objects of interest (at deeper subtlety still, it seems logical to consider numbers lifted from the first radical set of reals associated to a particular prime number, so that we are considering 'higher primes' that are derived from the base prime number sequence, and we have

$$L^{(1)}_{\begin{bmatrix} r_{p_0,p_0} & r_{p_1,p_0} & r_{p_0+p_1,p_0} \\ r_{p_0,p_1} & r_{p_1,p_1} & r_{p_0+p_1,p_1} \end{bmatrix}}$$

spaces, i.e. higher Lens spaces with parameters specified by the composition of the Fibonacci number sequence with elements from the first radical set of numbers, such that $r_{i,j}$ is 'select the $i$-th element from the first radicals associated to the integer $j$'). Anyway, a topic for a later paper, or potentially more than one future paper.).

## 3 Complexity measures on maps of structure

We might ask a seemingly naive question: why should the lower part of the cross in the first diagram above be longer than the others? The short answer turns out that the delta in complexity from null to foundations is larger than the delta in complexity from foundations to geometry, algebra, and topology.

The longer answer turns out to be more interesting, because it comes down to how we measure complexity as a metastructural metric, and how and when that might be a useful thing to consider. There are various measures of complexity that are interesting and worthy of investigation: Kolmogorov complexity, Chaitan complexity, etc. A further study of this area would be lacking if one did not also consider ideas regarding complexity, and how these provide weights or measures of linkages within structural maps as above.

# 4 Conclusions

A map has been sketched above, and some aesthetic remarks about structure theory provided. In order to make progress, it would be useful to make sure first that our foundations are solid. To do that, we should first make a more careful study of torus categories, and understand their connection with the theory of elliptic curves - either as a decoration, or as a core component of further structure seeking. The next step after that might be a study of Lens spaces and continued exploration of number theoretic consequences.

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