IDEAL TOPOLOGICAL VECTOR SPACES

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ABSTRACT. In this paper, we introduce and study the concept of ideal topological vector spaces.

1. Introduction

A topological vector space [7], [19] is a basic structure in topology in which a vector space X over a field F ($\mathbb{R}$ or $\mathbb{C}$) is endowed with a topology τ such that: the vector addition mapping $m : X \times X \to X$ defined by $m((x, y)) = x + y$ and the scalar multiplication mapping $M : F \times X \to X$ defined by $M((\lambda, x)) = \lambda \cdot x$ for all $\lambda \in F$ and $x, y \in X$ are continuous with respect to τ. Equivalently, $(X_{(F)}, \tau, \mathcal{I})$ is a topological vector space if for each $x, y \in X$, and for each open neighbourhood W of $x + y$ in X, there exist open neighbourhoods U of x and V of y in X such that $U + V \subset W$ and for each $\lambda \in F$, $x \in X$ and for each open neighbourhood W in X containing $\lambda \cdot x$, there exist open neighbourhoods U of λ in F and V of x in X such that $U \cdot V \subset W$. The concept of ideals in topological spaces has been introduced and studied by Kuratowski [13] and Vaidyanathaswamy, [20]. An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X and if $\mathcal{P}(X)$ is the set of all subsets of X, a set operator $(.)^\star$: $\mathcal{P}(X) \to \mathcal{P}(X)$, called the local function [20] of A with respect to τ and $\mathcal{I}$, is defined as follows: for $A \subset X$, $A^\star(\tau, \mathcal{I}) = \{x \in X | U \cap A \notin \mathcal{I} \text{ for every } U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau | x \in U\}$. A Kuratowski closure operator $\mathrm{Cl}^\star(.)$ for a topology $\tau^\star(\tau, \mathcal{I})$ called the $*$-topology, finer than τ is defined by $\mathrm{Cl}^\star(A) = A \cup A^\star(\tau, \mathcal{I})$ when there is no chance of confusion, $A^\star(\mathcal{I})$ is denoted by $A^\star$. If $\mathcal{I}$ is an ideal on X, then $(X, \tau, \mathcal{I})$ is called an ideal topological space. In 1990, Jankovic and Hamlett [11] introduced the notion of $\mathcal{I}$-open sets in topological spaces. In 1992, Abd El-Monsef et al. [1] further investigated $\mathcal{I}$-open sets and $\mathcal{I}$-continuous functions. Several characterizations and properties of $\mathcal{I}$-open sets were provided in [1, 15]. In this paper, we introduce and study the concept of ideal topological vector spaces.

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2. Preliminaries

If $X_{(F)}$ is a vector space then e denotes its identity element, and for a fixed $x \in X$, ${}_xT : X \to X; y \mapsto x+y$ and $T_x : X \times X; y \mapsto y+x$, denote the left and the right translation by x, respectively. The operator $+$ we call the addition mapping $m : X \times X \to X$ defined by $m((x,y)) = x+y$, and the scalar multiplication mapping $M_\lambda : F \times X \to X$ defined by $M((\lambda; x)) = \lambda \cdot x$. Recall that a topological vector space $(X_{(F)}, \tau, \mathcal{I})$ is a vector space over a field F (most often the real or complex numbers with their standard topologies) that is endowed with a topology such that: 1) Addition mapping $m : X \times X \to X$ defined by $m((x,y)) = x+y$, $x, y \in X$ is continuous function. 2) Multiplication mapping $M : F \times X \to X$ defined by $M((\lambda, x)) = \lambda \cdot x$, $\lambda \in F$, $x \in X$ is continuous function (where the domains of these functions are endowed with product topologies). Equivalently, we have a topological vector space X over a field F (most often the real or complex numbers with their standard topologies) that is endowed with a topology such that: 1) for each $x, y \in X$, and for each open neighbourhood W of $x+y$ in X, there exist neighbourhoods U and V of x and y, respectively in X such that $U + V \subset W$. 2) for each $\lambda \in F$, $x \in X$ and for each open neighbourhood W in X containing λx, there exist neighbourhoods U of λ in F and V of x in X such that $U \cdot V \subset W$ or equivalently, we have: topological vector space X over the field F ($\mathbb{R}$ or $\mathbb{C}$) with a topology on X such that $(X, +)$ is a topological group and $M : F \times X \to X$ is a continuous mapping. For a subset A of a topological space (X, τ), $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ denote the closure of A and the interior of A, respectively. A subset S of an ideal topological space $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-open [11] if $S \subset \mathrm{Int}(S^*)$. The complement of an $\mathcal{I}$-open set is said to be an $\mathcal{I}$-closed set. The $\mathcal{I}$-closure and the $\mathcal{I}$-interior, that can be defined in the same way as $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively, will be denoted by $\mathcal{I}\,\mathrm{Cl}(A)$ and $\mathcal{I}\,\mathrm{Int}(A)$, respectively. The family of all $\mathcal{I}$-open (resp. $\mathcal{I}$-closed) sets of $(X, \tau, \mathcal{I})$ is denoted by $\mathcal{I}O(X)$ (resp. $\mathcal{I}C(X)$). The family of all $\mathcal{I}$-open (resp. $\mathcal{I}$-closed) sets of $(X, \tau, \mathcal{I})$ containing a point $x \in X$ is denoted by $\mathcal{I}O(X, x)$ (resp. $\mathcal{I}C(X, x)$). A subset $M(x)$ of a topological space (X, τ) is called an $\mathcal{I}$-neighbourhood of a point $x \in X$ if there exists an $\mathcal{I}$-open set S such that $x \in S \subset M(x)$.

Definition 2.1. *A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is said to be:*

(1) *$\mathcal{I}$-continuous* [1] *if $f^{-1}(U) \in \mathcal{I}O(X)$ for every $U \in \sigma$.*
(2) *strongly $\mathcal{J}$-continuous if $f^{-1}(U) \in \tau$ for every $U \in \mathcal{J}O(Y)$.*
(3) *$\mathcal{J}$-open if $f(U) \in \mathcal{J}O(Y)$ for every $U \in \tau$.*

3. Ideal topological vector spaces

Definition 3.1. *An ideal topological vector space $(X_{(F)}, \tau, \mathcal{I})$ is a vector space X over the field F ($\mathbb{R}$ or $\mathbb{C}$) with an ideal topology $(X, \tau, \mathcal{I})$ defined on $X_{(F)}$ and standard topology on F such that: 1) for each*

$x, y \in X$, and for each open neighbourhood W of $x + y$ in X, there exist $\mathcal{I}$-open neighbourhoods U and V of x and y, respectively in X such that $U + V \subset W$
2) for each $\lambda \in F$, $x \in X$ and for each open neighbourhood W of $\lambda \cdot x$ in X, there exist $\mathcal{I}$-open neighbourhoods U of λ in F and V of x in X such that $U \cdot V \subset W$.

Theorem 3.2. *Let $(X_{(F)}, \tau, \mathcal{I})$ be an ideal topological vector space. Suppose $T_x : X \to X$ is a right translation and $M_\lambda : X \to X$ is multiplication mapping, then T_x and M_λ both are $\mathcal{I}$-continuous.*

Proof. Let y be an arbitrary element in X and let W be an open neighbourhood of $T_x(y) = y + x$. By definition of ideal topological vector space, there exist $\mathcal{I}$-open neighbourhoods U and V containing y and x, respectively, such that $U + V \subset W$. In particular, we have $U + x \subset W$ which means $T_x(U) \subset W$. The inclusion shows that T_x is $\mathcal{I}$-continuous at y. Hence T_x is $\mathcal{I}$-continuous on X. Now we prove the statement for multiplication mapping. Let $\lambda \in F$ and $x \in X$. Let W be an open neighbourhood of $M_\lambda(x) = \lambda \cdot x$. By definition of ideal topological vector spaces, there exist $\mathcal{I}$-open neighbourhoods U and V containing λ and x, respectively such that $U \cdot V \subset W$. In particular, we have $\lambda \cdot V \subset W$, which means $M_\lambda(V) \subset W$. The inclusion shows that M_λ is $\mathcal{I}$-continuous at x. Hence M_λ is $\mathcal{I}$-continuous on X. □

Theorem 3.3. *Let $(X_{(F)}, \tau, \mathcal{I})$ be an ideal topological vector space. If $A \in \tau$ then*

1. *for every $y \in X$, $A + y \in \mathcal{I}O(X)$,*
2. *for every nonzero $\lambda \in F$, $\lambda \cdot A \in \mathcal{I}O(X)$.*

Proof. 1). Let $z \in A + y$. We have to show that z is an $\mathcal{I}$-interior point of $A + y$. Now $z = x + y$, where x is some point in A. Then $T_{-y} : X \to X$ is $\mathcal{I}$-continuous for $z \in X$. Thus, for the open set A containing x; $x = T_{(-y)}(z)$, there exists an $\mathcal{I}$-open neighbourhood M_z of z such that $T_{(-y)}(M_z) = M_z + (-y) \subset A$. This implies $M_z \subset A + y$ which shows that z is an $\mathcal{I}$-interior point of $A + y$. Hence $A + y \in \mathcal{I}O(X)$.
2). Let $z \in \lambda \cdot A$. We have to show that z is an $\mathcal{I}$-interior point of $\lambda \cdot A$. Now $z = \lambda \cdot x$ for some x in A. We have multiplication mapping $M_{\lambda^{-1}} : X \to X$ is $\mathcal{I}$-continuous. Thus, for the set $A \in \tau$ containing $M_{\lambda^{-1}}(z) = \lambda^{-1} \cdot z = x$, there exists an $\mathcal{I}$-open neighbourhood U_z of z in X such that $M_{\lambda^{-1}}(U_z) = \lambda^{-1} \cdot U_z \subset A$. This implies $U_z \subset \lambda \cdot A$. This shows that z is an $\mathcal{I}$-interior point of $\lambda \cdot A$. Hence $\lambda \cdot A \in \mathcal{I}O(X)$. □

Theorem 3.4. *Let $(X_{(F)}, \tau, \mathcal{I})$ be an ideal topological vector space. If $A \in \tau$ and B is any subset of X, then $A + B \in \mathcal{I}O(X)$.*

Proof. We have by Theorem 3.3, $T_{xi}(A) = A + x_i \in \mathcal{I}O(X)$ for each $x_i \in B$. Since union of any number of $\mathcal{I}$-open sets is $\mathcal{I}$-open, $A + B = \bigcup_{x_i \in B} (A + x_i)$ is $\mathcal{I}$-open in X. □

Corollary 3.5. *Suppose $(X_{(F)}, \tau, \mathcal{I})$ is an ideal topological vector space and $A \in \tau$. Then the set $U = \overset{\infty}{\underset{n=1}{\cup}} nA$ is an $\mathcal{I}$-open set in X.*

Definition 3.6. *A bijective mapping f from an ideal topological space to itself is called $\mathcal{I}$-homeomorphism if it is $\mathcal{I}$-continuous and $\mathcal{I}$-open.*

Definition 3.7. *An ideal topological vector space $(X_{(F)}, \tau, \mathcal{I})$ is said to be $\mathcal{I}$-homogenous space if for all $x, y \in X$, there is an $\mathcal{I}$-homeomorphism f of the space X onto itself such that $f(x) = y$.*

Theorem 3.8. *Let $(X_{(F)}, \tau, \mathcal{I})$ be an ideal topological vector space. For given $y \in X$ and λ in F with $\lambda \neq \emptyset$, the right (left) translation map $T_y : x \mapsto x + y$ and multiplication map $M_\lambda : x \mapsto \lambda \cdot x$, where $x \in X$ are $\mathcal{I}$-homeomorphisms onto itself.*

Proof. It is obvious that right translations are bijective mappings. Then the translations T_y and M_λ are $\mathcal{I}$-continuous mappings. We prove that the translation T_y is $\mathcal{I}$-open. Let U be any open negihbourhood of x. Then $T_y(U) = U + y$. By Theorem 3.3, $U + y$ is $\mathcal{I}$-open in X. This proves that T_y is $\mathcal{I}$-open mapping. Similarly, we can prove that $M_\lambda : x \mapsto \lambda \cdot x$ is an $\mathcal{I}$-homeomorphism. □

Theorem 3.9. *Every ideal topological vector space $(X_{(F)}, \tau, \mathcal{I})$ is an $\mathcal{I}$-homogenous space.*

Proof. Take any $x, y \in X$ and put $z = (-x) + y$. Then $T_z : X \to X$ is an $\mathcal{I}$-homeomorphism and $T_z(x) = x + z = y$. □

Theorem 3.10. *Suppose $(X_{(F)}, \tau, \mathcal{I})$ is an ideal topological vector space and S is a subspace of X. If S contains a nonempty open set, then S is $\mathcal{I}$-open in (X, τ).*

Proof. Suppose $U \neq \emptyset$ is open subset in X such that $U \subset S$. For any $y \in S$ the set $T_y(U) = U + y$ is $\mathcal{I}$-open in X and is a subset of S. Therefore, the subspace $S = \underset{y \in S}{\cup} (U + y)$ is $\mathcal{I}$-open in X, as the union of $\mathcal{I}$-open sets. □

Definition 3.11. *A set A in a topological space X is called preopen* [16] *if $A \subset Int(Cl(A))$.*

Theorem 3.12. *Every preopen subspace S of an ideal topological vector space $(X_{(F)}, \tau, \mathcal{I})$ is also an ideal topological vector space (called ideal topological subspace of X).*

Proof. Let $x, y \in S$ and W be an open neighbourhood of $x + y$ in S. This gives W is an open neighbourhood of $x + y$ in X. Hence there exist $\mathcal{I}$-open neighbourhoods $U \subset X$ of x and $V \subset X$ of y such that $U + V \subset W$. Then the sets $A = U \cap S$ and $B = V \cap S$ are $\mathcal{I}$-open neighbourhoods of x and y, respectively in S. Also $A+B \subset U+V \subset W$. Again, let $\lambda \in F$ and $x \in S$. Let W be an open neighbourhood of $\lambda \cdot x$

in S. Since S is preopen in X, W is open neighbourhood of $\lambda \cdot x$ in X. Hence there exist $\mathcal{I}$-open neighbourhoods $U \subset F$ of λ and $V \subset X$ of x such that $U \cdot V \subset W$. Then the set $A = U \cap F$ is $\mathcal{I}$-open neighbourhood of λ in F and the set $B = V \cap S$ is $\mathcal{I}$-open neighbourhood of y in S. Also $A \cdot B \subset U \cdot V \subset W$, which means that S is an ideal topological vector space. □

Theorem 3.13. *In an ideal topological vector space, for any open neighbourhood U of e, there is an $\mathcal{I}$-open neighbourhood V of e such that $V + V \subset U$.*

Proof. Proof is simple and therefore omitted. □

Theorem 3.14. *Let A and B be subsets of an ideal topological vector space $(X_{(F)}, \tau, \mathcal{I})$. Then $\mathcal{I}\,\mathrm{Cl}(A) + \mathcal{I}\,\mathrm{Cl}(B) \subset \mathrm{Cl}(A + B)$.*

Proof. Suppose that $x \in \mathcal{I}\,\mathrm{Cl}(A)$, $y \in \mathcal{I}\,\mathrm{Cl}(B)$. Let W be an open neighbourhood of $x+y$. Then there are $\mathcal{I}$-open neighbourhoods U and V of x and y, respectively such that $U + V \subset W$. Since $x \in \mathcal{I}\,\mathrm{Cl}(A)$, $y \in \mathcal{I}\,\mathrm{Cl}(B)$, there are $a \in A \cap U$ and $b \in B \cap V$. Then $a + b \in (A + B) \cap (U + V) \subset (A + B) \cap W$. This means $x + y \in \mathrm{Cl}(A + B)$, that is $\mathcal{I}\,\mathrm{Cl}(A) + \mathcal{I}\,\mathrm{Cl}(B) \subset Cl(A + B)$. □

Theorem 3.15. *Let $f : X \to Y$ be a homomorphism of ideal topological vector spaces. If f is strongly $\mathcal{I}$-continuous at the identity e of $(X_{(F)}, \tau, \mathcal{I})$, then f is $\mathcal{I}$-continuous on X.*

Proof. Let $x \in X$. Suppose W is open neighbourhood of $y = f(x)$ in Y. Since $T_y : Y \to Y$ is $\mathcal{I}$-continuous, there is an $\mathcal{I}$-open neighbourhood V of e such that $T_y(V) = V + y \subset W$. Now from strong $\mathcal{I}$-continuity of f at e of X, there exists an open neighbourhood U of e in X such that $f(U) \subset V$. Since $T_x : X \to X$ is $\mathcal{I}$-open, the set $U + x$ is $\mathcal{I}$-open neighbourhood of x. So $f(U+x) = f(U)+f(x) = f(U)+y \subset V+y \subset W$. Therefore f is $\mathcal{I}$-continuous at x of X, and hence on X. □

Theorem 3.16. *Let $(X_{(F)}, \tau, \mathcal{I})$ be an ideal topological vector space. Then every open subspace of X is $\mathcal{I}$-closed in X.*

Proof. Let S be an open subspace of X. As right translation $T_x : X \to X$ is $\mathcal{I}$-open, $S + x$ is $\mathcal{I}$-open in X. Then $Y = \underset{x \in X \setminus S}{\cup} (S + x)$ is also $\mathcal{I}$-open. Hence $S = X \setminus Y$ is $\mathcal{I}$-closed. □

Question 1. *Are there proper examples of ideal topological vector spaces?*

Question 2. *What type of topology on an ideal finite-dimensional topological vector space makes it into a Hausdorff ideal topological space?*

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