ON PC-COMPACT SPACES *

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Abstract

In this paper we consider a new class of topological spaces, called *pc*-compact spaces. This class of spaces lies strictly between the classes of strongly compact spaces and C-compact spaces. Also, every *pc*-compact space is p-closed in the sense of Abo-Khadra. We will investigate the fundamental properties of *pc*-compact spaces, and consider their behaviour under certain mappings.

1 Introduction and Preliminaries

In 1989, Abo-Khadra [1] introduced a new type of compactness called p-closedness, which was further investigated by Dontchev et al. in [3]. It turned out that p-closedness is placed strictly between strong compactness [12] and quasi-H-closedness [17]. In [19], Viglino introduced and studied a subclass of the class of quasi-H-closed spaces, which he called C-compact spaces. By utilizing preopen sets, we obtain in an analogous manner a new class of spaces which we shall call *pc*-compact spaces. In this paper we will study the fundamental properties of *pc*-compact spaces and examine their behaviour under certain mappings.

Let (X, τ) be a topological space. $S \subseteq X$ is called *preopen* if $S \subseteq int(cl(S))$. $S \subseteq X$ is said to be *preclosed* if $X \setminus S$ is preopen, i.e. if $cl(int(S)) \subseteq S$. The *preclosure* of an arbitrary subset $A \subseteq X$ is the smallest preclosed set containing A, and will be denoted by $pcl(A)$. The *pre-interior* of a subset $A \subseteq X$ is the largest preopen set contained in A, and will be denoted by $pint(A)$. It is well known that $pcl(A) = A \cup cl(int(A))$ and $pint(A) = A \cap int(cl(A))$.

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Definition 1 A topological space (X, τ) is called

(i) *p-closed* [1] if every preopen cover of X has a finite subfamily whose preclosures cover X, i.e. if $\{V_\lambda : \lambda \in \Lambda\}$ is a preopen cover of X, there exists a finite subset $\Lambda_0 \subseteq \Lambda$ such that $X = \bigcup\{pcl(V_\lambda) : \lambda \in \Lambda_0\}$,

(ii) *quasi-H-closed* [17] if every open cover of X has a finite subfamily whose closures cover X ,

(iii) *strongly compact* [12] if every preopen cover of X has a finite subcover.

It is clear that every strongly compact space is p-closed, and that every p-closed space is quasi-H-closed. We also observe that a space (X, τ) is quasi-H-closed if and only if every preopen cover has a finite subfamily whose union is dense. Recall that a space (X, τ) is called *irresolvable* if it cannot be represented as a disjoint union of two dense subsets. (X, τ) is said to be *strongly irresolvable* [5] if every open subspace is irresolvable. (X, τ) is called *submaximal* if every dense subset is open, or, equivalently, if every preopen subset is open.

Theorem 1.1 *[3]* Let (X, τ) be a T_0 space. Then (X, τ) is p-closed if and only if (X, τ) is quasi-H-closed and strongly irresolvable.

Definition 2 A subset A of (X, τ) is called

(i) *p-closed relative to* (X, τ) [3] if every cover of A by preopen sets of (X, τ) has a finite subfamily whose preclosures cover A ,

(ii) *quasi-H-closed relative to* (X, τ) [17] if every cover of A by open sets of (X, τ) has a finite subfamily whose closures cover A.

2 PC-compact Spaces

Definition 3 A topological space (X, τ) is said to be

(i) *pc-compact* if every preclosed subset of (X, τ) is p-closed relative to (X, τ),

(ii) *C-compact* [19] if every closed subset of (X, τ) is quasi H-closed relative to (X, τ).

Clearly, every pc-compact (resp. C-compact) space is p-closed (resp. quasi-H-closed). It is easily checked that every strongly compact space is pc-compact. Moreover, since $pcl(V) = cl(V)$ for every open set V, we conclude that every pc-compact space must be C-compact.

Remark 2.1 So far we have observed the following implications for a space (X, τ) :

strongly compact	$\Rightarrow$	pc-compact	$\Rightarrow$	C-compact
$\Downarrow$		$\Downarrow$		$\Downarrow$
compact		p-closed	$\Rightarrow$	quasi H-closed

Next we will show that none of the implications above can be reversed.

Example 2.2 There exists a p-closed space which fails to be C-compact, hence cannot be pc-compact.

Let $\kappa\mathbb{N}$ denote the Katetov extension of the natural numbers $\mathbb{N}$. Recall that the points of $\kappa\mathbb{N}$ are the points of $\mathbb{N}$ and all free ultrafilters on $\mathbb{N}$. The topology of $\kappa\mathbb{N}$ is as follows : for each $n \in \mathbb{N}$, $\{n\}$ is open, and if $\alpha \in \kappa\mathbb{N} \setminus \mathbb{N}$, then a basic neighbourhood of α has the form $\{\alpha\} \cup U$ where $U \subseteq \mathbb{N}$ and $U \in \alpha$. It has been pointed out in [3] that $\kappa\mathbb{N}$ is p-closed.

We next show that $\kappa\mathbb{N}$ is not C-compact. Let $\{U_n : n \in \mathbb{N}\}$ be a partition of $\mathbb{N}$ where each U_n is infinite. For each $n \in \mathbb{N}$, let α_n be a free ultrafilter on $\mathbb{N}$ such that $U_n \in \alpha_n$, and let $A = \{\alpha_n : n \in \mathbb{N}\}$. Then $A \subseteq \kappa\mathbb{N} \setminus \mathbb{N}$ is closed in $\kappa\mathbb{N}$. Now let $S_n = \{\alpha_n\} \cup U_n$ for each $n \in \mathbf{N}$. Then each S_n is open in $\kappa\mathbb{N}$.

Suppose that $\kappa\mathbb{N}$ is C-compact. Since $\{S_n : n \in \mathbb{N}\}$ is an open cover of A, there exists a finite subset $F \subseteq \mathbb{N}$ such that $A \subseteq \bigcup\{cl(S_n) : n \in F\}$. Pick $m \in \mathbb{N} \setminus F$. Then $\alpha_m \in cl(S_n)$ for some $n \in F$. On the other hand, we clearly have that $(\{\alpha_m\} \cup U_m) \cap (\{\alpha_n\} \cup U_n) = \emptyset$, a contradiction. Thus $\kappa\mathbb{N}$ is not C-compact, hence cannot be pc-compact.

Example 2.3 (see [3]) The unit interval [0,1] with the usual topologly is compact, hence C-compact, but, by Theorem 1.1, not p-closed and hence not pc-compact.

Example 2.4 There exists a *pc*-compact space which fails to be compact, hence cannot be strongly compact.

Let $\mathbb{N}$ denote the set of natural numbers, let A be an infinite set disjoint from $\mathbb{N}$ and let $X = \mathbb{N} \cup A$. A topology τ on X is defined as follows : for each $n \in \mathbb{N}$, $\{n\}$ is open, and a basic open neighbourhood of $a \in A$ has the form $\{a\} \cup \mathbb{N}$. Clearly (X, τ) is not compact and hence not strongly compact. Observe that (X, τ) is submaximal, and thus preopen sets are open.

Let $\emptyset \neq C \subseteq X$ be preclosed (hence closed). If $n \in C$ for some $n \in \mathbb{N}$, then we have $a \in cl(C) = C$ for each $a \in A$, and thus we always have $A \cap C \neq \emptyset$ for each nonempty preclosed set C. If $S \subseteq X$ is preopen (hence open) and $a \in S$ for some $a \in A$, then $\{a\} \cup \mathbb{N} \subseteq S$ and so $pcl(S) = cl(S) = X$, since $\mathbb{N}$ is dense.

Now, if $\{S_\lambda : \lambda \in \Lambda\}$ is a preopen cover of some (nonempty) preclosed set C, then there exists $a \in A$ and $\mu \in \Lambda$ such that $a \in C$ and $a \in S_\mu$. Since $pcl(S_\mu) = X$, we have $C \subseteq pcl(S_\mu)$. Thus (X, τ) is *pc*-compact.

Recall that a subset A of a space (X, τ) is said to be *pre-regular p-open* [7] if $A = pint(pcl(A))$. One observes easily that $A \subseteq X$ is pre-regular p-open if and only if A is the pre-interior of some preclosed subset. Moreover, if $S \subseteq X$ is preopen and $T = pint(pcl(S))$, then $pcl(S) = pcl(T)$.

Proposition 2.5 *For a topological space (X, τ), the following are equivalent :*

(1) (X, τ) is PC-compact,

(2) If $A \subset X$ is preclosed and $\{D_\lambda : \lambda \in \Lambda\}$ is a family of preclosed sets such that $(\bigcap\{D_\lambda : \lambda \in \Lambda\}) \cap A = \emptyset$, then there exists a finite subset $\Lambda_0 \subseteq \Lambda$ such that $(\bigcap\{pint(D_\lambda) : \lambda \in \Lambda_0\}) \cap A = \emptyset$,

(3) For each preclosed set $A \subset X$ and each pre-regular p-open cover $\{U_\lambda : \lambda \in \Lambda\}$ of A, there exists a finite subset $\Lambda_0 \subseteq \Lambda$ such that $A \subseteq \bigcup\{pcl(U_\lambda) : \lambda \in \Lambda_0\}$.

Proof. (1) $\Leftrightarrow$ (2) and (1) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (1) : Let $\{U_\lambda : \lambda \in \Lambda\}$ be a preopen cover of $A \subseteq X$. For each $\lambda \in \Lambda$, let $S_\lambda = pint(pcl(U_\lambda))$. Then each S_λ is pre-regular p-open. Hence there exists a finite subset $\Lambda_0 \subseteq \Lambda$ such that $A \subseteq \bigcup\{pcl(S_\lambda) : \lambda \in \Lambda_0\} = \bigcup\{pcl(U_\lambda) : \lambda \in \Lambda_0\}$. □

Remark 2.6 There are, of course, also characterizations of *pc*-compact spaces in terms of certain filterbases and nets. We refer the interested reader to [7].

In our next result we provide a characterization of *pc*-compact spaces in terms of strong irresolvability.

Definition 4 A space (X,τ) is called *strongly C-compact* if every preclosed subset is quasi-H-closed relative to (X,τ) .

Theorem 2.7 Let (X,τ) be a T_0 space. Then (X,τ) is *pc*-compact if and only if (X,τ) is strongly C-compact and strongly irresolvable.

Proof. Suppose that (X,τ) is *pc*-compact. Then (X,τ) is p-closed and therefore strongly irresolvable by Theorem 1.1 . Let $A \subseteq X$ be preclosed and let $\{U_\lambda : \lambda \in \Lambda\}$ be an open cover of A . Then $A \subseteq \bigcup\{pcl(U_\lambda)\ : \lambda \in \Lambda_0\}$ for some finite subset $\Lambda_0 \subseteq \Lambda$. Since $pcl(U_\lambda) = cl(U_\lambda)$ for each $\lambda \in \Lambda$, we conclude that A is quasi-H-closed relative to (X,τ), and hence (X,τ) is strongly C-compact.

Conversely, let $A \subseteq X$ be preclosed and let $\{S_\lambda : \lambda \in \Lambda\}$ be a preopen cover of A . Let $U_\lambda = int(cl(S_\lambda))$ for each $\lambda \in \Lambda$. Then $\{U_\lambda : \lambda \in \Lambda\}$ is an open cover of A , and $cl(U_\lambda) = cl(S_\lambda)$ for each $\lambda \in \Lambda$. Since (X,τ) is strongly C-compact, there is a finite subset $\Lambda_0 \subseteq \Lambda$ such that $A \subseteq \bigcup\{cl(U_\lambda)\ : \lambda \in \Lambda_0\} = \bigcup\{cl(S_\lambda)\ : \lambda \in \Lambda_0\}$. Since (X,τ) is strongly irresolvable, S_λ is semi-open for each $\lambda \in \Lambda$ (see e.g. [6]), i.e. $S_\lambda \subseteq cl(int(S_\lambda))$ and thus $pcl(S_\lambda) = cl(S_\lambda)$. This proves that A is p-closed relative to (X,τ), and hence (X,τ) is *pc*-compact. □

We now consider subspaces of *pc*-compact spaces. We shall denote the family of preopen subsets of a subspace X_0 of a space (X,τ) by $PO(X_0)$.

Lemma 2.8 (see [13]) Let (X,τ) be a space and $A \subseteq X_0 \subseteq X$. If $A \in PO(X_0)$ and $X_0 \in PO(X)$, then $A \in PO(X)$.

Theorem 2.9 Let (X,τ) be *pc*-compact and let $X_0 \subseteq X$ be both preopen and preclosed in (X,τ) . Then the subspace X_0 is *pc*-compact.

Proof. Let $F \subseteq X_0$ be preclosed in X_0 . Then $X_0 \setminus F \in PO(X_0)$. By Lemma 2.8 , we have $X_0 \setminus F \in PO(X)$, and so $X \setminus F = (X \setminus X_0) \cup (X_0 \setminus F) \in PO(X)$, i.e. F is preclosed in (X, τ) and thus F is p-closed relative to (X, τ) . Let $\{S_\lambda : \lambda \in \Lambda\}$ be a cover of F where $S_\lambda \in PO(X_0)$ for each $\lambda \in \Lambda$. By Lemma 2.8, $S_\lambda \in PO(X)$ for each $\lambda \in \Lambda$, and so there exists a finite subset $\Lambda_0 \subseteq \Lambda$ such that $F \subseteq \bigcup\{pcl(S_\lambda)\ : \lambda \in \Lambda_0\}$. By Lemma 3.5 in [3] we have $pcl(S_\lambda) \subseteq pcl_{X_0}(S_\lambda)$. This proves that F is p-closed relative to $(X_0, \tau|_{X_0})$. Thus $(X_0, \tau|_{X_0})$ is pc-compact. □

Remark 2.10 Observe that we cannot drop the assumption that X_0 is preclosed. In Example 2.4, $\mathbb{N}$ is an open and discrete subspace of the pc-compact space (X, τ) , but neither C-compact nor pc-compact.

3 Some Mappings

Definition 5 A function $f : (X, \tau) \to (Y, \sigma)$ is said to be

(1) *almost p-continuous* [8] (or *p(Θ)-continuous* [2]) if for each $x \in X$ and each preopen set $V \subseteq Y$ containing $f(x)$, there exists an open set $U \subseteq X$ containing x such that $f(U) \subseteq pcl(V)$,

(2) *strongly M-precontinuous* [4] if for each $x \in X$ and each preopen set $V \subseteq Y$ containing $f(x)$, there exists an open set $U \subseteq X$ containing x such that $f(U) \subseteq V$,

(3) *preirresolute* [18] if for each $x \in X$ and each preopen set $V \subseteq Y$ containing $f(x)$, there exists preopen set $U \subseteq X$ containing x such that $f(U) \subseteq V$,

(4) *precontinuous* [11] if for each $x \in X$ and each open set $V \subseteq Y$ containing $f(x)$, there exists preopen set $U \subseteq X$ containing x such that $f(U) \subseteq V$,

(5) *strongly closed* [14] if $f(A) \subseteq Y$ is closed for each preclosed set $A \subseteq X$.

Remark 3.1 A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *weakly continuous* [10] if for each $x \in X$ and each open set $V \subseteq Y$ containing $f(x)$, there exists open set $U \subseteq X$ containing

x such that $f(U) \subseteq cl(V)$. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *almost weakly continuous* [9] if $f^{-1}(V) \subseteq int(cl(f^{-1}(cl(V))))$ for every open set $V \subseteq Y$. It is shown in Theorem 3.1 of [16] that a function $f : (X, \tau) \to (Y, \sigma)$ is almost weakly continuous if and only if if for each $x \in X$ and each open set $V \subseteq Y$ containing $f(x)$, there exists a preopen set $U \subseteq X$ containing x such that $f(U) \subseteq cl(V)$.

We observe that the following relations hold:

$$\begin{array}{ccccc} \text{strongly } M\text{-continuous} & \Rightarrow & \text{almost } p\text{-continuous} & \Rightarrow & \text{weakly continuous} \\ \Downarrow & & & & \Downarrow \\ \text{preirresolute} & \Rightarrow & \text{precontinuous} & \Rightarrow & \text{almost weakly continuous} \end{array}$$

Definition 6 The graph $G(f)$ of a function $f : (X, \tau) \to (Y, \sigma)$ is said to be *strongly p-closed* if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist an open set $U \subseteq X$ containing x and a preopen set $V \subseteq Y$ containing y such that $(U \times pcl(V)) \cap G(f) = \emptyset$ (or, equivalently, $f(U) \cap pcl(V) = \emptyset$) .

Recall that a space (X, τ) is called *pre-Urysohn* if for any two distinct points $x \neq y$ there exist preopen sets $U, V \subseteq X$ such that $x \in U, y \in V$ and $pcl(U) \cap pcl(V) = \emptyset$.

Theorem 3.2 Let $f : (X, \tau) \to (Y, \sigma)$ be a function.

(i) If f is almost p-continuous and (Y, σ) is pre-Urysohn, then $G(f)$ is strongly p-closed.

(ii) If $G(f)$ strongly p-closed, then $f^{-1}(K) \subseteq X$ is closed for each $K \subseteq Y$ which is p-closed relative to (Y, σ) .

Proof. (i) Let $(x, y) \in (X \times Y) \setminus G(F)$, i.e. $f(x) \neq y$. Since (Y, σ) is pre-Urysohn, there exist preopen sets $V, W \subseteq Y$ containing $f(x)$ and y, respectively, such that $pcl(V) \cap pcl(W) = \emptyset$. Since f is almost p-continuous, there exists an open set $U \subseteq X$ containing x such that $f(U) \subset pcl(V)$. Hence $f(U) \cap pcl(W) = \emptyset$, and so G(f) is strongly p-closed.

(ii) Let $K \subseteq Y$ be p-closed relative to (Y, σ) and let $x \in X \setminus f^{-1}(K)$. For each $y \in K$ we have $(x, y) \notin G(f)$ and so there exist a preopen set $V_y \subseteq Y$ containing y and an open set $U_y \subseteq X$ containing x such that $f(U_y) \cap pcl(V_y) = \emptyset$. Since $K \subseteq Y$ is p-closed relative to (Y,σ), there exists a finite subset $K_1 \subseteq K$ such that $K \subseteq \bigcup\{pcl(V_y) : y \in K_1\}$. If $U = \bigcap\{U_y : y \in K_1\}$, then U is an open neighbourhood of x satisfying $f(U) \cap K = \emptyset$. Hence $U \cap f^{-1}(K) = \emptyset$, and so $f^{-1}(K)$ is closed in (X,τ) . □

Corollary 3.3 Let $f : (X,\tau) \to (Y,\sigma)$ be a function where (Y,σ) is pre-Urysohn and *pc*-compact. Then the following properties are equivalent:

(1) f is strongly M-continuous,

(2) f is almost p-continuous,

(3) $G(f)$ is strongly p-closed,

(4) $f^{-1}(K)$ is closed for each subset $K \subseteq Y$ which is p-closed relative to (Y,σ) .

Proof. (1) ⇒ (2) ⇒ (3) ⇒ (4) follow from Theorem 3.2 .

(4) ⇒ (1): Let $F \subseteq Y$ be preclosed. Since (Y,σ) is *pc*-compact, F is *p*-closed relative to (Y,σ) and hence $f^{-1}(K)$ is closed in (X,τ). Therefore, f is strongly M-continuous. □

Theorem 3.4 If (X,τ) is *pc*-compact and $f : (X,\tau) \to (Y,\sigma)$ is a preirresolute (resp. precontinuous) surjection, then (Y,σ) is *pc*-compact (resp. C-compact).

Proof. Let $F \subseteq Y$ be preclosed (resp. closed). Since f is preirresolute (resp. precontinuous), $f^{-1}(F) \subseteq X$ is preclosed and therefore *p*-closed relative to (X,τ) . It follows from Theorem 4.14 of [3] that $F = f(f^{-1}(F))$ is *p*-closed relative to (Y,σ) (resp. quasi H-closed relative to (Y,σ)). Thus (Y,σ) is *pc*-compact (resp. C-compact). □

Corollary 3.5 If a product $\Pi\{X_\alpha : \alpha \in \Lambda\}$ is *pc*-compact, then each factor space (X_α,τ_α) is *pc*-compact.

Proof. Each projection map is an open and continuous surjection and therefore preirresolute. □

In conclusion, recall that Viglino [19] showed that every continuous function from a C-compact space into a Hausdorff space is closed. We are able to offer an analogous result.

Theorem 3.6 *Let $f : (X, \tau) \to (Y, \sigma)$ be precontinuous where (X, τ) is pc-compact and (Y, σ) is Hausdorff. Then f is strongly closed.*

Proof. Let $F \subseteq X$ be preclosed. Since (X, τ) is *pc*-compact, F is p-closed relative to X and by Theorem 4.14 of [3], $f(F)$ is quasi H-closed relative to (Y, σ). Since (Y, σ) is Hausdorff, $f(F)$ is closed. Hence f is strongly closed. $\square$

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