A Boundary Operator for Simplices

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Abstract

We generalize the very well known boundary operator of the ordinary singular homology theory, discussed in many books about algebraic topology.

We describe a variant of this ordinary simplicial boundary operator, where the usual boundary $(n-1)$-simplices of each n-simplex, i.e. the 'faces', are replaced by combinations of internal $(n-1)$-simplices parallel to the faces. This construction may lead to an infinite class of extraordinary non-isomorphic homology theories. Further, we show some interesting constructions on the standard simplex.

Contents

1 Introduction

In their famous book *Foundations of Algebraic Topology* [**2**] Samuel Eilenberg and Norman Steenrod presented a new method to distinguish topological spaces. Since this time the *singular homology theory* is a very useful and successful method in mathematics and in other fields of science. This 'theory' is in fact a sequence of functors from pairs of topological spaces to the category of Abelian groups. It begins with a definition of a *boundary operator* ∂_n of a standard simplex Δ_n. It is then shown that if T is any continuous map from Δ_n into a topological space X,

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the 'boundary' $\partial_n(T)$ can be considered as the generated map if we restrict T to the topological boundary of Δ_n as the domain of T instead of the entire Δ_n.

This construction is basicly done by elementary calculations in the n-dimensional real space $\mathbb{R}^n$. In this paper we generalize this construction. Here our 'boundary operator' ∂_n is determined not only by the topological boundary but also by parts of the interior of the standard simplex. The author took the idea from a similar work which deals with cubical homology, see **[8]**. This was the natural way, because cubes are easier to handle than simplices.

In the ordinary singular homology theory the boundary operator is constructed by taking the topological boundary of an n-dimensional standard simplex Δ_n as a linear combination of $n+1$ simplices of dimension $n-1$, (the *faces*), provided with alternating signs. We generalize this by taking a linear combination of a fixed number $L+1$ of $(n-1)$-dimensional simplices parallel to each of their $n+1$ faces, provided with a coefficient tuple $\vec{m} := (m_0, m_1, m_2, \ldots, m_L)$. Note that for a fixed $L > 0$, 'our' boundary operator ${}_{\vec{m}}\partial_n$ maps not only the topological boundary but also parts of the interior of the standard simplex, in contrast to the classical singular homology theory.

In the paper we use the customary notations $\mathbb{N} := \{1, 2, 3, \ldots\}, \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{Z} := \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$, and $\mathbb{R}$ for the real numbers. We shall use the brackets $(\ldots)$ and $[\ldots]$ for tuples and to structure text and formulas, $[\ldots]$ also for intervals. The brackets $\langle\ldots\rangle$ will be needed for the boundary operator.

Let for $n \in \mathbb{N}_0$ and all $j \in \{0, 1, 2, \ldots, n\}$: $e_j := (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$ (with a single 1 at the j^{th} place) be the j^{th}-standard unit vector of the $\mathbb{R}^{n+1}$. Let

$$\Delta_n \; := \; \left\{ (x_0, x_1, x_2, \; \ldots, \; x_n) \in \mathbb{R}^{n+1} \mid 0 \leq x_j \leq 1 \, \text{for} \;\; j \in \{0, 1, 2, \ldots, n\} \; \text{ and } \sum_{j=0}^{n} x_j = 1 \right\}.$$

This means that Δ_n is the *oriented n-dimensional standard simplex* with the usual Euclidian topology, i.e. the convex hull of the $n+1$ standard unit vectors $e_0, e_1, e_2, \ldots, e_n$ of the real vectorspace $\mathbb{R}^{n+1}$. The elements of $\{e_0, e_1, e_2, \ldots, e_n\}$ are called the *vertices* of Δ_n. Note that $\Delta_n \subset \mathbb{R}^{n+1}$. The space Δ_1 is homeomorphic to $\mathbf{I} := [0, 1]$, the unit interval, and $\Delta_0 = \{1\}$. Let for all $n \in \mathbb{N}_0$ and for all topological spaces X

$$\mathcal{C}_n(X) := \{ \, T : \Delta_n \to X \mid T \text{ is continuous} \, \} \, .$$

The symbol $\mathcal{C}$ stands for *Continuous maps*. Moreover, all maps we shall use are continuous. Let $\mathcal{R}$ be a commutative ring with unit $1_{\mathcal{R}}$. We set $\mathcal{F}(\mathcal{R})_{-1}(X) := \{0\}$, and for larger n we define

$$\mathcal{F}(\mathcal{R})_n(X) := \text{The free } \mathcal{R}\text{-module with the basis } \mathcal{C}_n(X) \, .$$

It means that $\mathcal{F}(\mathcal{R})_n(X)$ consists of finite $\mathcal{R}$-linear combinations from elements of $\mathcal{C}_n(X)$. Every $\mathsf{u} \in \mathcal{F}(\mathcal{R})_n(X)$ is called a *chain*. In the special case that the ring $\mathcal{R}$ is the ring of integers $\mathbb{Z}$ it means that $\mathcal{F}(\mathbb{Z})_n(X)$ is the free Abelian group with the basis $\mathcal{C}_n(X)$. In the following we omit mostly the ring $\mathcal{R}$ in the term $\mathcal{F}(\mathcal{R})_n(X)$, and we shall write instead $\mathcal{F}_n(X)$.

Let TOP be the category of topological spaces and continuous maps as morphisms. That means $(f : X \to Y) \in \mathsf{TOP}$ if and only if X and Y are topological spaces and f is continuous. Let $\mathcal{R}$-MOD be the category of $\mathcal{R}$-Modules, let AB be the category of Abelian groups.

To describe the singular homology theory in a very compact way, we say that it is a sequence $(\mathcal{H}_n)_{n \geq 0}$ of functors, $\mathcal{H}_n : \mathsf{TOP} \longrightarrow \mathsf{AB}$. This means that we have for each number $n \in \mathbb{N}_0$ a functor $\mathcal{H}_n$ from the topological spaces and continuous maps into the Abelian groups and group morphisms,

$$\left(X \stackrel{f}{\longrightarrow} Y \right) \;\longmapsto\; \left(\mathcal{H}_n(X) \stackrel{\mathcal{H}_n(f)}{\longrightarrow} \mathcal{H}_n(Y) \right) .$$

Further, it holds some additional properties, the 'axioms', which make the sequence $(\mathcal{H}_n)_{n\geq 0}$ to a useful tool in topology.

For more detailed information about singular homology theory see, for instance, **[3]**, **[5]**, **[6]**, **[7]**, **[9]**. For cubical singular homology theory see **[4]**.

Let X be a topological space. In this paper we create an infinite set of different boundary operators ${}_{\vec{m}}\partial_n$, i.e. for $L \in \mathbb{N}_0$ for each $(L+1)$-tuple $\vec{m} = (m_0, m_1, \ldots, m_L)$ from elements of the ring $\mathcal{R}$ we describe a map

$$ {}_{\vec{m}}\partial_n : \ \mathcal{C}_n(X) \longrightarrow \mathcal{F}_{n-1}(X) \ . $$

We extend the boundary operators ${}_{\vec{m}}\partial_n$ from $\mathcal{C}_n(X)$ to $\mathcal{F}_n(X)$ by linearity, and we get a chain of group morphisms

$$ \cdots\cdots \xrightarrow{{}_{\vec{m}}\partial_{n+1}} \mathcal{F}_n(X) \xrightarrow{{}_{\vec{m}}\partial_n} \mathcal{F}_{n-1}(X) \xrightarrow{{}_{\vec{m}}\partial_{n-1}} \cdots \xrightarrow{{}_{\vec{m}}\partial_2} \mathcal{F}_1(X) \xrightarrow{{}_{\vec{m}}\partial_1} \mathcal{F}_0(X) \xrightarrow{{}_{\vec{m}}\partial_0} \{0\} \, . \quad (1) $$

In the case of $L = 1$ we can prove that image$({}_{\vec{m}}\partial_{n+1})$ is a subgroup of kernel$({}_{\vec{m}}\partial_n)$, i.e.

$$ {}_{\vec{m}}\partial_n \circ {}_{\vec{m}}\partial_{n+1} = 0 \ . $$

It means that the above chain (1) of maps is actually a *chain complex*, and the Abelian group

$$ {}_{\vec{m}}\mathcal{H}_n(X) := \frac{kernel({}_{\vec{m}}\partial_n)}{image({}_{\vec{m}}\partial_{n+1})} $$

is well defined for each topological space X, for all $n \in \mathbb{N}_0$.

In this way we create an infinite set of different boundary operators, i.e. for $L = 1$ and for each coefficient pair $\vec{m} = (m_0, m_1)$ we construct a chain complex, and then we can take in each dimension the quotient module kernel/image. Hence we generate a sequence $({}_{\vec{m}}\mathcal{H}_n)_{n\geq 0}$ of functors. But to get an 'extraordinary homology theory' the *Excision Axiom* and the *Homotopy Axiom* are missing. The proofs of both axioms seem to be difficult. Assuming that the proofs have been established, it may nevertheless be uncertain whether there is any application of this extraordinary homology theory. An old paper **[1]** shows that the homology groups of an extraordinary homology which is defined by a chain complex can be expressed by a product of singular homology groups. This means that all informations about a topological space we can get from this new homology theory we already are able to derive from the known singular homology. Hence, the contribution of the present paper lies more in the related considerations about the standard simplices, which are made in the fourth and fifth section.

Very briefly we describe our work as follows. For a fixed $L \in \mathbb{N}_0$ and for all fixed tuples $\vec{m} = (m_0, m_1, m_2, \ldots, m_L) \in \mathcal{R}^{L+1}$ we try to construct a functor ${}_{\vec{m}}\mathcal{H}_n : \mathsf{TOP} \longrightarrow \mathcal{R}\text{-}\mathsf{MOD}$ for all $n \in \mathbb{N}_0$. We shall have a complete success for $L = 1$, while for $L > 1$ we are missing a set of homeomorphisms on the standard simplex Δ_n. Our construction is a generalization of the ordinary boundary operator in the usual singular homology, i.e. the case $L = 0, m_0 = 1$.

Further, for $L = 1$, we can also generate this well-known ordinary boundary operator if we choose the pair of integers $\vec{m} = (1, 0)$.

2 The Boundary Operator

Now we shall define for $L \in \mathbb{N}_0$ and a coefficient tuple $\vec{m} \in \mathcal{R}^{L+1}$ for all $n \in \mathbb{N}_0$ the 'boundary operators' ${}_{\vec{m}}\partial_n : \mathcal{F}_n(X) \to \mathcal{F}_{n-1}(X)$.

For each $n \in \mathbb{N}_0$ the maps ${}_{\vec{m}}\partial_n$ will use homeomorphisms $\Theta_{L,n,i}$ on Δ_n. For the time being the existence and some properties of these homeomorphisms are assumed, and with them we can prove ${}_{\vec{m}}\partial_n \circ {}_{\vec{m}}\partial_{n+1} = 0$. Later we construct them explicitly in the special case $L = 1$ (in the section 'Induction Step'), and this will be the most difficult part of this paper.

Assume that we have fixed a number $L \in \mathbb{N}_0$ and an $(L+1)$-tuple of ring elements $\vec{m} = (m_0, m_1, m_2, \dots, m_L) \in \mathcal{R}^{L+1}$. For all $n \in \mathbb{N}$ for all basis elements $T \in \mathcal{C}_n(X)$, for all $i \in \{0, 1, 2, \dots, L\}$ and $j \in \{0, 1, 2, \dots, n\}$ the map $\langle T \rangle_{L,\, n,\, i,\, j}$ will be an element of $\mathcal{C}_{n-1}(X)$,

$$\langle \,.\, \rangle_{L,\, n,\, i,\, j} \; : \mathcal{C}_n(X) \to \mathcal{C}_{n-1}(X) \;, \quad T \longmapsto \langle T \rangle_{L,\, n,\, i,\, j} \;.$$

Definition 1. We define for elements $(x_0, x_1, \, \dots \, , x_{n-1}) \in \Delta_{n-1} \subset \mathbb{R}^n$ and $T : \Delta_n \to X$

$$\langle T \rangle_{\,L,\, n,\, i,\, j} \; (\, x_0, x_1, \dots, x_{j-1}, x_j, \dots, x_{n-1}) \;:=\; T(y_0, y_1, \dots, y_{j-1}, v, y_j, \dots, y_{n-1}), \quad \text{with}$$

$$v := \frac{i}{(L+1)\cdot(n+1)} \;, \quad \text{and for all} \;\; k \in \{0, 1, 2, \dots, n-1\} \quad \text{let} \quad y_k := (1 - v) \cdot x_k \,.$$ □

We get $v + \sum_{i=0}^{n-1} y_i = 1$, hence $\langle T \rangle_{\,L,\, n,\, i,\, j}$ is an element of $\mathcal{C}_{n-1}(X)$. Note that if T is injective, the maps $\langle T \rangle_{\,L,\, n,\, i,\, j}$ have also this property.

Assume for all $L \in \mathbb{N}_0$ and $n \in \mathbb{N}_0$, and for all $i \in \{0, 1, 2, \dots, L\}$ the existence of a special homeomorphism $\Theta_{L,n,i}$ on Δ_n, which we shall construct later. (These homeomorphisms will be necessary for the proof of ${}_{\vec{m}}\partial_n \circ {}_{\vec{m}}\partial_{n+1} = 0$).

Definition 2. We define for all $n \in \mathbb{N}$ and for an arbitrary $T \in \mathcal{C}_n(X)$

$${}_{\vec{m}}\partial_n(T) \;:=\; \sum_{j=0}^{n} (-1)^j \cdot \sum_{i=0}^{L} m_i \cdot \Big[\langle T \rangle_{\,L,\, n,\, i,\, j} \,\circ \Theta_{L,n-1,i} \Big] \,,$$

and let ${}_{\vec{m}}\partial_0(T) := 0$ be the only possible map. Extend ${}_{\vec{m}}\partial_n$: $\mathcal{F}_n(X) \to \mathcal{F}_{n-1}(X)$ by linearity. □

See the following Figures, which illustrates the case $\mathcal{R} = \mathbb{Z}$, $L = 1$, $\vec{m} = (9, 4)$, and T is the identical map on Δ_1 or Δ_2, respectively.

Figure 1 shows on the left hand side the 1 -dimensional simplex Δ_1. In the middle simplex the four points are the locations of that what will be the 'boundary' ${}_{(9,4)}\partial_1(T)$. On the right hand side we see the 'boundary' ${}_{(9,4)}\partial_1(T)$. This 'boundary' ${}_{(9,4)}\partial_1(T)$ is a linear combination of four 0-dimensional simplices, i.e. of four points, with coefficients 9 and 4, and with alternating signs.

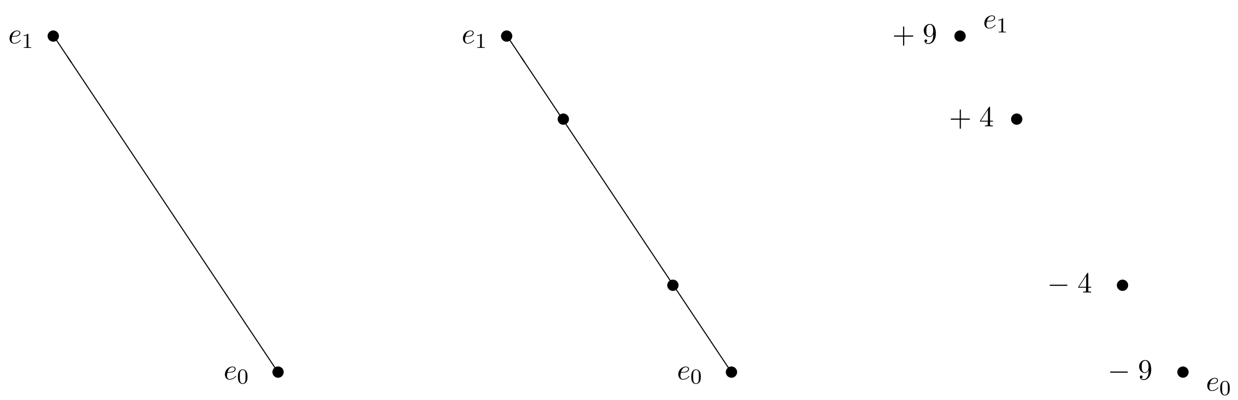

Figure 1:

See Figure 2. On the left hand side we see the 2 -dimensional standard simplex Δ_2. On the right hand side we see the subset of Δ_2 from which the 'boundary' ${}_{(9,4)}\partial_2(T)$ will be taken.

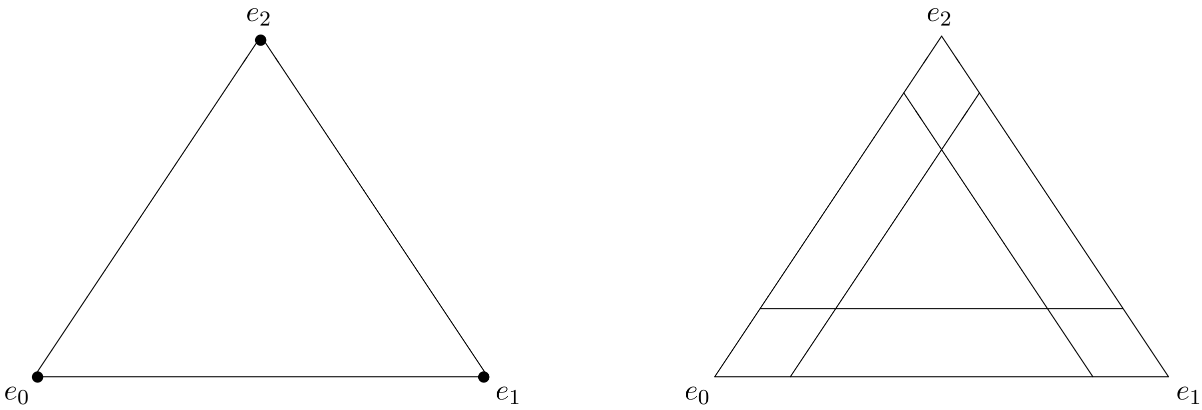

Figure 2:

Below on Figure 3 we see the 'boundary' ${}_{(9,4)}\partial_2(T)$, which consists of six 1-dimensional standard simplices $\langle T\rangle_{1,\,2,\,i,\,j}$, $i \in \{0,1\}$ and $j \in \{0,1,2\}$, multiplied by coefficients 9 and 4, elements of the ring $\mathbb{Z}$, and with alternating signs.

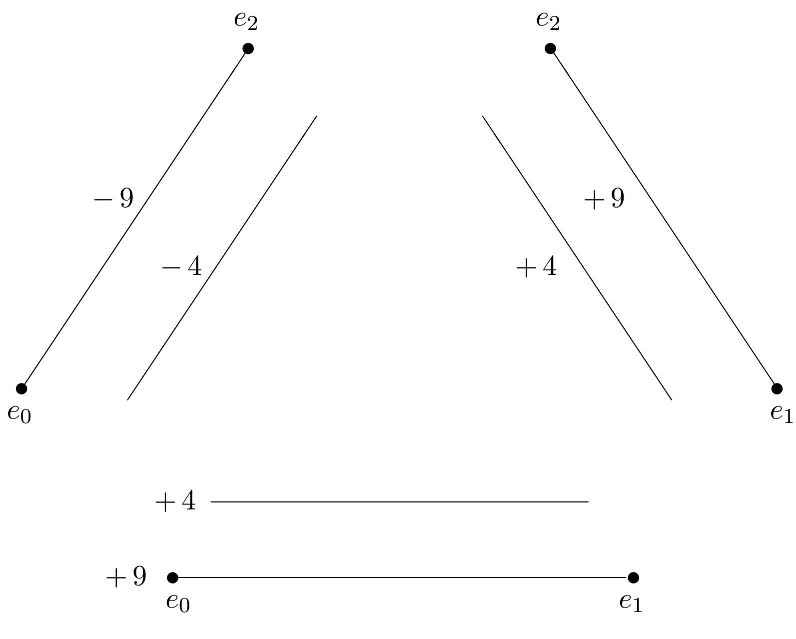

Figure 3:

Please compare the above figures with Definition (2) of the boundary ${}_{\vec{m}}\partial_n(T)$,

$$\begin{aligned} {}_{(9,4)}\partial_2(T) \;=\; & 9 \cdot [\langle T\rangle_{\,1,\,2,\,0,\,0} \circ \Theta_{1,1,0}] + 4 \cdot [\langle T\rangle_{\,1,\,2,\,1,\,0} \circ \Theta_{1,1,1}] - 9 \cdot [\langle T\rangle_{\,1,\,2,\,0,\,1} \circ \Theta_{1,1,0}] \\ & - 4 \cdot [\langle T\rangle_{\,1,\,2,\,1,\,1} \circ \Theta_{1,1,1}] + 9 \cdot [\langle T\rangle_{\,1,\,2,\,0,\,2} \circ \Theta_{1,1,0}] + 4 \cdot [\langle T\rangle_{\,1,\,2,\,1,\,2} \circ \Theta_{1,1,1}] \;. \end{aligned}$$

Now we define a set of important maps which will play a central part in the proof that ${}_{\vec{m}}\partial_n$ is a 'boundary operator', i.e. ${}_{\vec{m}}\partial_n \circ {}_{\vec{m}}\partial_{n+1} = 0$. As from now in this section we omit the parameter L in such expressions as $\langle T \rangle_{L,n,i,j}$ and $\Theta_{L,n,i}$. Note that still the maps $\Theta_{n,i}$ are not defined, we only are using some of their desired properties.

Definition 3. For every $n \in \mathbb{N}_0$ let id be the identical map on Δ_n. For all $n \in \mathbb{N}$ and for all $i, k \in \{0, 1, \ldots, L\}$ we look at some injective maps,

$$\langle id \rangle_{n+1,i,j} \circ \Theta_{n,i} \circ \langle id \rangle_{n,k,p} \circ \Theta_{n-1,k} \quad \text{for} \quad j \in \{0, 1, \ldots, n+1\}, p \in \{0, 1, \ldots, n\},$$

which are injective continuous maps from Δ_{n-1} to Δ_{n+1}.
Now let $j, p \in \{0, 1, \ldots, n\}$ with $j \leq p$. If we have the equality of the following two maps,

$$\langle id \rangle_{n+1,\, i,\, j} \circ \Theta_{n,i} \circ \langle id \rangle_{n,\, k,\, p} \circ \Theta_{n-1,k} \quad = \quad \langle id \rangle_{n+1,\, k,\, p+1} \circ \Theta_{n,k} \circ \langle id \rangle_{n,\, i,\, j} \circ \Theta_{n-1,i}\ , \tag{2}$$

then we abbreviate this important equation by $\texttt{EQUATION}_{n,j \leq p,i,k}$, for every fixed $n \in \mathbb{N}, j, p \in \{0, 1, \ldots, n\}$ with $j \leq p$, and $i, k \in \{0, 1, \ldots, L\}$. □

Remark 1. One will find a corresponding equation in every book about simplicial homology theory, e.g. in **[5, p.65]** it appears as 'If $k < j$, the face maps satisfy $\varepsilon_j^{n+1} \circ \varepsilon_k^n = \varepsilon_k^{n+1} \circ \varepsilon_{j-1}^n$'.

Now we state the theorem that if $\texttt{EQUATION}_{n,j \leq p,i,k}$ holds, the above construction leads to a 'boundary operator', this means ${}_{\vec{m}}\partial_n \circ {}_{\vec{m}}\partial_{n+1} = 0$.

Theorem 1. *Let L be a fixed element of $\mathbb{N}_0$, and let $\vec{m} := (m_0, m_1, m_2, \ldots, m_L)$ be a fixed tuple from $\mathcal{R}^{L+1}$. In addition we assume the following property: For every $n \in \mathbb{N}$, for all $i, k \in \{0, 1, \ldots, L\}$ the equation $\texttt{EQUATION}_{n,j \leq p,i,k}$ holds for all $j, p \in \{0, 1, \ldots, n\}$ with $j \leq p$. Then we have for all $n \in \mathbb{N}_0$ for all $T \in \mathcal{C}_{n+1}(X)$ (i.e. $T : \Delta_{n+1} \to X$ is continuous):*

$$ {}_{\vec{m}}\partial_n \circ {}_{\vec{m}}\partial_{n+1}(T) \ = \ 0\ .$$

Proof. The statement is trivial for $n = 0$, so let $n \in \mathbb{N}$. Note that $\langle T \rangle_{n+1,i,j}$ is a map with the domain Δ_n, and note $\langle T \rangle_{n+1,i,j} = T \circ \langle id \rangle_{n+1,i,j}$. We have

$$\begin{aligned}
&{}_{\vec{m}}\partial_n \circ {}_{\vec{m}}\partial_{n+1}(T) \ = \ {}_{\vec{m}}\partial_n \left(\sum_{j=0}^{n+1} (-1)^j \cdot \sum_{i=0}^{L} m_i \cdot \Big[\langle T \rangle_{n+1,\, i,\, j} \ \circ \ \Theta_{n,i} \Big] \right) \\
&= \ \sum_{p=0}^{n} (-1)^p \cdot \sum_{k=0}^{L} m_k \cdot \sum_{j=0}^{n+1} (-1)^j \cdot \sum_{i=0}^{L} m_i \cdot \Big[\Big\langle \langle T \rangle_{n+1,\, i,\, j} \ \circ \ \Theta_{n,i} \Big\rangle_{n,\, k,\, p} \ \circ \ \Theta_{n-1,k} \Big] \\
&= \ \sum_{p=0}^{n} (-1)^p \cdot \sum_{k=0}^{L} m_k \cdot \sum_{j=0}^{n+1} (-1)^j \cdot \sum_{i=0}^{L} m_i \cdot \Big[\langle T \rangle_{n+1,\, i,\, j} \ \circ \ \Theta_{n,i} \ \circ \ \langle id \rangle_{n,\, k,\, p} \ \circ \ \Theta_{n-1,k} \Big]
\end{aligned}$$

$$= \ \sum_{p=0}^{n} \sum_{j=0}^{n+1} \sum_{i,k=0}^{L} (-1)^{j+p} \cdot m_i \cdot m_k \cdot \Big[T \ \circ \ \langle id \rangle_{n+1,\, i,\, j} \ \circ \ \Theta_{n,i} \ \circ \ \langle id \rangle_{n,\, k,\, p} \ \circ \ \Theta_{n-1,k} \Big]\ . \tag{3}$$

The sign depends only on j and p.
The set $M := \{0, 1, 2, \ldots, n, n+1\} \times \{0, 1, 2, \ldots, n\}$ contains $(n+2) \cdot (n+1)$ elements. With

$M_{small} := \{(j,p) \in M | j \leq p\}$ and $M_{big} := \{(j,p) \in M | j > p\}$, we have $M = M_{small} \cup M_{big}$, and $M_{small} \cap M_{big} = \emptyset$. The map $\mathcal{B}: \; M_{small} \to M_{big}, \; (j,p) \mapsto (p+1, j)$ is bijective. We have

$$
\begin{aligned}
{}_{\vec{m}}\partial_n \circ {}_{\vec{m}}\partial_{n+1}(T) = & \sum_{(j,p)\in M_{small}} \sum_{i,k=0}^{L} (-1)^{j+p} \cdot m_i \cdot m_k \cdot \Big[T \circ \langle id \rangle_{n+1,i,j} \circ \Theta_{n,i} \circ \langle id \rangle_{n,k,p} \circ \Theta_{n-1,k} \Big] \\
& + \sum_{(j,p)\in M_{big}} \sum_{i,k=0}^{L} (-1)^{j+p} \cdot m_i \cdot m_k \cdot \Big[T \circ \langle id \rangle_{n+1,i,j} \circ \Theta_{n,i} \circ \langle id \rangle_{n,k,p} \circ \Theta_{n-1,k} \Big].
\end{aligned}
$$

We rename the elements $(j,p) \in M_{big}$ in $(p+1, j)$. Further, because of $\sum_{i,k=0}^{L}$, we can exchange the parts of i and k in the second sum. Hence we get the equation ${}_{\vec{m}}\partial_n \circ {}_{\vec{m}}\partial_{n+1}(T) =$

$$
\sum_{(j,p)\in M_{small}} \sum_{i,k=0}^{L} (-1)^{j+p} \cdot m_i \cdot m_k \cdot \Big[T \circ \langle id \rangle_{n+1,i,j} \circ \Theta_{n,i} \circ \langle id \rangle_{n,k,p} \circ \Theta_{n-1,k} \Big] \tag{4}
$$

$$
+ \sum_{(p+1,j)\in M_{big}} \sum_{i,k=0}^{L} (-1)^{(p+1)+j} \cdot m_k \cdot m_i \cdot \Big[T \circ \langle id \rangle_{n+1,k,p+1} \circ \Theta_{n,k} \circ \langle id \rangle_{n,i,j} \circ \Theta_{n-1,i} \Big] \tag{5}
$$

Because of the bijection $\mathcal{B}$ of M_{small} and M_{big} every summand in (4) corresponds to another in (5). Because of $\texttt{EQUATION}_{n,j\leq p,i,k}$ and because of different signs, the $(n+1) \cdot (n+2) \cdot (L+1)^2$ summands in term (3) cancel pairwise. It follows ${}_{\vec{m}}\partial_n \circ {}_{\vec{m}}\partial_{n+1}(T) = 0$. □

Remark 2. One may miss the idea behind the above definition of the boundary operator at first glance. Here is an attempt to explain it: For any $(n+1)$-simplex $\Delta \subset \mathbb{R}^{n+2}$, the 'boundary' ${}_{\vec{m}}\partial_{n+1}(\Delta)$ is a linear combination of n-simplices. We regard them as subsets of Δ. The set ${}_{\vec{m}}\partial_n \circ {}_{\vec{m}}\partial_{n+1}(\Delta)$ is a union of $(n-1)$-simplices. In fact it is the union of the intersections of two at a time of the n-simplices of ${}_{\vec{m}}\partial_{n+1}(\Delta)$. Every $(n-1)$-simplex of ${}_{\vec{m}}\partial_n \circ {}_{\vec{m}}\partial_{n+1}(\Delta)$ occurs twice. And, by factors m_i and different signs, they cancel each other. Hence ${}_{\vec{m}}\partial_n \circ {}_{\vec{m}}\partial_{n+1}(\Delta) = 0$.

A simple example with $L = 1$ is sketched in the next Figure 4, where ${}_{\vec{m}}\partial_1 \circ {}_{\vec{m}}\partial_2(\Delta_2)$ is a linear combination of 12 0-simplices, i.e. of 12 points. On the right hand side we see the remaining 12 points (without coefficients and signs). They are the intersections of the six 1-dimensional simplices of ${}_{\vec{m}}\partial_2(\Delta_2)$. By factors m_0 and m_1 and different signs they cancel each other, and we get ${}_{\vec{m}}\partial_1 \circ {}_{\vec{m}}\partial_2(\Delta_2) = 0$.

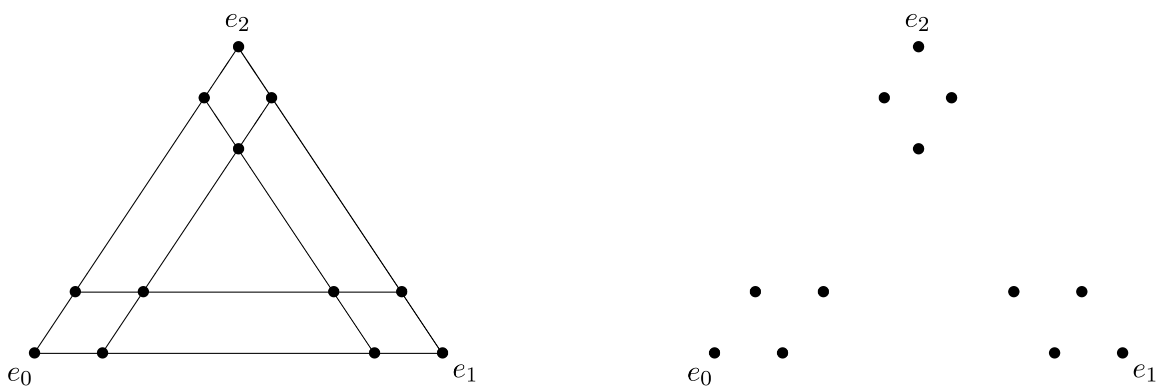

Figure 4:

The shortest way to complete this new boundary operator would be to state:
'The construction of the homeomorphisms $\Theta_{L,n,i}$ is left to the reader as an easy exercise!'

Admittedly this is just a joke; the construction of the $\Theta_{L,n,i}$'s is indeed the most difficult part in the proof of Theorem (1). (In this theorem we assumed the existence of these homeomorphisms). We shall mention a solution for $L = 0$ because it is well known, and in the next sections we shall present a solution for the special case $L = 1$. This may also show an idea of a general construction for the case of an arbitrary $L \in \mathbb{N}$. But this is still an open problem, and it is left to the reader 'as an easy exercise'.

3 Beginning of the Induction

In the main statement Theorem (1) we have assumed homeomorphisms $\Theta_{n,i}$ and the validity of some equations $\texttt{EQUATION}_{n,j\leq p,i,k}$ (for certain j, p, i, k) for each $n \in \mathbb{N}$, see Definition (3). Now we start with their constructions. The way is, as usual, by induction on n. In this section the beginning of the induction is carried out.

We formulate a trivial but important lemma.

Lemma 1. *The maps $\langle T\rangle_{L,n,i,j}$ have the important property that they respect permutations. As before, let us fix $L \in \mathbb{N}_0$, $n \in \mathbb{N}$, and $\vec{m} = (m_0, m_1, m_2, \ldots, m_L) \in \mathcal{R}^{L+1}$. Let $i \in \{0, 1, 2, \ldots, L\}$, and $j, \widetilde{j} \in \{0, 1, 2, \ldots, n\}$, with (for instance) $j < \widetilde{j}$. We take an arbitrary $T \in \mathcal{C}_n(X)$, and an n-tuple $(x_0, x_1, x_2, \ldots, x_{n-1}) \in \Delta_{n-1}$. If*

$$\langle T\rangle_{L,n,i,j}\,(x_0, x_1, \ldots, x_{j-1}, x_j, \ldots, x_{n-1}) = T\,(y_0, y_1, \ldots, y_{j-1}, v, y_j, \ldots, y_{n-1}),$$

as it is defined above in Definition (1), we have

$$\langle T\rangle_{L,n,i,\widetilde{j}}\,(x_0, x_1, \ldots, x_{n-1}) = T\Big(y_0, y_1, \ldots, y_{j-1}, y_j, \ldots, y_{\widetilde{j}-1}, v, y_{\widetilde{j}}, \ldots, y_{n-1}\Big).$$

And if ϑ is an arbitrary permutation of $\{0, 1, 2, \ldots, n-1\}$, then it holds

$$\langle T\rangle_{L,n,i,j}\,\big(x_{\vartheta(0)}, x_{\vartheta(1)}, \ldots, x_{\vartheta(n-1)}\big) = T\,\big(y_{\vartheta(0)}, \ldots, y_{\vartheta(j-1)}, v, y_{\vartheta(j)}, \ldots, y_{\vartheta(n-1)}\big)$$

and

$$\langle T\rangle_{L,n,i,\widetilde{j}}\Big(x_{\vartheta(0)}, x_{\vartheta(1)}, \ldots, x_{\vartheta(j-1)}, x_{\vartheta(j)}, \ldots, x_{\vartheta(\widetilde{j}-1)}, x_{\vartheta(\widetilde{j})}, \ldots, x_{\vartheta(n-1)}\Big)$$
$$= T\Big(y_{\vartheta(0)}, y_{\vartheta(1)}, \ldots, y_{\vartheta(j-1)}, y_{\vartheta(j)}, \ldots, y_{\vartheta(\widetilde{j}-1)}, v, y_{\vartheta(\widetilde{j})}, \ldots, y_{\vartheta(n-1)}\Big).$$

Proof. These facts are trivial, but it is necessary to mention them. □

Now we look at special cases of L.

The case L := 0. Let for all $n \in \mathbb{N}_0$: $\Theta_{0,n,0} := id(\Delta_n)$, the identity map on Δ_n, (or any other homeomorphism which leaves the topological boundary of Δ_n unchanged). With $\vec{m} := (1)$ we get the well known boundary operator of the ordinary singular homology theory with the coefficient module $\mathcal{R}$. A description can be found in **[2]**, and it is also introduced for instance in **[3]**, or **[5]**.

The case L := 1. This case is more complicated, and we shall need the rest of the paper to explain it.

We need to construct two homeomorphisms $\Theta_{1,n,0}, \Theta_{1,n,1}$ on Δ_n, for each dimension $n \in \mathbb{N}_0$. We have the singleton $\Delta_0 = \{1\}$, hence $\Theta_{1,0,0} = \Theta_{1,0,1} := id(\{1\})$, of course.

The homeomorphisms $\Theta_{1,n,0}$ will be described by a general construction, and the maps $\Theta_{1,n,1}$ will be defined by induction on n to let the equations $\texttt{EQUATION}_{n,j\leq p,i,k}$ be true.

Beginning of the induction. For $n = 1$ let us construct two homeomorphisms $\Theta_{1,1,0}, \Theta_{1,1,1}$ on Δ_1. We use the auxiliary homeomorphisms $\eta, \kappa : [0,1] \xrightarrow{\cong} [0,1]$, let η be the polygon through four points $\left\{(0,0), \left(\frac{1}{4}, \frac{1}{6}\right), \left(\frac{3}{4}, \frac{5}{6}\right), (1,1)\right\}$, and let κ be the polygon through four points $\left\{(0,0), \left(\frac{1}{4}, \frac{1}{5}\right), \left(\frac{3}{4}, \frac{4}{5}\right), (1,1)\right\}$. Note that both maps η, κ are symmetrical at the point $\left(\frac{1}{2}, \frac{1}{2}\right)$. Hence it holds $\eta(x) + \eta(1-x) = 1 = \kappa(x) + \kappa(1-x)$, for all $x \in [0,1]$. Further, note $\eta\left(\frac{1}{4}\right) = \frac{1}{6}$, and $\kappa\left(\frac{1}{4}\right) = \frac{1}{5}$.

Definition 4. Define for all pairs $(x, 1-x) \in \Delta_1$

$$\Theta_{1,1,0}(x, 1-x) := (\eta(x), \eta(1-x)) \quad \text{and} \quad \Theta_{1,1,1}(x, 1-x) := (\kappa(x), \kappa(1-x)) \ .$$

□

Now we have to consider $4 \cdot 3$ equations $\texttt{EQUATION}_{n=1,j\le p,i,k}$, for $i,k \in \{0,1\}$, and $j,p \in \{0,1\}$ with $j \le p$. But fortunately, because of Lemma (1), we can fix the positions $j = p = 0$. The other pairs $(j,p) \in \{(0,1),(1,1)\}$ work in the same manner. Up to now we omit the parameter $L = 1$ for better readability. We want to prove the correctness of

$$\langle id\rangle_{2,i,j=0} \circ \Theta_{1,i} \circ \langle id\rangle_{1,k,p=0} \circ \Theta_{0,k} = \langle id\rangle_{2,k,p+1=1} \circ \Theta_{1,k} \circ \langle id\rangle_{1,i,j=0} \circ \Theta_{0,i} \ .$$

To begin with we set $i = 0, k = 1$, i.e. we consider $\texttt{EQUATION}_{n=1,j=0\le p=0,i=0,k=1}$. We have to show the identity

$$\langle id\rangle_{2,0,0} \circ \Theta_{1,0} \circ \langle id\rangle_{1,1,0} \circ \Theta_{0,1} = \langle id\rangle_{2,1,1} \circ \Theta_{1,1} \circ \langle id\rangle_{1,0,0} \circ \Theta_{0,0} \ .$$

We need to map the set $\Delta_0 = \{1\}$, hence the left hand side of the equation is

$$\begin{aligned}\langle id\rangle_{2,0,0} \circ \Theta_{1,0} \circ \langle id\rangle_{1,1,0} \circ \Theta_{0,1}\ (1) &= \langle id\rangle_{2,0,0} \circ \Theta_{1,0} \circ \langle id\rangle_{1,1,0}\ (1) \\ = \langle id\rangle_{2,0,0} \circ \Theta_{1,0}\ \left(\tfrac{1}{4}, \tfrac{3}{4}\right) &= \langle id\rangle_{2,0,0}\ \left(\tfrac{1}{6}, \tfrac{5}{6}\right) = \left(0, \tfrac{1}{6}, \tfrac{5}{6}\right) \in \Delta_2 \ .\end{aligned}$$

The right hand side of the equation is

$$\begin{aligned}\langle id\rangle_{2,1,1} \circ \Theta_{1,1} \circ \langle id\rangle_{1,0,0} \circ \Theta_{0,0}\ (1) &= \langle id\rangle_{2,1,1} \circ \Theta_{1,1} \circ \langle id\rangle_{1,0,0}\ (1) \\ = \langle id\rangle_{2,1,1} \circ \Theta_{1,1}\ (0,1) &= \langle id\rangle_{2,1,1}\ (0,1) = \left(0, \tfrac{1}{6}, \tfrac{5}{6}\right), \quad \text{hence } \texttt{EQUATION}_{1,0\le 0,0,1} \text{ holds.}\end{aligned}$$

More beautiful is a comutative diagram (Figure 5):

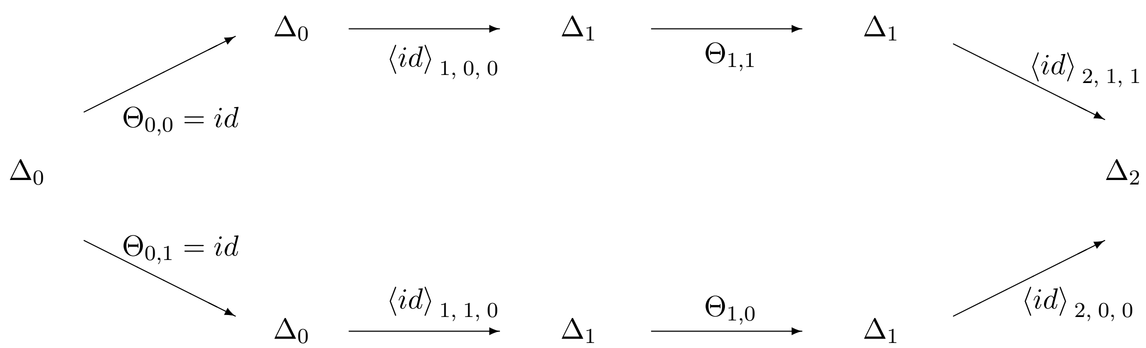

Figure 5:

We get a corresponding diagram if we replace the set Δ_0 by its single element (1), and in the following diagram (Figure 6) we show again the equation $\texttt{EQUATION}_{n=1,j=0\leq p=0,i=0,k=1}$.

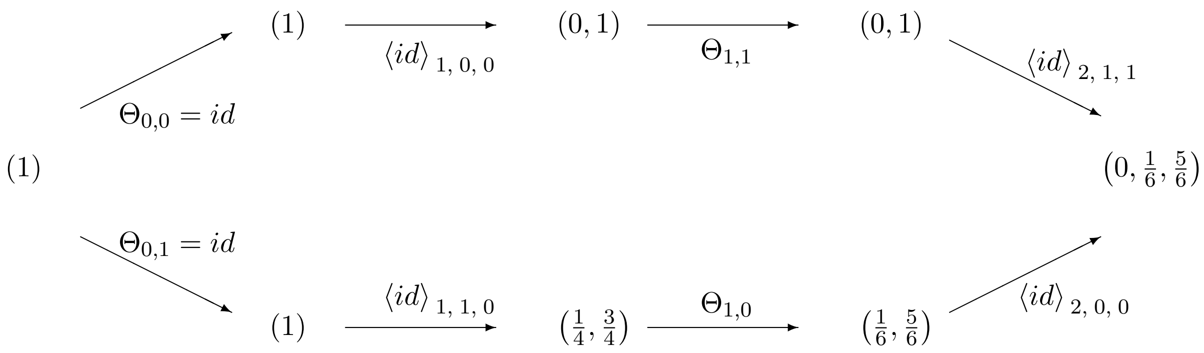

Figure 6:

If we exchange i and k we get a similar diagram, both sides of the $\texttt{EQUATION}_{1,0\leq 0,i=1,k=0}$ map the single element (1) to the point $\left(\frac{1}{6}, 0, \frac{5}{6}\right)$. Hence this equation also holds. Further, the reader may also establish that the case $i = k = 0$, i.e. the $\texttt{EQUATION}_{1,0\leq 0,i=0,k=0}$ is trivial. Only the $\texttt{EQUATION}_{1,0\leq 0,i=1,k=1}$ is missing. We map the single element $(1) \in \Delta_0$, and we prove $\texttt{EQUATION}_{1,0\leq 0,i=1,k=1}$, see the following Figure 7.

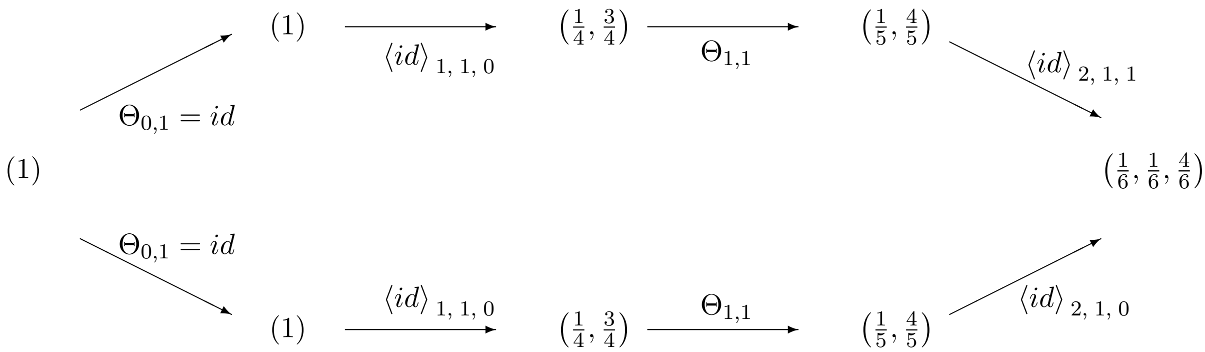

Figure 7:

As we have mentioned above, the other pairs of positions $(j, p) \in \{(0, 1), (1, 1)\}$ with $(i, k) \in \{(0, 0), (1, 1), (0, 1), (1, 0)\}$ work in the same manner. Hence we have done the beginning of the induction for $n = 1$.

Before we can continue the induction with the construction of $\Theta_{n,0}$ and $\Theta_{n,1}$ for larger n, we need to carry out some technical considerations, calculations and constructions on the standard simplices Δ_n.

4 The Geometry of the Standard Simplex

The constructions of this section will be needed in the next one. Some of the following definitions and propositions can be skipped over during the first reading. The reader may take a look on the next section first.

Some notations will be introduced, and subsets of the standard simplex will be defined. We define further $\mathcal{COMFORT}(\Delta_n)$, a subgroup of the group of all homeomorphisms of Δ_n. We discuss properties of elements of $\mathcal{COMFORT}(\Delta_n)$. At the end we talk about possibilities to extend homeomorphisms, which are defined on a subset of Δ_n onto the entire Δ_n.

Definition 5. For all $n \in \mathbb{N}_0$, let $\mathbf{Center}_n := \left(\frac{1}{n+1}, \frac{1}{n+1}, \ \ldots, \ \frac{1}{n+1}\right) \in \Delta_n$, the center of the n-dimensional standard simplex $\Delta_n \subset \mathbb{R}^{n+1}$. □

Definition 6. Let for fixed $n \in \mathbb{N}_0$ and $\alpha \in [0,1]$ the set $\clubsuit_{n,\alpha}$ be a subset of Δ_n,
$\clubsuit_{n,\alpha} := \{ \ (x_0, x_1, \ldots, x_n) \ \in \Delta_n \mid$ there is at least one $j \in \{0, \ldots, n\}$ such that $x_j = \alpha \ \}$.
$\clubsuit_{n,\alpha}$ is called the '$\alpha - Cross\ of\ \Delta_n$' .
□

See Figure 8 with two examples $\clubsuit_{2,\frac{1}{6}}$ and $\clubsuit_{2,\frac{5}{6}}$. Note that the $\frac{1}{6}$-cross $\clubsuit_{2,\frac{1}{6}}$ is connected, while the $\frac{5}{6}$-cross $\clubsuit_{2,\frac{5}{6}}$ is comprised of three components, i.e. $\clubsuit_{2,\frac{5}{6}}$ is not connected.

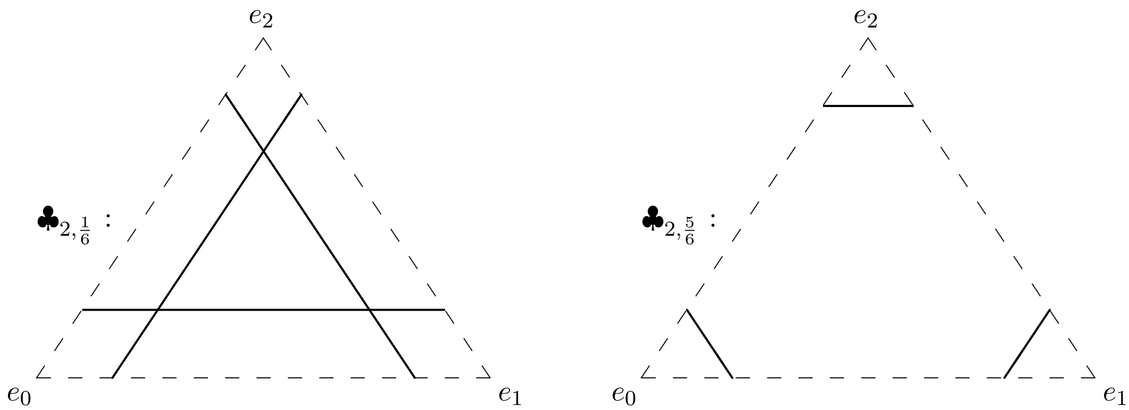

Figure 8:

Note $\clubsuit_{n,1} = \{e_0, e_1, \ldots, e_n\}$, the set of the vertices of Δ_n, and for positive n note that $\clubsuit_{n,0}$ is the topological boundary of Δ_n, while $\clubsuit_{0,0} = \emptyset$.

Lemma 2. *We have for* $0 < \alpha, \beta < \frac{1}{n+1}$ *that the α-cross $\clubsuit_{n,\alpha}$ is homeomorphic to the β-cross $\clubsuit_{n,\beta}$, i.e. there exists a homeomorphism* $\clubsuit_{n,\alpha} \xrightarrow{\cong} \clubsuit_{n,\beta}$, *for all* $n \in \mathbb{N}$.

Proof. To prove the lemma we need any increasing homeomorphism f on the interval $\left[0, \frac{1}{n+1}\right]$ with $f(\alpha) = \beta$. For instance take the polygon through three points $\left\{(0,0), (\alpha, \beta), \left(\frac{1}{n+1}, \frac{1}{n+1}\right)\right\}$. Then we use Proposition (3), which we shall deal a few pages later, and the restriction of $\mathbf{\Psi_n}(f)$ onto the subset $\clubsuit_{n,\alpha}$ of Δ_n yields the desired homeomorphism. See Remark (5). □

Definition 7. Let for all $n \in \mathbb{N}$
$\mathbf{Bound}_n := \{(x_0, x_1, \ldots, x_n) \in \Delta_n \mid \text{there is at least one } j \in \{0, 1, \ldots, n\} \text{ such that } x_j = 0\}$.
□

$\mathbf{Bound}_n$ is called the '*Boundary of* Δ_n', since $\mathbf{Bound}_n$ is the topological boundary of Δ_n. Note that $\mathbf{Bound}_n$ is homeomorphic to the $(n-1)$-sphere, and $\mathbf{Bound}_n = \clubsuit_{n,0}$.

Definition 8. Let for all $n \in \mathbb{N}$ and for an $\alpha \in \left[0, \frac{1}{n+1}\right]$, $\mathbf{Layer}_{n,\alpha} \subset \clubsuit_{n,\alpha}$,

$$\mathbf{Layer}_{n,\alpha} := \{(x_0, x_1, \ldots, x_n) \in \Delta_n \mid \alpha = \min\{x_0, x_1, \ldots, x_n\}\}.$$

$\mathbf{Layer}_{n,\alpha}$ is called the '$\alpha -$ *Layer of* Δ_n'. □

Note $\mathbf{Layer}_{n,\frac{1}{n+1}} = \{\mathbf{Center}_n\}$ and $\mathbf{Layer}_{n,0} = \mathbf{Bound}_n = \clubsuit_{n,0}$. For $0 \le \alpha < \frac{1}{n+1}$ we have that $\mathbf{Layer}_{n,\alpha}$ is homeomorphic to the $(n-1)$-sphere. There is a homeomorphism

$$\vec{x} \longmapsto \vec{x} - \mathbf{Center}_n \longmapsto \frac{\vec{x} - \mathbf{Center}_n}{\| \vec{x} - \mathbf{Center}_n \|} , \quad \text{for all } \vec{x} \in \mathbf{Layer}_{n,\alpha} .$$

We get a disjoint union $\Delta_n = \bigcup \{\mathbf{Layer}_{n,\alpha} \mid 0 \le \alpha \le \frac{1}{n+1}\}$.

Definition 9. Let for $n \in \mathbb{N}$ and for fixed $j \in \{0, 1, 2, \ldots, n\}$ $\mathbf{Section}_{n,j}$ be a subset of Δ_n,

$$\mathbf{Section}_{n,j} := \{(x_0, x_1, \ldots, x_n) \in \Delta_n \mid \min\{x_0, \ldots, x_n\} = x_j\}.$$

$\mathbf{Section}_{n,j}$ is called the '*j-Section of* Δ_n'. A subset of Δ_n of the form $\mathbf{Bound}_n \cap \mathbf{Section}_{n,j}$ is called a *face* of Δ_n. □

Note the union $\Delta_n = \bigcup \{\mathbf{Section}_{n,j} \mid j = 0, 1, 2, \ldots, n\}$ which is not disjoint, e.g. we have for all $j \in \{0, 1, \ldots, n\}$ that $\mathbf{Center}_n$ is an element of $\mathbf{Section}_{n,j}$. Further, note $\mathbf{Bound}_n \cap \mathbf{Section}_{n,j} = \{(x_0, x_1, \ldots, x_n) \in \Delta_n \mid \min\{x_0, \ldots, x_n\} = x_j = 0\}$, for $j = 0, 1, \ldots, n$.

Let n be a natural number. Now we will project every $\vec{x} \in \Delta_n$, $\vec{x} \neq \mathbf{Center}_n$ onto the α-Layer of Δ_n, for all $0 \le \alpha \le \frac{1}{n+1}$.
We will define a projection 'π' onto the boundary, $\pi := \pi_0 : \Delta_n \backslash \{\mathbf{Center}_n\} \longrightarrow \mathbf{Bound}_n$, and for $0 < \alpha \le \frac{1}{n+1}$ we will define projections $\pi_\alpha : \Delta_n \backslash \{\mathbf{Center}_n\} \longrightarrow \mathbf{Layer}_{n,\alpha}$.

Definition 10. Let $n \in \mathbb{N}$. For an arbitrary $\vec{x} \in \Delta_n \backslash \{\mathbf{Center}_n\}, \vec{x} = (x_0, x_1, x_2, \ldots, x_n)$, let ϑ be a permutation on $\{0, 1, 2, \ldots, n\}$ such that $0 \le x_{\vartheta(0)} \le x_{\vartheta(1)} \le x_{\vartheta(2)} \le \ldots \le x_{\vartheta(n)}$.

Then define $\pi(\vec{x}) := \vec{b} := (b_0, b_1, b_2, \ldots, b_n)$, we set for all $i \in \{0, 1, 2, \ldots, n\}$

$$b_i := \frac{1}{1 - (n+1) \cdot x_{\vartheta(0)}} \cdot \left(x_i - x_{\vartheta(0)}\right) .$$

We get $b_{\vartheta(0)} = 0$, hence $\pi(\vec{x}) \in \mathbf{Bound_n}$.
Generally, for $0 \le \alpha \le \frac{1}{n+1}$, let $\pi_\alpha(\vec{x}) := (y_0, y_1, y_2, \ldots, y_n)$, for $i \in \{0, 1, 2, \ldots, n\}$ we define

$$y_i := \alpha + (1 - (n+1) \cdot \alpha) \cdot b_i = \alpha + \frac{1 - (n+1) \cdot \alpha}{1 - (n+1) \cdot x_{\vartheta(0)}} \cdot \left(x_i - x_{\vartheta(0)}\right) .$$

We extend the definition of $\pi_{\frac{1}{n+1}}$ on the entire Δ_n, let $\pi_{\frac{1}{n+1}}(\mathbf{Center}_n) := \mathbf{Center}_n$. □

We get $y_{\vartheta(0)} = \alpha$, and for all $i \in \{0, \ldots, n\}$ we have $y_i \ge \alpha$, hence $\pi_\alpha(\vec{x}) \in \mathbf{Layer}_{n,\alpha}$. Note that the projection $\pi_{\frac{1}{n+1}} : \Delta_n \longrightarrow \{\mathbf{Center}_n\}$ is a constant map. Further, the maps π_α are surjective on $\mathbf{Layer}_{n,\alpha}$, and they have the property $\pi_\alpha \circ \pi_\alpha = \pi_\alpha$, for all fixed $\alpha \in \left[0, \frac{1}{n+1}\right]$.

Consider also the following continuous surjective map $\mathcal{A} : \Delta_n \longrightarrow \left[0, \frac{1}{n+1}\right]$.

Definition 11. Let $n \in \mathbb{N}$. Let $\vec{x} = (x_0, x_1, \dots, x_n)$ be an arbitrary element of Δ_n. We define the number $\mathcal{A}(\vec{x}) \in \left[0, \frac{1}{n+1}\right]$, let $\mathcal{A}(\vec{x}) := \min\{x_0, x_1, \dots, x_n\}$. $\square$

We abbreviate $t := \mathcal{A}(\vec{x}) \cdot (n+1)$, i.e. $t \in [0,1]$. Then we have for $\vec{x} \neq \mathbf{Center}_n$ the representations

$$\begin{aligned} \vec{x} &= \pi(\vec{x}) + \mathcal{A}(\vec{x}) \cdot (n+1) \cdot (\mathbf{Center}_n - \pi(\vec{x})) = \pi(\vec{x}) + t \cdot (\mathbf{Center}_n - \pi(\vec{x})) \\ &= t \cdot \mathbf{Center}_n + (1-t) \cdot \pi(\vec{x}) . \end{aligned}$$

For $\vec{x} \in \Delta_n$, we have $\vec{x} = \pi_{\mathcal{A}(\vec{x})}(\vec{x}) \in \mathbf{Layer}_{n,\mathcal{A}(\vec{x})}$, and for $0 < \alpha, \mathcal{A}(\vec{x}) < \frac{1}{n+1}$ and $\mathcal{A}(\vec{x}) \neq \alpha$, there are four collinear points $\{\mathbf{Center}_n, \vec{x}, \pi_\alpha(\vec{x}), \pi(\vec{x})\}$.

Note that if we restrict $\mathcal{A}$ on an α-layer, for all $0 \leq \alpha \leq \frac{1}{n+1}$, $\mathcal{A}$ will be a constant map, $\mathcal{A}(\vec{x}) = \alpha$ for all $\vec{x} \in \mathbf{Layer}_{n,\alpha}$, because $\alpha = \min\{x_0, \dots, x_n\}$.

Definition 12. Let $n \in \mathbb{N}$ and $\vec{a}, \vec{b} \in \mathbb{R}^n$. We define $\left[\vec{a}, \vec{b}\right] \subset \mathbb{R}^n$. Let $\left[\vec{a}, \vec{b}\right] := \left\{ t \cdot \vec{a} + (1-t) \cdot \vec{b} \mid t \in [0,1] \right\}$ be the line segment confined by $\vec{a}$ and $\vec{b}$. $\square$

Note that for any standard n-simplex Δ_n we have the (nearly disjoint) union $\Delta_n = \bigcup \left\{ \left[\mathbf{Center}_n, \vec{b}\right] \mid \vec{b} \in \mathbf{Bound}_n \right\}$, all the lines intersect only in $\mathbf{Center}_n$. Further, note that for all $\vec{b} \in \mathbf{Bound}_n$ we have a constant projection

$$\pi|_{[\mathbf{Center}_n, \vec{b}] \setminus \{\mathbf{Center}_n\}} : \left[\mathbf{Center}_n, \vec{b}\right] \setminus \{\mathbf{Center}_n\} \longrightarrow \vec{b} .$$

For a point $\vec{x} = (x_0, \dots, x_n) \in \left[\mathbf{Center}_n, \vec{b}\right]$, $\vec{x} \neq \mathbf{Center}_n$, i.e. $\pi(\vec{x}) = \vec{b}$, we have the unique number $t = \mathcal{A}(\vec{x}) \cdot (n+1) \in [0,1]$ such that $\vec{x} = t \cdot \mathbf{Center}_n + (1-t) \cdot \vec{b}$. Let $\vec{b} = (b_0, \dots, b_n)$. For a component x_j we get the notation $x_j = t \cdot \frac{1}{n+1} + (1-t) \cdot b_j = b_j + t \cdot \left(\frac{1}{n+1} - b_j\right)$.

Definition 13. Let $n \in \mathbb{N}$. For any subset $M \subset \mathbb{R}^n$, $M \neq \emptyset$, let $\mathsf{Sponge}(M) := \{(x_1, x_2, \dots, x_n) \in M \mid x_i = x_j$ if and only if $i = j$, for $1 \leq i, j \leq n\}$. $\square$

This means that $\mathsf{Sponge}(M)$ contains n-tuples $(x_1, x_2, \dots, x_n)$ only with pairwise different components.

Remark 3. A continuous map $f : \mathsf{Sponge}(\Delta_n) \to \Delta_n$ is uniquely extendable to a continuous map $F : \Delta_n \to \Delta_n$, since $\text{closure}(\mathsf{Sponge}(\Delta_n)) = \Delta_n$. This means $f = F_{|\mathsf{Sponge}(\Delta_n)}$.

Now we define a subgroup of the group of all homeomorphisms on Δ_n, which has some comfortable properties.

Definition 14. For a fixed $n \in \mathbb{N}_0$ let $\mathcal{COMFORT}(\Delta_n) \subset \{F : \Delta_n \to \Delta_n\}$. A map F is an element of $\mathcal{COMFORT}(\Delta_n)$ if an only if F fulfils the following conditions $\widehat{[\mathbf{1}]}, \widehat{[\mathbf{2}]}, \widehat{[\mathbf{3}]}$.

$\widehat{[\mathbf{1}]}$: F is a homeomorphism on Δ_n .

$\widehat{[\mathbf{2}]}$: F respects permutations, that means if $\vec{x} = (x_0, x_1, \dots\dots, x_n) \in \Delta_n$ and if $F(\vec{x}) = \vec{y} = (y_0, y_1, \dots\dots, y_n) \in \Delta_n$, and if ϑ is a permutation on $\{0, 1, 2, \dots, n\}$, then it holds

$$F\left(x_{\vartheta(0)}, x_{\vartheta(1)}, x_{\vartheta(2)}, \dots\dots, x_{\vartheta(n)}\right) = \left(y_{\vartheta(0)}, y_{\vartheta(1)}, y_{\vartheta(2)}, \dots\dots, y_{\vartheta(n)}\right).$$

$\widehat{[\mathbf{3}]}$: F keeps the order. Trivially, for every $\vec{x} = (x_0, x_1, \dots\dots, x_n) \in \Delta_n$ exists a permutation ϑ on $\{0, 1, 2, \dots, n\}$ and a number $\mathtt{r} \in \{0, 1, 2, \dots, n\}$ (we introduce here the number $\mathtt{r}$, it will play a major part not until Proposition (2)) such that

$$0 \leq x_{\vartheta(0)} \leq x_{\vartheta(1)} \leq \ \dots \ \leq x_{\vartheta(\mathtt{r})} \leq \frac{1}{n+1} < x_{\vartheta(\mathtt{r}+1)} \leq \ \dots \ \leq x_{\vartheta(n)} \leq 1.$$

If $F(\vec{x}) = \vec{y} = (y_0, y_1, \ldots\ldots, y_n)$, we demand that
$$0 \le y_{\vartheta(0)} \le y_{\vartheta(1)} \le \ \ldots \ \le y_{\vartheta(j)} \le y_{\vartheta(j+1)} \le \ \ldots \ \le y_{\vartheta(n)} \le 1$$
holds for all $j \in \{0, 1, 2, \ldots, n-1\}$.

For any subset $\mathbf{S} \subset \Delta_n$ we say that a homeomorphism $F : \mathbf{S} \to \mathbf{S}$ is an element of $\mathcal{COMFORT}(\mathbf{S})$ if and only if F fulfils both $\widehat{[\mathbf{2}]}$ and $\widehat{[\mathbf{3}]}$. □

It follows for an $F \in \mathcal{COMFORT}(\Delta_n)$ that the homeomorphism F yields a homeomorphism on each $\mathbf{Section}_{n,k}$, $F|_{\mathbf{Section}_{n,k}} \in \mathcal{COMFORT}(\mathbf{Section}_{n,k})$ for $k \in \{0, 1, 2, \ldots, n\}$.

For $F, G \in \mathcal{COMFORT}(\Delta_n)$ it holds that $F^{-1} \in \mathcal{COMFORT}(\Delta_n)$, and furthermore $F \circ G \in \mathcal{COMFORT}(\Delta_n)$, hence $(\mathcal{COMFORT}(\Delta_n), \circ)$ is a subgroup of the group of all homeomorphisms on Δ_n.

We remark that a homeomorphism $f \in \mathcal{COMFORT}(\mathsf{Sponge}(\Delta_n))$ can be uniquely extended to a map $F \in \mathcal{COMFORT}(\Delta_n)$ with $f = F_{|\mathsf{Sponge}(\Delta_n)}$.

Note that each element of the set $\left\{\pi_\alpha \,|\, \alpha \in \left[0, \frac{1}{n+1}\right]\right\}$ of projections fulfils the conditions $\widehat{[\mathbf{2}]}$ and $\widehat{[\mathbf{3}]}$ of the above Definition (14), and all π_α except $\pi_{\frac{1}{n+1}}$ are elements of the following monoid of continuous functions
$$\left(\left\{f : \Delta_n \backslash \{\mathbf{Center}_n\} \to \Delta_n \backslash \{\mathbf{Center}_n\} \mid f \text{ fulfils the conditions } \widehat{[\mathbf{2}]} \text{ and } \widehat{[\mathbf{3}]} \text{ of Definition (14)}\right\}, \circ\right).$$

The next few lemmas deal with the behaviour of homeomorphisms $\Phi \in \mathcal{COMFORT}(\Delta_n)$.

Lemma 3. *Let Φ be any homeomorphism on Δ_n. Then $\Phi|_{\mathbf{Bound}_n} : \mathbf{Bound}_n \overset{\cong}{\to} \mathbf{Bound}_n$. For $F \in \mathcal{COMFORT}(\Delta_n)$ it follows that $F|_{\mathbf{Bound_n}}$ is an element of $\mathcal{COMFORT}(\mathbf{Bound_n})$.*

Proof. This is trivial, because $\mathbf{Bound}_n$ is the topological boundary of Δ_n . □

The following three lemmas describe the fact that a map $\Phi \in \mathcal{COMFORT}(\Delta_n)$ preserves equalities and inequalities of the components of an $\vec{x} \in \Delta_n$.

Lemma 4. *Let Φ be an element of $\mathcal{COMFORT}(\Delta_n)$. Trivially, for every $\vec{b} \in \mathbf{Bound}_n$ with components $\vec{b} = (b_0, b_1, \ldots, b_n)$ there is a permutation ϑ on $\{0, 1, \ldots, n\}$ and there are two natural numbers $\mathtt{q}, \mathtt{r}$, with $0 \le \mathtt{q} \le \mathtt{r} \le n-1$ such that $\Phi(\vec{b}) =: \vec{c} =: (c_0, c_1, \ldots, c_n)$, and*
$$0 = b_{\vartheta(0)} = b_{\vartheta(1)} = \ldots = b_{\vartheta(\mathtt{q})} = 0 < b_{\vartheta(\mathtt{q}+1)} \le \ldots \le b_{\vartheta(\mathtt{r})} \le \frac{1}{n+1} < b_{\vartheta(\mathtt{r}+1)} \le \ldots \le b_{\vartheta(n)} \le 1 \ .$$
(The case $\mathtt{q} = \mathtt{r}$ is possible). Because $\Phi \in \mathcal{COMFORT}(\Delta_n)$ we have $\vec{c} \in \mathbf{Bound}_n$ and
$$0 = c_{\vartheta(0)} \le c_{\vartheta(1)} \le \ldots \le c_{\vartheta(\mathtt{q})} \le c_{\vartheta(\mathtt{q}+1)} \le \ldots \le c_{\vartheta(\mathtt{r})} \le c_{\vartheta(\mathtt{r}+1)} \le \ \ldots \ \le c_{\vartheta(n)} \le 1 \ .$$
Then the claim of this lemma is
$$c_{\vartheta(0)} = c_{\vartheta(1)} = \ldots = c_{\vartheta(\mathtt{q})} = 0 < c_{\vartheta(\mathtt{q}+1)} \ .$$

Proof. $\Phi \in \mathcal{COMFORT}(\Delta_n)$ means that Φ keeps the order. Because of $\Phi(\vec{b}) = \vec{c}$ and because of $0 = b_{\vartheta(\mathtt{q})} \le b_{\vartheta(0)} = 0$, it follows $c_{\vartheta(\mathtt{q})} \le c_{\vartheta(0)} = 0$, hence $c_{\vartheta(\mathtt{q})} = 0$.

Now assume $c_{\vartheta(\mathtt{q}+1)} = 0$. As we described in Definition (14), Φ^{-1} is an element of $\mathcal{COMFORT}(\Delta_n)$. We have $\Phi^{-1}(\vec{c}) = \vec{b}$, and $c_{\vartheta(\mathtt{q}+1)} = 0 \le c_{\vartheta(0)} = 0$, and we get a contradiction to $b_{\vartheta(\mathtt{q}+1)} > b_{\vartheta(0)} = 0$. Hence the only possibility is $c_{\vartheta(\mathtt{q}+1)} > 0$. □

Lemma 5. *Let Φ be an element of $\mathcal{COMFORT}(\Delta_n)$. Trivially, for $\vec{x} = (x_0, x_1, \ldots, x_n) \in \Delta_n$ there is a permutation ϑ on $\{0, 1, \ldots, n\}$, and there are two natural numbers $\mathtt{q}, \mathtt{r}$, with $0 \le \mathtt{q} \le \mathtt{r} \le n$ such that $\Phi(\vec{x}) =: \vec{y} =: (y_0, y_1, \ldots, y_n) \in \Delta_n$, and*
$$x_{\vartheta(0)} = x_{\vartheta(1)} = \ldots = x_{\vartheta(\mathtt{q})} < x_{\vartheta(\mathtt{q}+1)} \le \ldots \le x_{\vartheta(\mathtt{r})} \le \frac{1}{n+1} < x_{\vartheta(\mathtt{r}+1)} \le \ldots \le x_{\vartheta(n)} \ .$$

($\mathtt{q} = \mathtt{r}$ *is possible. If* $\vec{x} = \mathbf{Center}_n$ *we have* $\mathtt{q} = \mathtt{r} = n$). *Since* $\Phi \in \mathcal{COMFORT}(\Delta_n)$ *we get*

$$y_{\vartheta(0)} \leq y_{\vartheta(1)} \leq \cdots \leq y_{\vartheta(\mathtt{q})} \leq y_{\vartheta(\mathtt{q}+1)} \leq \cdots \leq y_{\vartheta(\mathtt{r})} \leq y_{\vartheta(\mathtt{r}+1)} \leq \ \cdots \ \leq y_{\vartheta(n)} \ .$$

Then we claim

$$y_{\vartheta(0)} = y_{\vartheta(1)} = \ldots = y_{\vartheta(\mathtt{q})} < y_{\vartheta(\mathtt{q}+1)} \ .$$

Proof. The proof can be established in similarity to the proof of Lemma (4). Again we use the fact that Φ^{-1} is an element of $\mathcal{COMFORT}(\Delta_n)$. □

Lemma 6. *Let* Φ *be an element of* $\mathcal{COMFORT}(\Delta_n)$. *Let* $\vec{x} \in \Delta_n \backslash \{\mathbf{Center}_n\}$, $\vec{x} = (x_0, x_1, \ldots, x_n)$. *Let* $\Phi(\vec{x}) =: \vec{y} =: (y_0, y_1, \ldots, y_n) \in \Delta_n$. *There is a permutation* ϑ *on* $\{0, 1, \ldots, n\}$, *and there are two natural numbers* $\mathtt{q}, \mathtt{t}$, *with* $0 \leq \mathtt{q} \leq \mathtt{t} \leq n$, *and*

$$x_{\vartheta(0)} \leq x_{\vartheta(1)} \leq \ldots \leq x_{\vartheta(\mathtt{q})} < x_{\vartheta(\mathtt{q}+1)} = x_{\vartheta(\mathtt{q}+2)} = \ \ldots\ldots \ = x_{\vartheta(\mathtt{t})} < x_{\vartheta(\mathtt{t}+1)} \leq \ldots\ldots \leq x_{\vartheta(n)} \, .$$

(*The case* $\mathtt{t} = n$ *is possible*). *Because of* $\Phi \in \mathcal{COMFORT}(\Delta_n)$ *we have*

$$y_{\vartheta(0)} \leq y_{\vartheta(1)} \leq \cdots \leq y_{\vartheta(\mathtt{q})} \leq y_{\vartheta(\mathtt{q}+1)} \leq y_{\vartheta(\mathtt{q}+2)} \ \leq \cdots \leq y_{\vartheta(\mathtt{t})} \leq y_{\vartheta(\mathtt{t}+1)} \leq \cdots \leq \ y_{\vartheta(n)} .$$

We claim

$$y_{\vartheta(\mathtt{q})} < y_{\vartheta(\mathtt{q}+1)} = y_{\vartheta(\mathtt{q}+2)} \ = \ldots\ldots \ = y_{\vartheta(\mathtt{t})} < y_{\vartheta(\mathtt{t}+1)} \, .$$

Proof. Again, essentially it is the same proof as for Lemma (4). □

Corollary 1. *It follows from the previous lemmas that an element* $F \in \mathcal{COMFORT}(\Delta_n)$ *yields an* $F_{|\mathsf{Sponge}(\Delta_n)} \in \mathcal{COMFORT}(\mathsf{Sponge}(\Delta_n))$.

We continue the investigations of maps $\Phi \in \mathcal{COMFORT}(\Delta_n)$ with an important statement. Note that in the following Lemma (7) we assume a homeomorphism $\clubsuit_{n,\alpha} \xrightarrow{\cong} \clubsuit_{n,\beta}$ for some α and β, which is not always possible, e.g. for $\alpha > 0$ and $\beta = 0$. We believe, but we have no proof that a homeomorphism $\clubsuit_{n,\alpha} \xrightarrow{\cong} \clubsuit_{n,\beta}$ is possible in the following cases of α and β:

$\frac{1}{k+1} < \alpha, \beta < \frac{1}{k}$, for all natural numbers k with $1 \leq k \leq n$.

But we know that there exists a homeomorphism $\clubsuit_{n,\alpha} \xrightarrow{\cong} \clubsuit_{n,\beta}$ if $\alpha = \beta$, of course, and also if $0 < \alpha, \beta < \frac{1}{n+1}$. This fact is mentioned in Lemma (2), and we use it now.

Lemma 7. *Let* $n \in \mathbb{N}$. *Let either* $\alpha = \beta = 0$ *or* $0 < \alpha, \beta < \frac{1}{n+1}$. *We take a map* $\Phi \in \mathcal{COMFORT}(\Delta_n)$, *and we assume that* Φ *induces a homeomorphism* $\Phi|_{\clubsuit_{n,\alpha}} : \clubsuit_{n,\alpha} \xrightarrow{\cong} \clubsuit_{n,\beta}$. *Let* $\vec{x} = (x_0, x_1, \ldots, x_n) \in \clubsuit_{n,\alpha} \cap \mathsf{Sponge}(\Delta_n)$, *i.e. we have a single* α *in the components* $\{x_0, x_1, \ldots, x_n\}$ *of* $\vec{x}$. *Trivially, there is a permutation* ϑ *on* $\{0, 1, \ldots, n\}$, *and a natural number* $\mathtt{q}$, *with* $0 \leq \mathtt{q} \leq n-1$ *such that the* $\vartheta(\mathtt{q})^{th}$ *component of* $\vec{x}$ *is the single* α, *i.e.* $\alpha = x_{\vartheta(\mathtt{q})}$, *and*

$$x_{\vartheta(0)} < x_{\vartheta(1)} < \ldots\ldots < x_{\vartheta(\mathtt{q}-1)} < \alpha < x_{\vartheta(\mathtt{q}+1)} < x_{\vartheta(\mathtt{q}+2)} < \ldots\ldots < x_{\vartheta(n)} \ .$$

Let $\Phi(\vec{x}) =: \vec{y} =: (y_0, y_1, \ldots, y_n)$. *We claim that the* $\vartheta(\mathtt{q})^{th}$ *component of* $\vec{y}$ *is* β, *i.e.* $\beta = y_{\vartheta(\mathtt{q})}$, *and this is the only* β *in the components* $\{y_0, y_1, \ldots, y_n\}$ *of* $\vec{y}$, *i.e. we claim*

$$y_{\vartheta(0)} < y_{\vartheta(1)} < \ldots\ldots < y_{\vartheta(\mathtt{q}-1)} < \beta < y_{\vartheta(\mathtt{q}+1)} < \ldots\ldots < y_{\vartheta(n)} \ .$$

Proof. The case $n = 1$ is easy. It is $\alpha, \beta < \frac{1}{2}$, hence $\alpha < 1 - \alpha$, and $\beta < 1 - \beta$. We have the α-cross $\clubsuit_{1,\alpha} = \{(\alpha, 1-\alpha), (1-\alpha, \alpha)\}$, it follows

$$\Phi(\alpha, 1-\alpha) = (\beta, 1-\beta).$$

Let $n \geq 2$. For $\alpha = \beta = 0$ we apply Lemma (4), with $\mathtt{q} = 0$.

Let us assume $0 < \alpha, \beta < \frac{1}{n+1}$. Because $\vec{x} \in \clubsuit_{n,\alpha} \cap \mathsf{Sponge}(\Delta_n)$ and Corollary (1) the components $\{y_0, y_1, \ldots, y_n\}$ of $\vec{y}$ contain a single β. Since $\Phi \in \mathcal{COMFORT}(\Delta_n)$, we can deduce from the inequality

$$x_{\vartheta(\mathtt{q}-1)} < \alpha = x_{\vartheta(\mathtt{q})} < x_{\vartheta(\mathtt{q}+1)}$$

and Corollary (1) the inequality

$$y_{\vartheta(\mathtt{q}-1)} < y_{\vartheta(\mathtt{q})} < y_{\vartheta(\mathtt{q}+1)}.$$

(The case $\mathtt{q} = 0$ is posssible.) We want to show $y_{\vartheta(\mathtt{q})} = \beta$. We call '$\vartheta(\mathtt{k})$' the index of β, i.e. $\beta = y_{\vartheta(\mathtt{k})}$, for a suitable $\mathtt{k} \in \{0, 1, \ldots, n-1\}$, and we have

$$y_{\vartheta(0)} < y_{\vartheta(1)} < \ldots < y_{\vartheta(\mathtt{k}-1)} < \beta < y_{\vartheta(\mathtt{k}+1)} < \ldots < y_{\vartheta(n)} \ .$$

We want to show $\mathtt{k} = \mathtt{q}$. To prove this we consider the two other cases $\mathtt{k} < \mathtt{q}$ and $\mathtt{k} > \mathtt{q}$, and we seek for contradictions. We shall find a contradiction in the case of $\mathtt{k} < \mathtt{q}$. The case $\mathtt{k} > \mathtt{q}$ can be treated in the same way. (Because $\Phi^{-1} \in \mathcal{COMFORT}(\Delta_n)$ we can exchange the parts of α and β, note $\Phi^{-1}|_{\clubsuit_{n,\beta}} : \clubsuit_{n,\beta} \xrightarrow{\cong} \clubsuit_{n,\alpha}$).

The case $\mathtt{k} < \mathtt{q}$:
We have $0 < \mathtt{q}$ in this case. Further, note that we can exclude the case $x_{\vartheta(0)} = 0, x_{\vartheta(1)} = \alpha$, since from Lemma (4) would follow $y_{\vartheta(0)} = 0$. Since $\mathtt{k} < \mathtt{q} = 1$ we would get $\beta = y_{\vartheta(0)}$, this contradicts $0 < \beta$. This means from $\mathtt{q} = 1$ follows $x_{\vartheta(0)} > 0$.

We define an infinite connected subset $Subset[\clubsuit_{n,\alpha}] \subset \clubsuit_{n,\alpha}$ to use a topological argument. Let for all $\varepsilon \in [0,1]$ the element $\vec{a}_\varepsilon := (a_0, a_1, \ldots, a_n) \in \clubsuit_{n,\alpha}$ by setting

$$a_j := \begin{cases} x_j \cdot (1-\varepsilon) & \text{for} \quad j \in \{\vartheta(0), \vartheta(1), \ldots, \vartheta(\mathtt{q}-1)\} \\ x_j & \text{for} \quad j \in \{\vartheta(\mathtt{q}), \vartheta(\mathtt{q}+1), \ldots, \vartheta(n-1)\} \\ x_{\vartheta(n)} + \varepsilon \cdot \sum_{i=0}^{\mathtt{q}-1} x_{\vartheta(i)} & \text{for} \quad j = \vartheta(n). \end{cases}$$

We have $a_{\vartheta(\mathtt{q})} = \alpha$, hence $\vec{a}_\varepsilon \in \clubsuit_{n,\alpha}$ for all $\varepsilon \in [0,1]$. For $\varepsilon \neq 1$ we have $\vec{a}_\varepsilon \in \mathsf{Sponge}(\Delta_n)$. Let

$$Subset[\clubsuit_{n,\alpha}] \ := \{\vec{a}_\varepsilon \mid \varepsilon \in [0,1]\} \ .$$

For $\varepsilon = 0$ we have $\vec{a}_0 = \vec{x}$, and for $\varepsilon = 1$ we get $\vec{a}_1 =: (\overline{a}_0, \overline{a}_1, \ldots, \overline{a}_n) \in \mathbf{Bound}_n$, and we have

$$0 = \overline{a}_{\vartheta(0)} = \overline{a}_{\vartheta(1)} = \ldots = \overline{a}_{\vartheta(\mathtt{q}-1)} = 0 < \alpha < \overline{a}_{\vartheta(\mathtt{q}+1)} < \overline{a}_{\vartheta(\mathtt{q}+2)} < \ldots\ldots < \overline{a}_{\vartheta(n)} \ .$$

Let $\Phi(\vec{a}_\varepsilon) =: \vec{d}_\varepsilon =: (d_0, d_1, \ldots, d_n)$ for $\varepsilon \in [0,1]$. All $\vec{d}_\varepsilon$ are elements of $\clubsuit_{n,\beta}$ since $\vec{a}_\varepsilon \in \clubsuit_{n,\alpha}$. All $\vec{a}_\varepsilon$ have a single α at the component with the index $\vartheta(\mathtt{q})$, this means $a_{\vartheta(\mathtt{q})} = \alpha$. Because $\Phi \in \mathcal{COMFORT}(\Delta_n)$ and because of Lemma (4) and Lemma (6) it follows that $\vec{d}_\varepsilon$ has a single β, for all $\varepsilon \in [0,1]$. With $\Phi(\vec{a}_1) = \vec{d}_1 =: (\overline{d}_0, \overline{d}_1, \ldots, \beta, \ldots, \overline{d}_n) \in \clubsuit_{n,\beta}$ we get a single component β in $\vec{d}_1$, i.e. there is an index $\mathtt{j}$ such that $\beta = \overline{d}_{\vartheta(\mathtt{j})}$, and

$$0 = \overline{d}_{\vartheta(0)} = \ldots = \overline{d}_{\vartheta(\mathtt{q}-1)} = 0 < \overline{d}_{\vartheta(\mathtt{q})} < \overline{d}_{\vartheta(\mathtt{q}+1)} < \ldots < \overline{d}_{\vartheta(\mathtt{j}-1)} < \beta < \overline{d}_{\vartheta(\mathtt{j}+1)} \ < \ldots < \overline{d}_{\vartheta(n)} \, .$$

Obviously it follows $\mathtt{q} \leq \mathtt{j}$. Now we use the canonical projections PROJ_i, for $i = 0, 1, \ldots, n$,

$$\mathsf{PROJ}_i : \Delta_n \to [0,1] \, , \ (z_0, z_1, z_2, \ldots, z_i, \ldots, z_n) \mapsto z_i \, .$$

PROJ_i is continuous. Note that both $Subset[\clubsuit_{n,\alpha}]$ and its homeomorphic image $\Phi(Subset[\clubsuit_{n,\alpha}])$ are connected subsets of Δ_n, and also note that the set $\{\beta\}$ is closed in $[0,1]$. If we restrict PROJ_i for $i = 0, 1, \ldots, n$ to the set $\Phi(Subset[\clubsuit_{n,\alpha}]) = \{d_\varepsilon | \varepsilon \in [0,1]\}$, i.e.

$$\mathsf{PROJ}_i : \Phi(Subset[\clubsuit_{n,\alpha}]) \longrightarrow [0,1] \, ,$$

we can express $\Phi(Subset[\clubsuit_{n,\alpha}])$ as a disjoint union of domains $\mathsf{PROJ}_i{}^{-1}(\{\beta\})$, that means

$$\Phi(Subset[\clubsuit_{n,\alpha}]) \ = \ \bigcup \left\{ \mathsf{PROJ}_i{}^{-1}(\{\beta\}) \,|\, i = 0, 1, \ldots, n \right\} \ ,$$

and the union is disjoint.
Because $\Phi(Subset[\clubsuit_{n,\alpha}])$ is a connected set, and since $\mathsf{PROJ}_i{}^{-1}(\{\beta\})$ is closed, we see that $\mathsf{PROJ}_i{}^{-1}(\{\beta\})$ either is empty or the entire set, for each $i \in \{0,1,\dots,n\}$. We have $\vec{a}_0 = \vec{x}$, and

$$\Phi(\vec{x}) = \vec{y} = \vec{d}_0 \in \mathsf{PROJ}_{\vartheta(\mathtt{k})}{}^{-1}(\{\beta\}) \quad \text{and} \quad \Phi(\vec{a}_1) = \vec{d}_1 \in \mathsf{PROJ}_{\vartheta(\mathtt{j})}{}^{-1}(\{\beta\}) \ ,$$
$$\text{hence it follows} \quad \mathsf{PROJ}_{\vartheta(\mathtt{k})}{}^{-1}(\{\beta\}) \neq \emptyset \neq \mathsf{PROJ}_{\vartheta(\mathtt{j})}{}^{-1}(\{\beta\}) \, .$$

We get $\mathtt{k} = \mathtt{j}$, which contradicts $\mathtt{k} < \mathtt{q} \leq \mathtt{j}$!

As we already mentioned above, the case $\mathtt{k} > \mathtt{q}$ can be treated in the same way by exchanging the parts of α and β and considering Φ^{-1} instead of Φ. Note that Φ^{-1} induces a homeomorphism $\clubsuit_{n,\beta} \xrightarrow{\cong} \clubsuit_{n,\alpha}$. Again we would find a contradiction. As the only possibility remains $\mathtt{k} = \mathtt{q}$. The proof of Lemma (7) is finished. □

From the previous lemma we can deduce an important corollary.

Corollary 2. *Let $n \in \mathbb{N}$. Let $0 \leq \alpha, \beta < \frac{1}{n+1}$, let $\Phi \in \mathcal{COMFORT}(\Delta_n)$, and assume that Φ yields a homeomorphism $\Phi|_{\clubsuit_{n,\alpha}} : \clubsuit_{n,\alpha} \xrightarrow{\cong} \clubsuit_{n,\beta}$. (It follows $\alpha = \beta = 0$ or $0 < \alpha, \beta < \frac{1}{n+1}$). For a point $\vec{x} = (x_0, x_1, \dots, x_n) \in \Delta_n$ let the image be $\Phi(\vec{x}) = \vec{y} = (y_0, y_1, \dots, y_n)$. It holds*
$$x_i = \alpha \ \text{ if and only if } \ y_i = \beta \ \text{ for all indices } \ i = 0,1,\dots,n.$$

Proof. Use the previous Lemma (7) and note $\text{closure}(\mathsf{Sponge}(\Delta_n)) = \Delta_n$. □

Lemma 8. *Let $n \in \mathbb{N}$. Let $0 \leq \alpha, \beta < \frac{1}{n+1}$, and let $\Phi \in \mathcal{COMFORT}(\Delta_n)$ such that Φ yields a homeomorphism $\Phi|_{\clubsuit_{n,\alpha}} : \clubsuit_{n,\alpha} \xrightarrow{\cong} \clubsuit_{n,\beta}$. Let $\vec{x} = (x_0, x_1, \dots, x_n) \in \Delta_n \backslash \clubsuit_{n,\alpha}$, i.e. $\vec{x}$ is not an element of the α-cross. Trivially, either $\alpha < \min\{x_0, x_1, \dots, x_n\}$ or there is a permutation ϑ on $\{0,1,\dots,n\}$ and there are two natural numbers $\mathtt{q}, \mathtt{r}$, with*
$$0 \leq \mathtt{q} \leq \mathtt{r} < n \ (\mathtt{q} = \mathtt{r} \text{ is possible}), \text{ and}$$

$$x_{\vartheta(0)} \leq x_{\vartheta(1)} \leq \ldots \leq x_{\vartheta(\mathtt{q})} < \alpha < x_{\vartheta(\mathtt{q}+1)} \leq \ldots \leq x_{\vartheta(\mathtt{r})} \leq \frac{1}{n+1} < x_{\vartheta(\mathtt{r}+1)} \leq \ldots \leq x_{\vartheta(n)} \, .$$

Let $\Phi(\vec{x}) =: \vec{y} =: (y_0, y_1, \dots, y_n)$. Since $\Phi|_{\clubsuit_{n,\alpha}} : \clubsuit_{n,\alpha} \xrightarrow{\cong} \clubsuit_{n,\beta}$ we get $\vec{y} \in \Delta_n \backslash \clubsuit_{n,\beta}$.
We claim that either

$$\beta < \min\{y_0, y_1, \dots, y_n\} \quad \text{or}$$

$$y_{\vartheta(0)} \leq \ldots \leq y_{\vartheta(\mathtt{q}-1)} \leq y_{\vartheta(\mathtt{q})} < \beta < y_{\vartheta(\mathtt{q}+1)} \leq \ldots \leq y_{\vartheta(\mathtt{r})} < y_{\vartheta(\mathtt{r}+1)} \leq \ldots \leq y_{\vartheta(n)} \, .$$

Proof. The proof follows the line of the previous Lemma (7). We have either $\alpha = \beta = 0$ or $0 < \alpha, \beta < \frac{1}{n+1}$. The case $\alpha = \beta = 0$ is trivial since $\clubsuit_{n,0} = \mathbf{Bound}_n$. Please see Lemma (3). We have $\alpha = 0 < \min\{x_0, x_1, \dots, x_n\}$. It follows $\beta < \min\{y_0, y_1, \dots, y_n\}$.

Let $0 < \alpha, \beta < \frac{1}{n+1}$. We assume the second alternative, i.e. we have a suitable number $\mathtt{q}$ with $0 \leq \mathtt{q} \leq \mathtt{r} < n$ such that

$$x_{\vartheta(0)} \leq x_{\vartheta(1)} \leq \ldots \leq x_{\vartheta(\mathtt{q})} < \alpha < x_{\vartheta(\mathtt{q}+1)} \leq \ldots \leq x_{\vartheta(\mathtt{r})} \leq \frac{1}{n+1} < x_{\vartheta(\mathtt{r}+1)} \leq \ldots \leq x_{\vartheta(n)} \ .$$

We have to show that $\underline{(A)}$: $y_{\vartheta(\mathtt{q})} < \beta$, and $\underline{(B)}$: $\beta < y_{\vartheta(\mathtt{q}+1)}$.

$\underline{(A)}$: If $x_{\vartheta(\mathtt{q})} = 0$ it follows from Lemma (4) $y_{\vartheta(\mathtt{q})} = 0$, hence $y_{\vartheta(\mathtt{q})} < \beta$.
Assume $x_{\vartheta(\mathtt{q})} > 0$. Similar as in the proof of Lemma (7) we define a subset of Δ_n,
$$Subset[\Delta_n] := \{\vec{a}_\varepsilon \mid \varepsilon \in [0,1]\}.$$
For all $\varepsilon \in [0,1]$ let $\vec{a}_\varepsilon := (a_0, a_1, \dots, a_n) \in Subset[\Delta_n]$ by setting

$$a_j := \begin{cases} x_j \cdot (1-\varepsilon) & \text{for} \quad j \in \{\vartheta(0), \vartheta(1), \dots, \vartheta(\mathtt{q}-1), \vartheta(\mathtt{q})\} \\ x_j & \text{for} \quad j \in \{\vartheta(\mathtt{q}+1), \vartheta(\mathtt{q}+2), \dots, \vartheta(n-1)\} \\ x_{\vartheta(n)} + \varepsilon \cdot \sum_{i=0}^{\mathtt{q}} x_{\vartheta(i)} & \text{for} \quad j = \vartheta(n) \ . \end{cases}$$

Note that the second set $\{\vartheta(\mathtt{q}+1), \vartheta(\mathtt{q}+2), \ldots, \vartheta(n-1)\}$ of indices may be empty.
We have $a_{\vartheta(\mathtt{q})} < \alpha < a_{\vartheta(\mathtt{q}+1)}$, hence $\vec{a}_\varepsilon \in \Delta_n \backslash \clubsuit_{n,\alpha}$ for all $\varepsilon \in [0,1]$. We get $\vec{a}_0 = \vec{x}$, and $\vec{a}_1 =: (\overline{a}_0, \overline{a}_1, \ldots, \overline{a}_n) \in \mathbf{Bound}_n$ with

$$0 = \overline{a}_{\vartheta(0)} = \overline{a}_{\vartheta(1)} = \ldots = \overline{a}_{\vartheta(\mathtt{q})} = 0 < \alpha < \overline{a}_{\vartheta(\mathtt{q}+1)} \leq \ldots \leq \overline{a}_{\vartheta(\mathtt{r})} < \overline{a}_{\vartheta(\mathtt{r}+1)} \leq \ \ldots \ \leq \overline{a}_{\vartheta(n)} \ .$$

Let $\Phi(\vec{a}_\varepsilon) =: \vec{d}_\varepsilon =: (d_0, d_1, \ldots, d_n)$. Since $\vec{a}_\varepsilon \notin \clubsuit_{n,\alpha}$ it holds $\vec{d}_\varepsilon \notin \clubsuit_{n,\beta}$, for all $\varepsilon \in [0,1]$. Because $\Phi \in \mathcal{COMFORT}(\Delta_n)$, with $\Phi(\vec{a}_1) = \vec{d}_1 =: (\overline{d}_0, \overline{d}_1, \ldots, \overline{d}_n) \notin \clubsuit_{n,\beta}$ it follows that there is an index $\mathtt{j} \geq \mathtt{q}$ such that $\overline{d}_{\vartheta(\mathtt{j})} < \beta < \overline{d}_{\vartheta(\mathtt{j}+1)}$, see Lemma (4), and

$$0 = \overline{d}_{\vartheta(0)} = \ldots = \overline{d}_{\vartheta(\mathtt{q})} = 0 < \overline{d}_{\vartheta(\mathtt{q}+1)} \leq \ldots \leq \overline{d}_{\vartheta(\mathtt{j})} < \beta < \overline{d}_{\vartheta(\mathtt{j}+1)} \leq \ldots \leq \overline{d}_{\vartheta(\mathtt{r})} < \ldots \leq \overline{d}_{\vartheta(n)} \ .$$

Note $Subset[\Delta_n] \cap \clubsuit_{n,\alpha} = \emptyset$, and since $\Phi|_{\clubsuit_{n,\alpha}} : \clubsuit_{n,\alpha} \xrightarrow{\cong} \clubsuit_{n,\beta}$ it also holds

$$\Phi\left(Subset[\Delta_n]\right) \cap \clubsuit_{n,\beta} \ = \ \emptyset \ .$$

Further, note that both sets $Subset[\Delta_n]$ and its continous image $\Phi\left(Subset[\Delta_n]\right)$ are connected subsets of Δ_n. It follows that the projection onto the $\vartheta(\mathtt{q})^{th}$ component is a connected set, i.e. $\mathsf{PROJ}_{\vartheta(\mathtt{q})}(\Phi(Subset[\Delta_n]))$ is a connected subset of $[0,1]$, i.e. an interval.

Now we use $\Phi(\vec{a}_0) = \vec{y}$ and $\Phi(\vec{a}_1) = \vec{d}_1$, i.e. $\vec{y}, \vec{d}_1$ are elements of $\Phi\left(Subset[\Delta_n]\right)$. They have the $\vartheta(\mathtt{q})^{th}$ component $y_{\vartheta(\mathtt{q})}$ and $\overline{d}_{\vartheta(\mathtt{q})}$, respectively. We have shown $\overline{d}_{\vartheta(\mathtt{q})} = 0$, hence

$$\{0, y_{\vartheta(\mathtt{q})}\} \subset \mathsf{PROJ}_{\vartheta(\mathtt{q})}(\Phi(Subset[\Delta_n])),$$

hence the closed interval $[0, y_{\vartheta(\mathtt{q})}]$ is a subset of $\mathsf{PROJ}_{\vartheta(\mathtt{q})}(\Phi(Subset[\Delta_n]))$. Since we have the empty set $\Phi(Subset[\Delta_n]) \cap \clubsuit_{n,\beta} = \emptyset$, it follows $y_{\vartheta(\mathtt{q})} < \beta$, and (A) is shown.

(B): Since $\Phi^{-1} \in \mathcal{COMFORT}(\Delta_n)$ we use the same argument as in Lemma (7) to show $\beta < y_{\vartheta(\mathtt{q}+1)}$. We can exchange the parts of α and β since $\Phi^{-1}|_{\clubsuit_{n,\beta}} : \clubsuit_{n,\beta} \xrightarrow{\cong} \clubsuit_{n,\alpha}$. With $\Phi^{-1}(y_0, y_1, \ldots, y_n) = (x_0, x_1, \ldots, x_n)$ we can deduce that $y_{\vartheta(\mathtt{q}+1)} < \beta$ means $x_{\vartheta(\mathtt{q}+1)} < \alpha$, and we get a contradiction. Hence the only possibility is $y_{\vartheta(\mathtt{q}+1)} > \beta$.

The first alternative of the lemma is $\alpha < \min\{x_0, x_1, \ldots, x_n\}$. It is treated correspondingly. If we assume $\beta > \min\{y_0, y_1, \ldots, y_n\}$ we can exchange the parts of α and β and consider Φ^{-1}, and we deduce $\alpha > \min\{x_0, x_1, \ldots, x_n\}$. We find a contradiction, and this ends the proof of Lemma (8). □

With the previous rather technical lemmas we are able to discuss some possibilities to extend a map φ, which is defined on a subset of Δ_n, onto the entire simplex.

Proposition 1. *Let $n \in \mathbb{N}$, and let us assume either $\alpha = 0 = \beta$, or $0 < \alpha, \beta < \frac{1}{n+1}$, or $\alpha = \frac{1}{n+1} = \beta$. We assume further a homeomorphism $\varphi : \mathbf{Layer}_{n,\alpha} \xrightarrow{\cong} \mathbf{Layer}_{n,\beta}$.*

We claim that the map φ can be extended to a homeomorphism on Δ_n, i.e. there is a homeomorphism $\Phi : \Delta_n \xrightarrow{\cong} \Delta_n$ such that $\Phi_{|\mathbf{Layer}_{n,\alpha}} = \varphi$. The constructed Φ has the property that for any $0 \leq \gamma \leq \frac{1}{n+1}$ there is a $0 \leq \delta \leq \frac{1}{n+1}$ such that we get a homeomorphism $\Phi_{|\mathbf{Layer}_{n,\gamma}} : \mathbf{Layer}_{n,\gamma} \xrightarrow{\cong} \mathbf{Layer}_{n,\delta}$.

If φ has the properties $\widehat{[\mathbf{2}]}$ (respecting permutations) and $\widehat{[\mathbf{3}]}$ (keeping the order) from Definition (14), the homeomorphism Φ is an element of $\mathcal{COMFORT}(\Delta_n)$.

For a better understanding see the next Figure 9. This figure shows on the left hand side the two dimensional simplex Δ_2. The inner triangle symbolizes $\mathbf{Layer}_{2,\alpha}$, for $\alpha = \frac{1}{6}$.

We assume that the homeomorphism φ maps $\mathbf{Layer}_{2,\frac{1}{6}}$ homeomorphicly onto $\mathbf{Layer}_{2,\frac{1}{4}}$. On the right hand side we see the the image $\Phi(\Delta_2)$, with $\Phi|_{\mathbf{Layer}_{2,\frac{1}{6}}} = \varphi$. The little triangle is $\mathbf{Layer}_{2,\frac{1}{4}} = \Phi\left(\mathbf{Layer}_{2,\frac{1}{6}}\right)$.

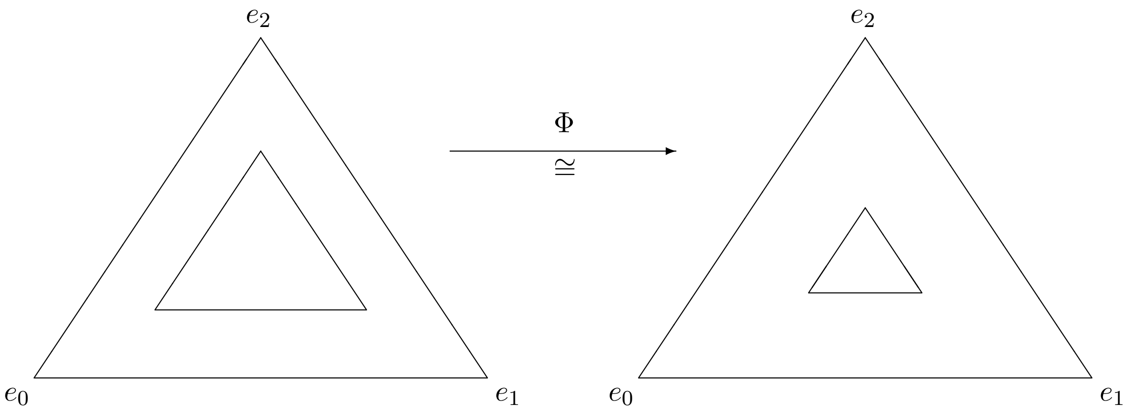

Figure 9:

Proof. The trivial case is $\alpha = \frac{1}{n+1} = \beta$, i.e. $\mathbf{Layer}_{n,\alpha} = \{\mathbf{Center}_n\}$. Let $\Phi := id(\Delta_n)$.

For the other cases we need an auxiliary function σ, it must be any increasing homeomorphism on the interval $\left[0, \frac{1}{n+1}\right]$ with $\sigma(\alpha) = \beta$. For instance σ can be the polygon through three points $\left\{(0,0), (\alpha, \beta), \left(\frac{1}{n+1}, \frac{1}{n+1}\right)\right\}$. Define $\Phi(\mathbf{Center}_n) := \mathbf{Center}_n$.
We assume $0 < \alpha, \beta < \frac{1}{n+1}$. For this case it is useful to repeat Definition (10) and Definition (11). As we noted already before and after theses definitions, for any $\vec{x} = (x_0, x_1, \ldots x_n) \in \Delta_n \backslash \{\mathbf{Center}_n\}$ we have an unique projection $\pi(\vec{x}) \in \mathbf{Bound}_n$ and an unique number $\mathcal{A}(\vec{x}) = \min\{x_0, x_1, \ldots x_n\}$, $0 \leq \mathcal{A}(\vec{x}) < \frac{1}{n+1}$, such that $\vec{x} \in \mathbf{Layer}_{n,\mathcal{A}(\vec{x})}$, and $\vec{x} \in [\mathbf{Center}_n, \pi(\vec{x})]$, and a representation

$$\vec{x} = \pi(\vec{x}) + \mathcal{A}(\vec{x}) \cdot (n+1) \cdot (\mathbf{Center}_n - \pi(\vec{x}))\ .$$

We define (we use the brackets '[...]' for a better view)

$$\Phi(\vec{x}) := \pi(\varphi(\pi_\alpha(\vec{x}))) + \sigma[\mathcal{A}(\vec{x})] \cdot (n+1) \cdot [\mathbf{Center}_n - \pi(\varphi(\pi_\alpha(\vec{x})))]\,,$$

we get that $\Phi(\vec{x})$ is in the line segment $[\mathbf{Center}_n, \pi(\varphi(\pi_\alpha(\vec{x})))]$. With $\delta := \sigma(\gamma)$ the reader can confirm that Φ has all demanded properties.

To complete the proof we deal with the last case. If we have $\alpha = \beta = 0$ we can take $\sigma := id\left(\left[0, \frac{1}{n+1}\right]\right)$, and for $\vec{x} \in \Delta_n \backslash \{\mathbf{Center}_n\}$ we use the corresponding formula

$$\Phi(\vec{x}) := \varphi(\pi(\vec{x})) + \mathcal{A}(\vec{x}) \cdot (n+1) \cdot [\mathbf{Center}_n - \varphi(\pi(\vec{x}))]\ .$$

The proof of Proposition (1) is done. □

Now we consider another possibility to extend a map on $\mathbf{Bound}_n$ to a map on Δ_n with some properties preserved. This proposition will be very important in the next section.

Proposition 2. *Let* $n \in \mathbb{N}$ *and* $0 < \alpha, \beta < \frac{1}{n+1}$. *Further, let* $\varphi : \mathbf{Bound}_n \overset{\cong}{\to} \mathbf{Bound}_n$ *be a homeomorphism such that* φ *yields a homeomorphism*

$$\varphi_{|\mathbf{Bound}_n \cap \clubsuit_{n,\alpha}} :\ \mathbf{Bound}_n \cap \clubsuit_{n,\alpha} \overset{\cong}{\longrightarrow} \mathbf{Bound}_n \cap \clubsuit_{n,\beta}\,.$$

Further, let φ *have the properties* $\widehat{[\mathbf{2}]}$ *(respecting permutations) and* $\widehat{[\mathbf{3}]}$ *(keeping the order), i.e.* $\varphi \in \mathcal{COMFORT}(\mathbf{Bound}_n)$.

We claim that φ *can be extended to a homeomorphism on* Δ_n*, more precisely there is a homeomorphism* $\Phi : \Delta_n \overset{\cong}{\longrightarrow} \Delta_n$ *such that* $\Phi_{|\mathbf{Bound}_n} = \varphi$*, the map* Φ *is an element of* $\mathcal{COMFORT}(\Delta_n)$*, and* Φ *induces a homeomorphism* $\Phi_{|\clubsuit_{n,\alpha}} : \clubsuit_{n,\alpha} \overset{\cong}{\longrightarrow} \clubsuit_{n,\beta}$.

For a better understanding see the following Figure 10 and Figure 11.
Figure 10 shows on the left hand side $\mathbf{Bound_2}$, the boundary of Δ_2. The six points symbolize the α-cross $\clubsuit_{2,\alpha}$ for $\alpha = \frac{1}{6}$, restricted on $\mathbf{Bound_2}$.

We assume a homeomorphism φ on $\mathbf{Bound}_2$ such that φ induces a homeomorphism $\mathbf{Bound}_2 \cap \clubsuit_{2,\frac{1}{6}} \xrightarrow{\cong} \mathbf{Bound}_2 \cap \clubsuit_{2,\frac{1}{4}}$. On the right hand side we see the image $\varphi(\mathbf{Bound}_2)$. The six points symbolize the $\frac{1}{4}$-cross of $\mathbf{Bound}_2$, which is the image $\varphi\left(\mathbf{Bound}_2 \cap \clubsuit_{2,\frac{1}{6}}\right)$.

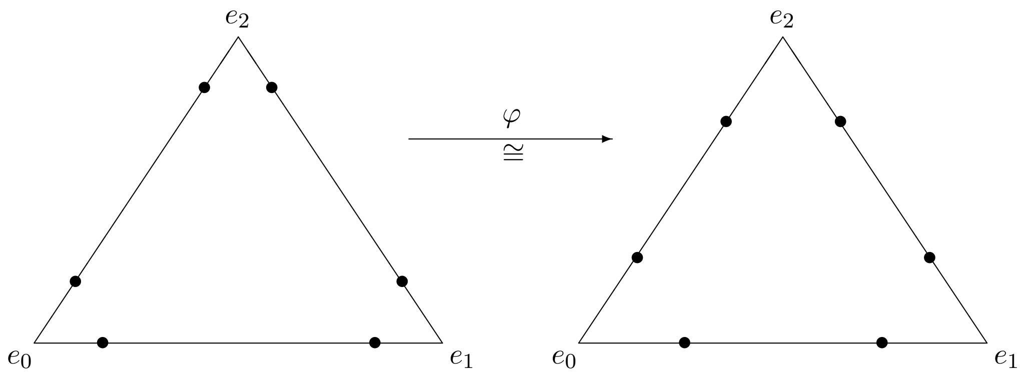

Figure 10:

In Figure 11 we see on the left hand side the standard simplex Δ_2 with the $\frac{1}{6}$-cross $\clubsuit_{2,\frac{1}{6}}$. The right hand side shows the image $\Phi(\Delta_2)$, with $\Phi\left(\clubsuit_{2,\frac{1}{6}}\right) = \clubsuit_{2,\frac{1}{4}}$.

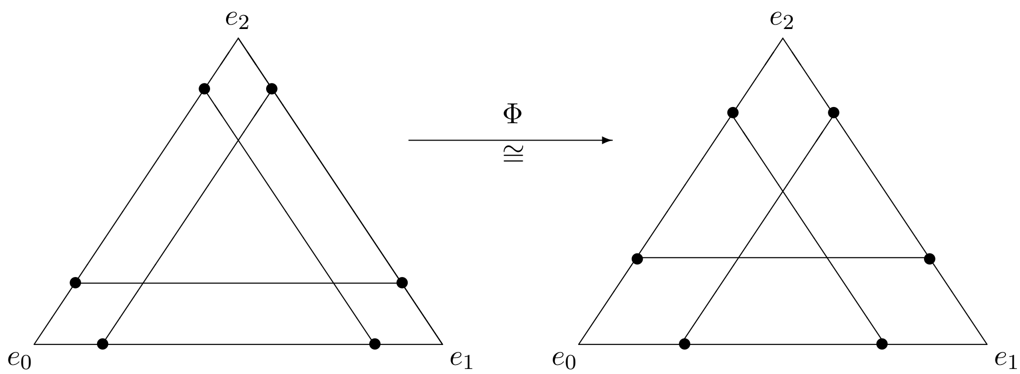

Figure 11:

Proof. Recall the almost disjoint union $\Delta_n = \bigcup \left\{ \left[\mathbf{Center}_n, \vec{b}\right] \mid \vec{b} \in \mathbf{Bound}_n \right\}$, all the lines intersect only in $\mathbf{Center}_n$. Since φ is a homeomorphism, we get the union

$$\Delta_n = \bigcup \left\{ \left[\mathbf{Center}_n, \varphi(\vec{b})\right] \mid \vec{b} \in \mathbf{Bound}_n \right\}.$$

We shall construct the homeomorphism Φ by mapping every line segment $\left[\mathbf{Center}_n, \vec{b}\right]$ homeomorphicly onto the line segment $\left[\mathbf{Center}_n, \varphi(\vec{b})\right]$, such that if $\vec{x} = (x_0, x_1, \ldots, x_n) \in \left[\mathbf{Center}_n, \vec{b}\right]$ contains a component α, the image $\Phi(\vec{x}) \in \left[\mathbf{Center}_n, \varphi(\vec{b})\right]$ will contain a component β at the same position.

Trivially, for $\vec{b} = (b_0, b_1, \ldots, b_n) \in \mathbf{Bound}_n$ there is a permutation ϑ on $\{0, 1, \ldots, n\}$ and there are two natural numbers $\mathtt{q},\mathtt{r}$, with $0 \le \mathtt{q} \le \mathtt{r} < n$ such that

$$0 = b_{\vartheta(0)} \le b_{\vartheta(1)} \le \ldots \le b_{\vartheta(\mathtt{q})} \le \alpha < b_{\vartheta(\mathtt{q}+1)} \le \ldots \le b_{\vartheta(\mathtt{r})} \le \frac{1}{n+1} < b_{\vartheta(\mathtt{r}+1)} \le \ldots \le b_{\vartheta(n)} \ . \quad (6)$$

(The case $\mathtt{q} = \mathtt{r}$ is possible).
Let $\varphi(\vec{b}) =: \vec{c} =: (c_0, c_1, \ldots, c_n) \in \mathbf{Bound}_n$. Because $\varphi \in \mathcal{COMFORT}(\mathbf{Bound}_n)$ we have

$$0 = c_{\vartheta(0)} \le c_{\vartheta(1)} \le \ldots \le c_{\vartheta(\mathtt{q})} < c_{\vartheta(\mathtt{q}+1)} \le \ldots \le c_{\vartheta(\mathtt{r})} < c_{\vartheta(\mathtt{r}+1)} \le \ \ldots \ \le c_{\vartheta(n)} \ .$$

We need the following two lemmas.

Lemma 9. *With the just assumed properties of φ in this proposition it holds*

$$c_{\vartheta(\mathtt{q})} \le \beta < c_{\vartheta(\mathtt{q}+1)}, \ \textit{and } c_{\vartheta(\mathtt{j})} = \beta \ \textit{ if and only if } \ b_{\vartheta(\mathtt{j})} = \alpha, \ \textit{for } 1 \le \mathtt{j} \le \mathtt{q}.$$

Further, we consider the case of identical components. Assume that we have
$b_{\vartheta(\mathtt{j})} < b_{\vartheta(\mathtt{j}+1)} = b_{\vartheta(\mathtt{j}+2)} = \ \ldots\ldots = b_{\vartheta(\mathtt{t})} < b_{\vartheta(\mathtt{t}+1)}$ *for any* $\mathtt{j},\mathtt{t}$ *with* $0 \le \mathtt{j} < \mathtt{t} \le n$.
We get $c_{\vartheta(\mathtt{j})} < c_{\vartheta(\mathtt{j}+1)} = c_{\vartheta(\mathtt{j}+2)} = \ \ldots\ldots = c_{\vartheta(\mathtt{t})} < c_{\vartheta(\mathtt{t}+1)}$, *and vice versa.*

Proof. We can argue as we did it in Lemma (6), and in the lemmas (7) and (8). There the case that φ maps an α-cross onto a β-cross is discussed. Please, look at Corollary (2), too. □

For the next lemma we still assume any $\vec{b} \in \mathbf{Bound}_n$ with a representation as in the above expression (6).

Lemma 10. *For all $\vec{b} = (b_0, b_1, \ldots, b_n) \in \mathbf{Bound}_n$ the number of intersections of $\left[\mathbf{Center}_n, \vec{b}\right]$ with the α-cross $\clubsuit_{n,\alpha}$ is at least 1 and at most $\mathtt{q}+1$ (some of them may be equal), more precisely*

$$\text{cardinality}\left(\left[\mathbf{Center}_n, \vec{b}\right] \cap \clubsuit_{n,\alpha}\right) \ = \ \text{cardinality}\left(\left\{b_{\vartheta(0)}, b_{\vartheta(1)}, \ldots, b_{\vartheta(\mathtt{q})}\right\}\right) \ .$$

Proof. Recall that for an $\vec{x} = (x_0, x_1, \ldots, x_n) \in \left[\mathbf{Center}_n, \vec{b}\right]$ we have an unique number $t \in [0,1]$ such that $\vec{x} = t \cdot \mathbf{Center}_n + (1-t)\cdot \vec{b} = \vec{b} + t\cdot(\mathbf{Center}_n - \vec{b})$, i.e. for a component x_j we have

$$x_j \ = \ t \cdot \frac{1}{n+1} + (1-t)\cdot b_j \ = \ b_j + t \cdot \left(\frac{1}{n+1} - b_j\right) \ .$$

If we use the canonical projections PROJ_i, for $i = 0,1,\ldots,n$, we have for a fixed index $j \in \{0,1,\ldots,\mathtt{r}\}$, i.e. $b_{\vartheta(j)} \le \frac{1}{n+1}$, that the continuous map

$$[0,1] \longrightarrow \left[\mathbf{Center}_n, \vec{b}\right] \longrightarrow [0,1] \ , \ t \mapsto \vec{x} \mapsto \mathsf{PROJ}_{\vartheta(\mathsf{j})}(\vec{x}), \quad \text{i.e.}$$
$$t \longmapsto t \cdot \mathbf{Center}_n + (1-t)\cdot\vec{b} \longmapsto b_{\vartheta(j)} + t\cdot\left(\tfrac{1}{n+1} - b_{\vartheta(j)}\right),$$

is monotonic increasing (strictly increasing for $j \in \{0,1,\ldots,\mathtt{q}\}$), while for the other indices $j \in \{\mathtt{r}+1, \mathtt{r}+2, \ldots, n\}$ (i.e. $\frac{1}{n+1} < b_{\vartheta(j)}$) the map $[0,1] \longrightarrow [0,1]$, $t \mapsto b_{\vartheta(j)} + t \cdot \left(\frac{1}{n+1} - b_{\vartheta(j)}\right)$ is strictly monotonic decreasing. Hence all components $\left\{b_{\vartheta(0)}, b_{\vartheta(1)}, \ldots, b_{\vartheta(\mathtt{q})}\right\}$ meet α any time while they are continuously increasing to $\frac{1}{n+1}$. The other components $\left\{b_{\vartheta(\mathtt{q}+1)}, \ldots, b_{\vartheta(\mathtt{r})}, \ldots, b_{\vartheta(n)}\right\}$ of $\vec{b}$ remain larger than α while they are changing into $\frac{1}{n+1}$.
Lemma (10) is proven. □

Now we need for $\vec{b} = (b_0, b_1, \ldots, b_n) \in \mathbf{Bound}_n$ an increasing homeomorphism $\mathbf{\Gamma}[\vec{b}]$ on $[0,1]$. We had $\varphi(\vec{b}) = \vec{c} = (c_0, c_1, \ldots, c_n)$. Let $\mathcal{SPO}(\vec{b})$ be the following set of points, it has a cardinality between 3 and $\mathtt{q}+3$,

$$\mathcal{SPO}(\vec{b}) := \left\{ \ (0,0),\ \ldots\ ,\ \left(\frac{\alpha - b_{\vartheta(\mathtt{q}-j)}}{\frac{1}{n+1} - b_{\vartheta(\mathtt{q}-j)}}, \frac{\beta - c_{\vartheta(\mathtt{q}-j)}}{\frac{1}{n+1} - c_{\vartheta(\mathtt{q}-j)}} \right),\ \ldots\ ,\ (1,1) \right\},\ \text{ for } j = 0, 1, \ldots, \mathtt{q}\,.$$

Noting the order of $\mathcal{SPO}(\vec{b})$, let $\mathbf{\Gamma}[\vec{b}]$ be the polygon defined by $\mathcal{SPO}(\vec{b})$.

Lemma 11. *For all* $\vec{b} = (b_0, b_1, \ldots, b_n) \in \mathbf{Bound}_n$ *the just constructed object* $\mathbf{\Gamma}[\vec{b}]$ *is a well defined increasing homeomorphism on the interval* $[0,1]$*, i.e.* $\mathbf{\Gamma}[\vec{b}] : [0,1] \xrightarrow{\cong} [0,1]$.

Proof. Easy. We have to ensure that (1) both the first components and the second components of the set $\mathcal{SPO}(\vec{b})$ are monotonic increasing, which can be confirmed by a calculation, and (2) that some of the first components of $\mathcal{SPO}(\vec{b})$ are equal if and only if the corresponding second components are equal, too.

Note $b_{\vartheta(0)} \leq \ldots \leq b_{\vartheta(\mathtt{q})} \leq \alpha < b_{\vartheta(\mathtt{q}+1)}$ and $c_{\vartheta(0)} \leq \ldots \leq c_{\vartheta(\mathtt{q})} \leq \beta < c_{\vartheta(\mathtt{q}+1)}$. Please see the previous Lemma (9). For any index $\mathtt{j} \in \{0, 1, \ldots, n\}$ we have $b_{\mathtt{j}} = \alpha$ if and only if $c_{\mathtt{j}} = \beta$. And also $b_{\mathtt{j}} = b_{\mathtt{k}}$ if and only if $c_{\mathtt{j}} = c_{\mathtt{k}}$, for any pair $\mathtt{j}, \mathtt{k}$ of indices. $\square$

Let $\vec{x}$ be an element of the line segment $\left[\mathbf{Center}_n, \vec{b}\right]$, i.e. $\vec{x} = t \cdot \mathbf{Center}_n + (1-t) \cdot \vec{b} = \vec{b} + t \cdot (\mathbf{Center}_n - \vec{b})$, for a suitable $t \in [0,1]$. We get for a number $0 \leq \widehat{t_j} < 1$ with

$$\widehat{t_j} := \frac{\alpha - b_{\vartheta(j)}}{\frac{1}{n+1} - b_{\vartheta(j)}} \quad \text{for } j = 0, 1, \ldots, \mathtt{q}\ ,$$

that the $\vartheta(j)^{th}$ component of $\vec{x}$ is α, i.e. for $\vec{x} = \vec{b} + \widehat{t_j} \cdot \left(\mathbf{Center}_n - \vec{b}\right)$ we have

$$x_{\vartheta(j)} = b_{\vartheta(j)} + \widehat{t_j} \cdot \left(\frac{1}{n+1} - b_{\vartheta(j)} \right) = \alpha, \quad \text{for} \quad j = 0, 1, \ldots, \mathtt{q}.$$

Now we describe the map Φ. For $\vec{b} \in \mathbf{Bound}_n$ we define for all points $\vec{x} \in \left[\mathbf{Center}_n, \vec{b}\right]$, i.e. $\vec{x} = \vec{b} + t \cdot \left(\mathbf{Center}_n - \vec{b}\right)$ for a suitable $t \in [0,1]$, with $\varphi(\vec{b}) = \vec{c}$ the image $\Phi(\vec{x})$. Let

$$\Phi(\vec{x}) \ :=\ \vec{c} + \mathbf{\Gamma}[\vec{b}](t) \cdot (\mathbf{Center}_n - \vec{c}) \ \in [\mathbf{Center}_n, \vec{c}]\ ,$$

and the map Φ fulfils all the properties which are demanded in Proposition (2). The details can be left to the reader. Hence the proof of Proposition (2) is finished. $\square$

We continue our investigations with a further interesting proposition.

Proposition 3. *Let* $n \in \mathbb{N}$. *There is an injective group morphism*
$\mathbf{\Psi_n} : \left(\left\{ f | f \text{ is an increasing homeomorphism on } \left[0, \frac{1}{n+1}\right] \right\}, \circ\right) \longrightarrow (\mathcal{COMFORT}(\Delta_n), \circ)$.

Remark 4. Obviously, we can replace $\left[0, \frac{1}{n+1}\right]$ by any closed interval.

Proof. Let f be a map with the above conditions, i.e. f is an increasing homeomorphism on $\left[0, \frac{1}{n+1}\right]$. It follows $f(0) = 0$ and $f\left(\frac{1}{n+1}\right) = \frac{1}{n+1}$. Let $\vec{x} := (x_0, x_1, \ldots, x_n) \in \Delta_n$. Trivially, there is a permutation ϑ of $\{0, 1, 2, \ \ldots\ , n\}$ and there is an index $\mathtt{r} \in \{0, 1, 2, \ \ldots, n\}$ such that

$$0 \leq x_{\vartheta(0)} \leq x_{\vartheta(1)} \leq \ \ldots\ \leq x_{\vartheta(\mathtt{r})} \leq \frac{1}{n+1} < x_{\vartheta(\mathtt{r}+1)} \leq x_{\vartheta(\mathtt{r}+2)} \ \leq\ \ldots\ \leq x_{\vartheta(n-1)} \leq x_{\vartheta(n)} \leq 1\ .$$

We define the map $\mathbf{\Psi_n}(f) := F : \Delta_n \longrightarrow \Delta_n$. Let $F(x_0, x_1, \ldots, x_n) := \vec{y} := (y_0, y_1, \ldots, y_n)$. For all $i \in \{0, 1, \ldots, \mathtt{r}\}$ we set

$$y_{\vartheta(i)} := f\left(x_{\vartheta(i)}\right) .$$

Note that in the case $\mathtt{r} = n$ we have $F(\vec{x}) = \vec{x} = \left(\frac{1}{n+1}, \; \ldots \; , \frac{1}{n+1}\right) = \mathbf{Center}_n$. For $\mathtt{r} < n$ we define two real numbers D, δ by

$$D \; := \; \sum_{i=0}^{\mathtt{r}} \left(x_{\vartheta(i)} - y_{\vartheta(i)}\right) \quad \text{and} \quad \delta \; := \; \frac{D}{\sum_{i=\mathtt{r}+1}^{n} \left(x_{\vartheta(i)} - \frac{1}{n+1}\right)} \; . \tag{7}$$

Finally let for all $i \in \{\mathtt{r}+1, \mathtt{r}+2, \ldots, n\}$

$$y_{\vartheta(i)} \; := \; x_{\vartheta(i)} + \delta \cdot \left(x_{\vartheta(i)} - \frac{1}{n+1}\right) \; = \; x_{\vartheta(i)} \cdot (1+\delta) - \frac{\delta}{n+1} \, ,$$

and the definition of F is complete. To prove Proposition (3) we still have to verify that

- F is a map to Δ_n
- F is injective
- F is surjective
- F is continuous
- F fulfils the conditions $\widehat{[\mathbf{2}]}$ and $\widehat{[\mathbf{3}]}$ of Definition (14).

F is a map to Δ_n : We have for $\vec{x} = (x_0, x_1, \ldots, x_n) \neq \mathbf{Center}_n$, i.e. $\mathtt{r} < n$:

$$\begin{aligned} 1 - \sum_{i=0}^{n} y_{\vartheta(i)} \;&=\; \sum_{i=0}^{n} (x_{\vartheta(i)} - y_{\vartheta(i)}) \;=\; \sum_{i=0}^{\mathtt{r}} (x_{\vartheta(i)} - y_{\vartheta(i)}) + \sum_{i=\mathtt{r}+1}^{n} (x_{\vartheta(i)} - y_{\vartheta(i)}) \\ &=\; D + \sum_{i=\mathtt{r}+1}^{n} (x_{\vartheta(i)} - y_{\vartheta(i)}) \\ &=\; D - \sum_{i=\mathtt{r}+1}^{n} \delta \cdot \left(x_{\vartheta(i)} - \frac{1}{n+1}\right) \qquad \text{(see the definition of the } y_{\vartheta(i)}\text{'s)} \\ &=\; D - \frac{D}{\sum_{i=\mathtt{r}+1}^{n} \left(x_{\vartheta(i)} - \frac{1}{n+1}\right)} \cdot \left[\sum_{i=\mathtt{r}+1}^{n} \left(x_{\vartheta(i)} - \frac{1}{n+1}\right)\right] \;=\; 0 \; . \end{aligned}$$

Hence $\sum_{i=0}^{n} y_{\vartheta(i)} = 1$.

Now we show that $y_{\vartheta(j)} \geq 0$ for all $j \in \{0, \ldots, n\}$. This is trivial for $j \in \{0, \ldots, \mathtt{r}\}$ or for $D \geq 0$. Thus let j be a fixed element from $\{\mathtt{r}+1, \mathtt{r}+2, \ldots, n\}$, and let $D < 0$, hence $\delta < 0$. We prove the even stronger inequality $y_{\vartheta(j)} > \frac{1}{n+1}$. Note $\sum_{i=0}^{n} x_i = 1$.

Proof. We have $y_{\vartheta(0)} < \frac{1}{n+1}$ and $y_{\vartheta(i)} \leq \frac{1}{n+1}$ for $i = 1, 2, \ldots \mathtt{r}$. We get equivalences

$$\begin{aligned}
\frac{\mathtt{r}+1}{n+1} > \sum_{i=0}^{\mathtt{r}} y_{\vartheta(i)} &\iff 1 - \frac{n-\mathtt{r}}{n+1} > \sum_{i=0}^{\mathtt{r}} y_{\vartheta(i)} \\
&\iff 1 - \frac{n-\mathtt{r}}{n+1} - \sum_{i=0}^{\mathtt{r}} x_{\vartheta(i)} > \sum_{i=0}^{\mathtt{r}} y_{\vartheta(i)} - \sum_{i=0}^{\mathtt{r}} x_{\vartheta(i)} = -D \\
&\iff \left(\sum_{i=\mathtt{r}+1}^{n} x_{\vartheta(i)} \right) - \frac{n-\mathtt{r}}{n+1} > -D \\
&\iff 1 > \frac{-D}{\left(\sum_{i=\mathtt{r}+1}^{n} x_{\vartheta(i)}\right) - \frac{n-\mathtt{r}}{n+1}} = -\delta \\
&\iff 1 > -\delta \iff x_{\vartheta(j)} - \frac{1}{n+1} > (-\delta) \cdot \left(x_{\vartheta(j)} - \frac{1}{n+1} \right) \\
&\iff x_{\vartheta(j)} + \delta \cdot \left(x_{\vartheta(j)} - \frac{1}{n+1} \right) = y_{\vartheta(j)} > + \frac{1}{n+1} .
\end{aligned}$$

□

Hence we have proven that the image of F is Δ_n, i.e. we have a map $F : \Delta_n \longrightarrow \Delta_n$.

Lemma 12. *(1) We can see from the above equivalences the inequality* $1 > -\delta$. *If we look at the definition of the components* $y_{\vartheta(i)}$ *with indices* $i \in \{\mathtt{r}+1, \mathtt{r}+2, \ldots, n\}$, *the fact* $1 + \delta > 0$ *easily leads to the conclusion that F keeps the order, i.e. the condition* $\widehat{[\mathbf{3}]}$ *in Definition* (14) *is fulfilled.*
(2) F does not change components $\{x_0, \ldots, x_n\}$ *of the set* $\left\{0, \frac{1}{n+1}, 1\right\} \cup \{$*the fixed points of f*$\}$.
(3) F respects permutations on Δ_n, *that means if* ϑ *is any permutation of* $\{0, 1, 2, \ldots, n\}$, *and if* $F(x_0, x_1, \ldots, x_n) = (y_0, y_1, \ldots, y_n)$, *we get* $F\left(x_{\vartheta(0)}, x_{\vartheta(1)}, \ldots, x_{\vartheta(n)}\right) = \left(y_{\vartheta(0)}, y_{\vartheta(1)}, \ldots, y_{\vartheta(n)}\right)$. *This means that* $\widehat{[\mathbf{2}]}$ *in Definition* (14) *is fulfilled.*

Proof. (1): Let $0 \leq x_{\vartheta(0)} \leq x_{\vartheta(1)} \leq \ldots \leq x_{\vartheta(\mathtt{r})} \leq \frac{1}{n+1} < x_{\vartheta(\mathtt{r}+1)} \leq \ldots \leq x_{\vartheta(n)} \leq 1$. For $x_i < x_j \leq \frac{1}{n+1}$ we have $y_i < y_j \leq \frac{1}{n+1}$, because f is an homeomorphism. If $\frac{1}{n+1} < x_i < x_j$ we defined $y_i = x_i \cdot (1+\delta) - \frac{\delta}{n+1}$ and $y_j = x_j \cdot (1+\delta) - \frac{\delta}{n+1}$. With $1 + \delta > 0$ it follows $y_i < y_j$. We get

$$0 \leq y_{\vartheta(0)} \leq y_{\vartheta(1)} \leq \ldots \leq y_{\vartheta(\mathtt{r})} \leq \tfrac{1}{n+1} < y_{\vartheta(\mathtt{r}+1)} \leq \ldots \leq y_{\vartheta(n)} \leq 1.$$

(2) and (3): Both points follow easily from the construction of F. □

<u>F is injective :</u> The injectivity is a consequence of the fact that we are able to construct the inverse map F^{-1}.

<u>F is surjective :</u> We construct F^{-1}. Trivially, for an element $\vec{y} = (y_0, y_1, \ldots, y_n) \in \Delta_n$ there is a permutation ϑ and an index $\mathtt{r} \in \{0, 1, 2, \ldots, n\}$ such that

$$0 \leq y_{\vartheta(0)} \leq y_{\vartheta(1)} \leq \ldots \leq y_{\vartheta(\mathtt{r})} \leq \tfrac{1}{n+1} < y_{\vartheta(\mathtt{r}+1)} \leq y_{\vartheta(\mathtt{r}+2)} \leq \ldots \leq y_{\vartheta(n-1)} \leq y_{\vartheta(n)} \leq 1.$$

Then we define the inverse map $F^{-1}(y_0, y_1, \ldots, y_n) := (x_0, x_1, \ldots, x_n)$ by $x_{\vartheta(j)} := f^{-1}\left(y_{\vartheta(j)}\right)$ for $j \in \{0, 1, \ldots, \mathtt{r}\}$, and (in the case of $\mathtt{r} < n$) we define $D := \sum_{i=0}^{\mathtt{r}} \left(x_{\vartheta(i)} - y_{\vartheta(i)}\right)$ and

$$\delta := \frac{D}{\left[\sum_{i=\mathtt{r}+1}^{n} \left(y_{\vartheta(i)} - \frac{1}{n+1}\right)\right] - D} .$$

Of course, the values of D and δ coincide with those values from the terms in (7).

We define for the other indices $j \in \{\mathtt{r}+1, \mathtt{r}+2, \ldots, n\}$ the components

$$x_{\vartheta(j)} := \left(y_{\vartheta(j)} + \frac{\delta}{n+1} \right) \cdot \left(\frac{1}{1+\delta} \right) .$$

Then we get $F(\vec{x}) = \vec{y}$, as well as $F \circ F^{-1} = F^{-1} \circ F = id(\Delta_n)$.

<u>F is continuous :</u> Let PROJ_j be the canonical projection for all $j \in \{0, \ldots, n\}$, $\mathsf{PROJ}_j : \Delta_n \longrightarrow [0,1]$, $(x_0, x_1, \ldots, x_j, \ldots, x_n) \mapsto x_j$. PROJ_j is continuous. Let $F_j := \mathsf{PROJ}_j \circ F$. Consider the following commutative diagram (Figure 12):

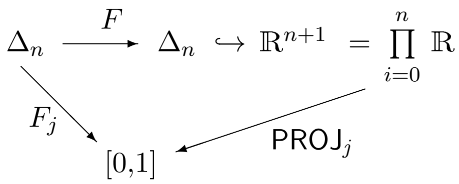

Figure 12:

We have that F is continuous if and only if F_j is continuous, for all $j \in \{0, \ldots, n\}$. The continuity of the F_j's is rather easy. We have to turn some attention to the case when components of $(x_0, x_1, \ldots, x_n) \in \Delta_n$ cross the value $\frac{1}{n+1}$.

This finishes the construction of $\mathbf{\Psi_n}(f) = F$, and F is an element of $\mathcal{COMFORT}(\Delta_n)$, since it was already shown in Lemma (12) that $\mathbf{\Psi_n}(f)$ fulfils $\widehat{[\mathbf{2}]}$ and $\widehat{[\mathbf{3}]}$ of Definition (14). Furthermore, for two increasing homeomorphisms f, g on $\left[0, \frac{1}{n+1}\right]$ we can confirm with the aid of technical calculations (which are omitted here), that we have the identities

$$\mathbf{\Psi_n}(g \circ f) = \mathbf{\Psi_n}(g) \circ \mathbf{\Psi_n}(f), \text{ and } \mathbf{\Psi_n}\left(f^{-1}\right) = \left(\mathbf{\Psi_n}(f)\right)^{-1}, \text{ and } \mathbf{\Psi_n}\left(id\left([0, \tfrac{1}{n+1}]\right)\right) = id(\Delta_n).$$

This proves that $\mathbf{\Psi_n}$ is a group morphism. The injectivity of $\mathbf{\Psi_n}$ is trivial. Now the proof of Proposition (3) is complete. □

Remark 5. Note that for all $0 \leq \alpha \leq \frac{1}{n+1}$ the homeomorphism $\mathbf{\Psi_n}(f)$ from Proposition (3) maps the α-cross homeomorphicly onto the $f(\alpha)$-cross,

$$\mathbf{\Psi_n}(f)_{|\clubsuit_{n,\alpha}} : \clubsuit_{n,\alpha} \xrightarrow{\cong} \clubsuit_{n,f(\alpha)} .$$

Lemma 13. *The map* $\mathbf{\Psi_n}$ *from* Proposition (3) *is not surjective.*

Proof. We need an element $F \in \mathcal{COMFORT}(\Delta_n)$ which is not in the image of $\mathbf{\Psi_n}$, for one $n \in \mathbb{N}$. We construct a homeomorphism $F : \Delta_2 \xrightarrow{\cong} \Delta_2$. We begin the definition of F on a face of Δ_2.

$$F_{|\mathbf{Bound_2} \cap \mathbf{Section_{2,0}}}(0, x, 1-x) := \begin{cases} \left(0, \frac{1}{2} \cdot x, 1 - \frac{1}{2} \cdot x\right) & \text{for } \; x \in \left[0, \frac{1}{4}\right] \\ \left(0, \frac{5}{2} \cdot x - \frac{1}{2}, \frac{3}{2} - \frac{5}{2} \cdot x\right) & \text{for } \; x \in \left[\frac{1}{4}, \frac{1}{3}\right] \\ (0, x, 1-x) & \text{for } \; x \in \left[\frac{1}{3}, \frac{1}{2}\right] \end{cases} ,$$

and $F_{|\mathbf{Bound_2} \cap \mathbf{Section_{2,0}}}$ can uniquely be extended on $\mathbf{Bound_2}$ by the properties $\widehat{[\mathbf{2}]}$ and $\widehat{[\mathbf{3}]}$. With Proposition (1) (we have the case $\alpha = \beta = 0$) $F_{|\mathbf{Bound_2}}$ can be extended to a homeomorphism $F \in \mathcal{COMFORT}(\Delta_2)$. This map F has the property that for $0 \leq \mu \leq \frac{1}{3}$ we have a homeomorphism $F_{|\mathbf{Layer}_{2,\mu}} : \mathbf{Layer}_{2,\mu} \xrightarrow{\cong} \mathbf{Layer}_{2,\mu}$. That means for an element $(\mu, v, w) \in \mathbf{Layer}_{2,\mu}$ (i.e. μ is the smallest element of $\{\mu, v, w\}$) that $F(\mu, v, w) = (\mu, \tilde{v}, \tilde{w})$ for suitable numbers $\tilde{v}, \tilde{w} \geq \mu$. If there would exist a homeomorphism $f : \left[0, \frac{1}{3}\right] \stackrel{\cong}{\rightarrow} \left[0, \frac{1}{3}\right]$ with $\mathbf{\Psi_2}(f) = F$, it follows $f(\mu) = \mu$ for all $\mu \in \left[0, \frac{1}{3}\right]$, i.e. f would be the identity. Hence F would be the identity on Δ_2. This contradicts the above definition of F on $\mathbf{Bound}_2 \cap \mathbf{Section}_{2,0}$. □

5 Induction Step

Here we define the homeomorphisms $\Theta_{1,n,0}$ explicitly for all n. Thereafter, we can conduct the induction step n to $n+1$ to get $\Theta_{1,n,1}$ and to approve $\texttt{EQUATION}_{n,j\leq p,i,k}$, i.e. we prove Theorem (2). But we work out before the details of the induction step from $n = 1$ to $n = 2$. We think that this is helpful, since the general step is rather intricate.

The reason for introducing the map $\mathbf{\Psi_n}$ of Proposition (3) is that we will take the postulated homeomorphism $\Theta_{1,n,0}$ from the image of $\mathbf{\Psi_n}$, for each n. After that we use Proposition (2) to construct the maps $\Theta_{1,n,1}$ by induction on n.

We still are considering the case $L = 1$. In the following we omit this constant $L = 1$ for better readability. Recall that we have defined in Definition (4) the maps for $n = 1$:

$$\Theta_{1,0}(x, 1-x) = (\eta(x), \eta(1-x)) \quad \text{and} \quad \Theta_{1,1}(x, 1-x) = (\kappa(x), \kappa(1-x)) \text{ , for } (x, 1-x) \in \Delta_1.$$

Note $\Theta_{1,0}\left(\frac{1}{4}, \frac{3}{4}\right) = \left(\frac{1}{6}, \frac{5}{6}\right)$, and $\Theta_{1,1}\left(\frac{1}{4}, \frac{3}{4}\right) = \left(\frac{1}{5}, \frac{4}{5}\right)$.

Now we can continue the constructions of the homeomorphisms $\Theta_{n,0}$ and $\Theta_{n,1}$. They will be elements of $\mathcal{COMFORT}(\Delta_n)$. Four of them, $\Theta_{n-1,0}, \Theta_{n-1,1}, \Theta_{n,0}, \Theta_{n,1}$, will be needed to fulfil the equations $\texttt{EQUATION}_{n,j\leq p,i,k}$, for all $i, k \in \{0, 1\}$ and for $j, p \in \{0, 1, \ldots, n\}$ with $j \leq p$, for all fixed $n \in \mathbb{N}$.

Definition 15. Let $n \in \mathbb{N}_0$. First we define homeomorphisms $\phi_{n,0}$ on the interval $\left[0, \frac{1}{n+1}\right]$. Let $\phi_{n,0}$ be the polygon through three points

$$\left\{(0,0), \left(\frac{1}{2\cdot(n+1)}, \frac{1}{2\cdot(n+2)}\right), \left(\frac{1}{n+1}, \frac{1}{n+1}\right)\right\} .$$

Now we define for positive n the homeomorphism $\Theta_{n,0} := \mathbf{\Psi_n}(\phi_{n,0})$, and by Proposition (3), $\Theta_{n,0}$ is an element of $\mathcal{COMFORT}(\Delta_n)$ for all numbers $n \in \mathbb{N}_0$. □

Note $\phi_{1,0} = \eta_{|[0,\frac{1}{2}]}$, hence the definition of $\Theta_{1,0} = \mathbf{\Psi_1}(\phi_{1,0})$ corresponds with those we already have given in Definition (4), i.e. $\Theta_{1,0}(x, 1-x) = (\eta(x), \eta(1-x))$. And note

$$\phi_{n,0}\left(\frac{1}{2\cdot(n+1)}\right) = \frac{1}{2\cdot(n+2)}, \quad \text{for all } n \in \mathbb{N}_0.$$

Lemma 14. *For all $n \in \mathbb{N}$, the homeomorphisms $\Theta_{n,0}$ yield homeomorphisms*

$$\Theta_{n,0|\clubsuit_{n,\frac{1}{2\cdot(n+1)}}} : \clubsuit_{n,\frac{1}{2\cdot(n+1)}} \stackrel{\cong}{\longrightarrow} \clubsuit_{n,\frac{1}{2\cdot(n+2)}} .$$

Proof. See Definition (6) of the α-cross $\clubsuit_{n,\alpha}$, and please repeat Proposition (3), where the maps $\mathbf{\Psi_n}$ have been defined. □

Now we construct the maps $\Theta_{n,1}$ for $n \in \mathbb{N}$ by induction to fulfil $\texttt{EQUATION}_{n,j\leq p,i,k}$. Note that $\Theta_{0,1}$ and $\Theta_{1,1}$ already exist. Because of $i, k \in \{0, 1\}$, we have to look at four possibilities, for $(i, k) \in \{(0,0), (0,1), (1,0), (1,1)\}$. The reader should notice that in the case $i = k = 0$ the $\texttt{EQUATION}_{n,j\leq p,0,0}$ is trivial. (The maps $\langle id\rangle_{n,0,j}$ only add a component 0 to the components of an element $(x_0, \ldots, x_{n-1})$ of Δ_{n-1}.) Let $i := 0$, $k := 1$.

We fix the positions $j := p := 0$, and and we construct the maps $\Theta_{n,1}$ by induction on n. At first we define $\Theta_{n,1}$ on the topological boundary $\mathbf{Bound_n}$ to fulfil $\texttt{EQUATION}_{n,0\leq 0,i=0,k=1}$. After that we shall extend the map $\Theta_{n,1}$ on Δ_n to fulfil $\texttt{EQUATION}_{n,0\leq 0,i=1,k=1}$, too.

To verify the equation $\texttt{EQUATION}_{n,j\leq p,i=0,k=1}$ we have to show the commutativity of Figure 13 for all natural numbers n:

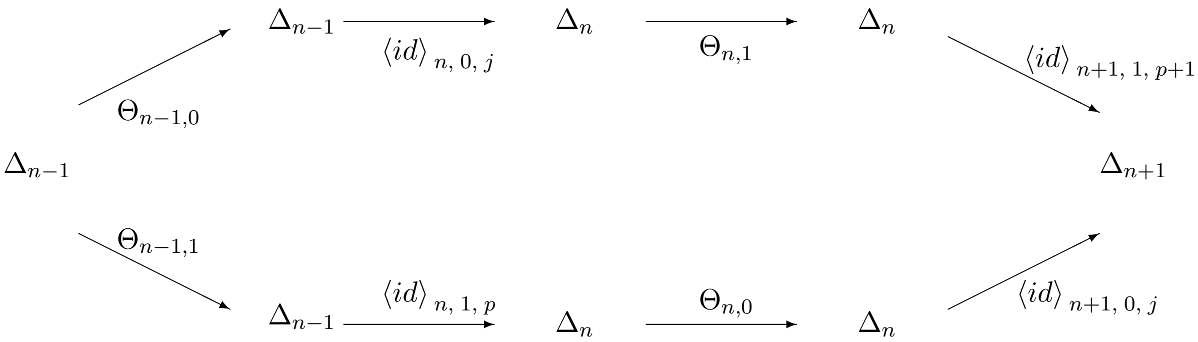

Figure 13:

We describe the way of the induction by writing down the case $n = 2$ explicitly. We consider $\texttt{EQUATION}_{n=2,j=0\leq p=0,i=0,k=1}$. We have to show the equation

$$\langle id \rangle_{3,0,0} \circ \Theta_{2,0} \circ \langle id \rangle_{2,1,0} \circ \Theta_{1,1} = \langle id \rangle_{3,1,1} \circ \Theta_{2,1} \circ \langle id \rangle_{2,0,0} \circ \Theta_{1,0}\,,$$

this means that we need to demonstrate the commutativity of Figure 14:

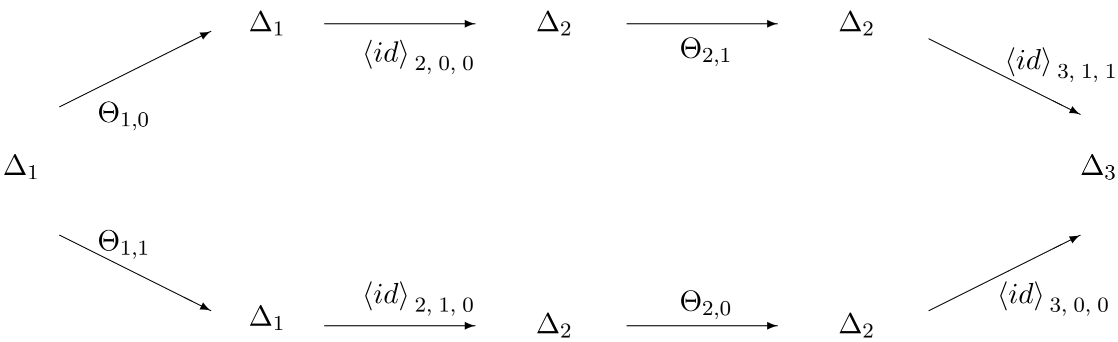

Figure 14:

Take an arbitrary element $(x, 1-x) \in \Delta_1$, $x \in [0,1]$. Note $\phi_{2,0}\left(\frac{1}{6}\right) = \frac{1}{8}$, and therefore that $\Theta_{2,0}$ yields a homeomorphism $\Theta_{2,0}|_{\clubsuit_{2,\frac{1}{6}}} : \clubsuit_{2,\frac{1}{6}} \xrightarrow{\cong} \clubsuit_{2,\frac{1}{8}}$. We have $\Theta_{1,1}\ (x, 1-x) = (\kappa(x), \kappa(1-x))$, and $\langle id \rangle_{2,1,0}\ (\kappa(x), \kappa(1-x)) = \left(\frac{1}{6}, \frac{5}{6} \cdot \kappa(x), \frac{5}{6} \cdot \kappa(1-x)\right)$. This element will be mapped by $\Theta_{2,0}$, we write $\Theta_{2,0}\left(\frac{1}{6}, \frac{5}{6} \cdot \kappa(x), \frac{5}{6} \cdot \kappa(1-x)\right) =: \left(\frac{1}{8}, y, z\right)$ for suitable real numbers y, z. In short, an element $(x, 1-x) \in \Delta_1$ will be mapped in the bottom path of the above Figure 14 in the following way:

$$\langle id \rangle_{\,3,\,0,\,0} \circ \Theta_{2,0} \circ \langle id \rangle_{\,2,\,1,\,0} \circ \Theta_{1,1}\ (x, 1-x) \ = \ \left(0, \frac{1}{8}, y, z\right) \ .$$

Since the diagram in Figure 14 shall commute, the map $\langle id \rangle_{\,3,\,1,\,1}$ from the upper path needs an element $\left(0, \frac{8}{7} \cdot y, \frac{8}{7} \cdot z\right)$ to map it to $\left(0, \frac{1}{8}, y, z\right)$.

Hence we have to define $\Theta_{2,1}$ in a way that the following diagram (Figure 15) commutes:

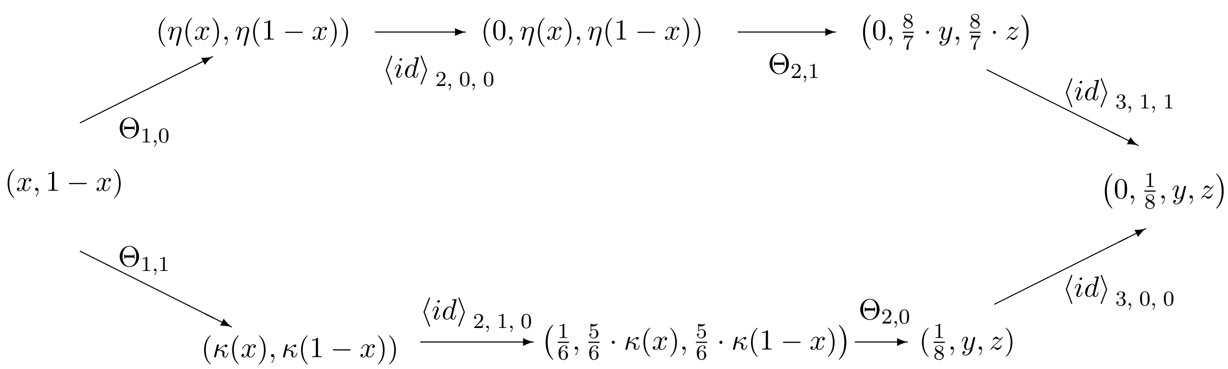

Figure 15:

Therefore, we must define $\Theta_{2,1}\,(0, \eta(x), \eta(1-x)) := \left(0, \frac{8}{7} \cdot y, \frac{8}{7} \cdot z\right)$. This will be explained now in detail.

We are able to change the directions of the maps $\Theta_{1,0}$, $\langle id \rangle_{\,2,\,0,\,0}$, and $\langle id \rangle_{\,3,\,1,\,1}$, respectively. We call the generated maps $\Theta_{1,0}^{-1}$, $\langle id \rangle_{\,2,\,0,\,0}^{-1}$, and $\langle id \rangle_{\,3,\,1,\,1}^{-1}$, respectively. Since $\Theta_{1,0}$ is a homeomorphism, the meaning of $\Theta_{1,0}^{-1}$ is clear. The map $\langle id \rangle_{\,2,\,0,\,0}^{-1}$ will map certain subsets of Δ_2 homeomorphicly onto Δ_1, and $\langle id \rangle_{\,3,\,1,\,1}^{-1}$ will map subsets of Δ_3 homeomorphicly onto Δ_2. Let $\langle id \rangle_{\,2,\,0,\,0}^{-1}$ be act on a 3-tuple $(0, \eta(x), \eta(1-x))$ by deleting the 0 at the first place,

$$\langle id \rangle_{\,2,\,0,\,0}^{-1}\ (0, \eta(x), \eta(1-x)) := (\eta(x), \eta(1-x))\ ,$$

$\langle id \rangle_{\,3,\,1,\,1}^{-1}$ acts on $\left(0, \frac{1}{8}, y, z\right)$ by deleting the fraction $\frac{1}{8}$ at the second place and multiplying the other components with $\frac{8}{7}$,

$$\langle id \rangle_{\,3,\,1,\,1}^{-1}\ \left(0, \frac{1}{8}, y, z\right) := \left(0, \frac{8}{7} \cdot y, \frac{8}{7} \cdot z\right) .$$

Note that the maps $\langle id \rangle_{\,2,\,0,\,0}^{-1}$ and $\langle id \rangle_{\,3,\,1,\,1}^{-1}$ are injective left inverse maps, it holds

$$\langle id \rangle_{\,2,\,0,\,0}^{-1} \circ \langle id \rangle_{\,2,\,0,\,0} = id(\Delta_1) \qquad \text{and as well} \qquad \langle id \rangle_{\,3,\,1,\,1}^{-1} \circ \langle id \rangle_{\,3,\,1,\,1} = id(\Delta_2)\ .$$

And note that we also have the identities $\langle id \rangle_{\,2,\,0,\,0} \circ \langle id \rangle_{\,2,\,0,\,0}^{-1}\ (0, a, b) \ = \ (0, a, b)$, and $\langle id \rangle_{\,3,\,1,\,1} \circ \langle id \rangle_{\,3,\,1,\,1}^{-1}\ \left(0, \frac{1}{8}, y, z\right) = \left(0, \frac{1}{8}, y, z\right)$ for suitable numbers $a, b, y, z \in [0, 1]$.

If we turn around the directions of the maps $\Theta_{1,0}$, $\langle id \rangle_{\,2,\,0,\,0}$, and $\langle id \rangle_{\,3,\,1,\,1}$, respectively, in the described way, and if we demand the commutativity of the above diagram (Figure 15), the map $\Theta_{2,1}$ is uniquely defined on the face $\mathbf{Bound}_2 \cap \mathbf{Section}_{2,0}$.

This means that we define $\Theta_{2,1}$ for every 3-tuple $(0, \eta(x), \eta(1-x)) \in \mathbf{Bound}_2 \cap \mathbf{Section}_{2,0}$, i.e. we define

$$\Theta_{2,1}|_{\mathbf{Bound}_2 \cap \mathbf{Section}_{2,0}} := \langle id \rangle^{-1}_{3,\,1,\,1} \circ \langle id \rangle_{3,\,0,\,0} \circ \Theta_{2,0} \circ \langle id \rangle_{2,\,1,\,0} \circ \Theta_{1,1} \circ \Theta^{-1}_{1,0} \circ \langle id \rangle^{-1}_{2,\,0,\,0} \ ,$$

$$\Theta_{2,1}|_{\mathbf{Bound}_2 \cap \mathbf{Section}_{2,0}} (0, \eta(x), \eta(1-x)) \ := \ \left(0, \frac{8}{7} \cdot y, \frac{8}{7} \cdot z\right) \ ,$$

see the following commutative diagram (Figure 16). There we start at $(0, \eta(x), \eta(1-x))$:

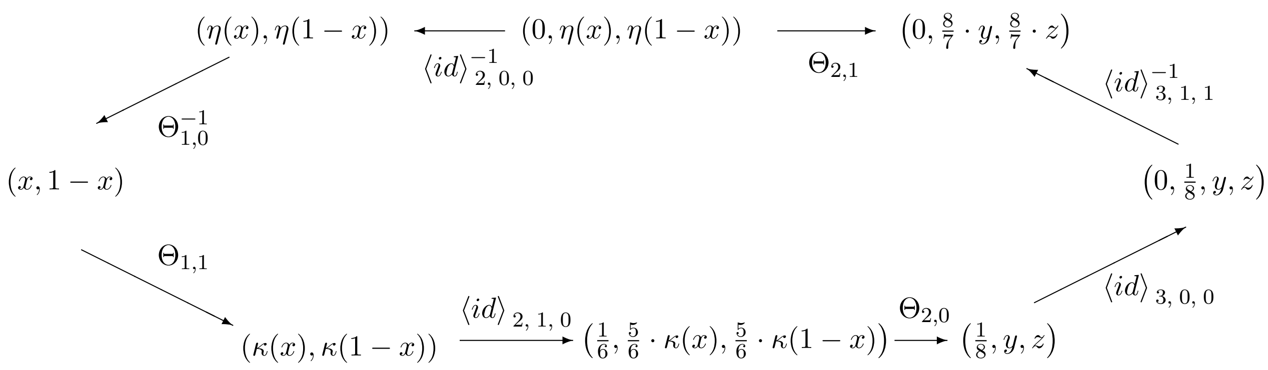

Figure 16:

Up to now the map $\Theta_{2,1}$ is a homeomorphism on $\mathbf{Bound}_2 \cap \mathbf{Section}_{2,0}$, i.e. it maps triples $(0, a, b) \in \Delta_2$. We had fixed the positions $j = p = 0$. If we vary $j, p \in \{0, 1, 2\}$ with $j \leq p$, we are able to define $\Theta_{2,1}$ on the other faces $\mathbf{Bound}_2 \cap \mathbf{Section}_{2,1}$ and $\mathbf{Bound}_2 \cap \mathbf{Section}_{2,2}$, respectively. All maps $\Theta_{1,0}$, $\Theta_{1,1}$, $\langle id \rangle_{2,\,0,\,j}$, $\langle id \rangle_{2,\,1,\,p}$, $\Theta_{2,0}$, $\langle id \rangle_{3,\,0,\,j}$, $\langle id \rangle_{3,\,1,\,p+1}$ respect permutations, see Lemma (1), hence there are no contradictions in the definition of $\Theta_{2,1}$ on $\mathbf{Bound}_2$, the topological boundary of Δ_2 .

Lemma 15. *The just constructed map $\Theta_{2,1}|_{\mathbf{Bound}_2}$ is a homeomorphism on $\mathbf{Bound}_2$. Further, it satisfies the conditions $\widehat{[\mathbf{2}]}$ (respecting permutations) and $\widehat{[\mathbf{3}]}$ (keeping the order).*

Proof.

- Continuity: $\Theta_{2,1}|_{\ \mathbf{Bound}_2 \cap \mathbf{Section}_{2,j}}$ is a product of seven continuous maps $\langle id \rangle^{-1}_{2,\,0,\,j}$, $\Theta^{-1}_{1,0}$, $\Theta_{1,1}$, $\langle id \rangle_{2,\,1,\,p}$, $\Theta_{2,0}$, $\langle id \rangle_{3,\,0,\,j}$, $\langle id \rangle^{-1}_{3,\,1,\,p+1}$, for $j, p \in \{0, 1, 2\}$, $j \leq p$.
- $\Theta_{2,1}|_{\mathbf{Bound}_2}$ is injective, because all seven maps are injective.
- $\Theta_{2,1}|_{\mathbf{Bound}_2}$ is surjective on $\mathbf{Bound}_2$.
- The properties $\widehat{[\mathbf{2}]}$ and $\widehat{[\mathbf{3}]}$ are easy to verify because of the construction of $\Theta_{2,1}|_{\mathbf{Bound}_2}$. See Figure 16, and note that all maps $\Theta_{1,0}, \Theta_{1,1}, \Theta_{2,0}$ are from $\mathcal{COMFORT}(\Delta_1)$ or $\mathcal{COMFORT}(\Delta_2)$, respectively. Please note Lemma (1).

□

Starting with the element $\left(\frac{1}{4}, \frac{3}{4}\right) \in \clubsuit_{1,\frac{1}{4}}$ in the last but one diagram in Figure 15, we see

$$\Theta_{2,1|\mathbf{Bound}_2 \cap \clubsuit_{2,\frac{1}{6}}} \ : \ \mathbf{Bound}_2 \cap \clubsuit_{2,\frac{1}{6}} \stackrel{\cong}{\longrightarrow} \mathbf{Bound}_2 \cap \clubsuit_{2,\frac{1}{7}} \ ,$$

e.g. $\left(0, \frac{1}{6}, \frac{5}{6}\right) \longmapsto \left(0, \frac{1}{7}, \frac{6}{7}\right)$. Now the reader should recall Proposition (2) and the attached figures Figure 10 and Figure 11. By this proposition the map $\Theta_{2,1}|_{\mathbf{Bound_2}}$ can be extended to a homeomorphism $\Theta_{2,1}$ on Δ_2 , even $\Theta_{2,1} \in \mathcal{COMFORT}(\Delta_2)$, such that

$$\Theta_{2,1|\clubsuit_{2,\frac{1}{6}}} : \clubsuit_{2,\frac{1}{6}} \xrightarrow{\cong} \clubsuit_{2,\frac{1}{7}} .$$

By the construction of $\Theta_{2,1}|_{\mathbf{Bound_2}}$, the $\texttt{EQUATION}_{2,j\leq p,i,k}$ is satisfied for the two pairs $(i,k) \in \{(0,1),(1,0)\}$. For $i = k = 0$, $\texttt{EQUATION}_{2,j\leq p,0,0}$ is trivial. (The maps $\langle id \rangle_{2,0,j}$ only add a third component 0 to the components of an element $(y, 1-y)$ of Δ_1).

As the remaining step, the case $i = k = 1$ needs to be considered, i.e. we must show $\texttt{EQUATION}_{2,j\leq p,1,1}$. With the property $\Theta_{2,1|\clubsuit_{2,\frac{1}{6}}} : \clubsuit_{2,\frac{1}{6}} \xrightarrow{\cong} \clubsuit_{2,\frac{1}{7}}$ it is easy to see that the following diagram (Figure 17)

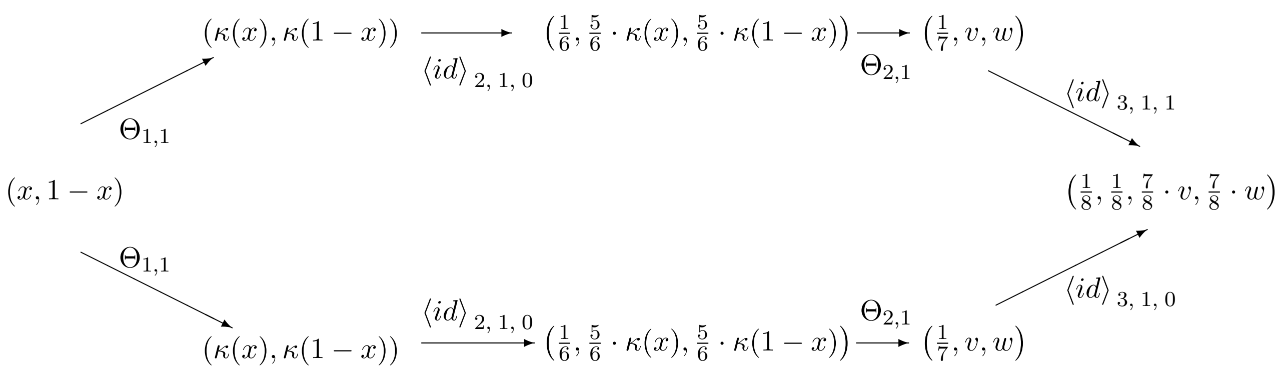

Figure 17:

commutes, with suitable numbers v, w. We also have confirmed that $\texttt{EQUATION}_{2,0\leq 0,i=1,k=1}$ holds. We get similar commutative diagrams if we take other pairs (j,p), for $j,p \in \{0,1,2\}$ with $j \leq p$. Hence, for $n = 2$ all 24 cases of $\texttt{EQUATION}_{2,j\leq p,i,k}$ are proven, for $i,k \in \{0,1\}$, and $j,p \in \{0,1,2\}$ with $j \leq p$.

At this point we take a summary of the results that we have got so far. We have two trivial homeomorphisms $\Theta_{0,0}, \Theta_{0,1}$ on $\Delta_0 = \{1\}$, and we have defined two homeomorphisms $\Theta_{1,0}, \Theta_{1,1}$ on Δ_1 and two homeomorphisms $\Theta_{2,0}, \Theta_{2,1}$ on Δ_2, respectively. Further, all four homeomorphisms are actually from $\mathcal{COMFORT}(\Delta_1)$ or $\mathcal{COMFORT}(\Delta_2)$, respectively. Furthermore, the four homeomorphisms $\Theta_{0,0}, \Theta_{0,1}, \Theta_{1,0}, \Theta_{1,1}$ satisfy the equations $\texttt{EQUATION}_{n=1,j\leq p,i,k}$, for $j,p \in \{0,1\}$ with $j \leq p$, and the four homeomorphisms $\Theta_{1,0}, \Theta_{1,1}, \Theta_{2,0}$ and $\Theta_{2,1}$ satisfy the equations $\texttt{EQUATION}_{n=2,j\leq p,i,k}$, for $j,p \in \{0,1,2\}$ with $j \leq p$, and always for all $i,k \in \{0,1\}$.

Theorem 2. *We are able to construct a homeomorphism $\Theta_{n,1} \in \mathcal{COMFORT}(\Delta_n)$ for all natural numbers n, such that, together with the already defined $\Theta_{n,0} = \mathbf{\Psi_n}(\phi_{n,0}) \in \mathcal{COMFORT}(\Delta_n)$, the four maps $\Theta_{n-1,0}, \Theta_{n-1,1}, \Theta_{n,0}, \Theta_{n,1}$ satisfy all equations $\texttt{EQUATION}_{n,j\leq p,i,k}$, for positions $j,p \in \{0,1,2,\ldots,n\}$ with $j \leq p$, and $i,k \in \{0,1\}$.*
Moreover, if we restrict $\Theta_{n,1}$ to the $\frac{1}{2\cdot(n+1)}$-cross of Δ_n, the restricted $\Theta_{n,1}$ maps $\clubsuit_{n,\frac{1}{2\cdot(n+1)}}$ homeomorphicly onto the $\frac{1}{2\cdot(n+1)+1}$-cross,

$$\Theta_{n,1}|_{\clubsuit_{n,\frac{1}{2\cdot(n+1)}}} : \quad \clubsuit_{n,\frac{1}{2\cdot(n+1)}} \xrightarrow{\cong} \clubsuit_{n,\frac{1}{2\cdot(n+1)+1}} .$$

With Theorem (2) we would get the validity of $\texttt{EQUATION}_{n,j\leq p,i,k}$. For instance for $i = 0$ and $k = 1$, the following diagram (Figure 18) would commute, for positions $j, p \in \{0, 1, 2, \ldots, n\}$ with $j \leq p$, for all integers n.

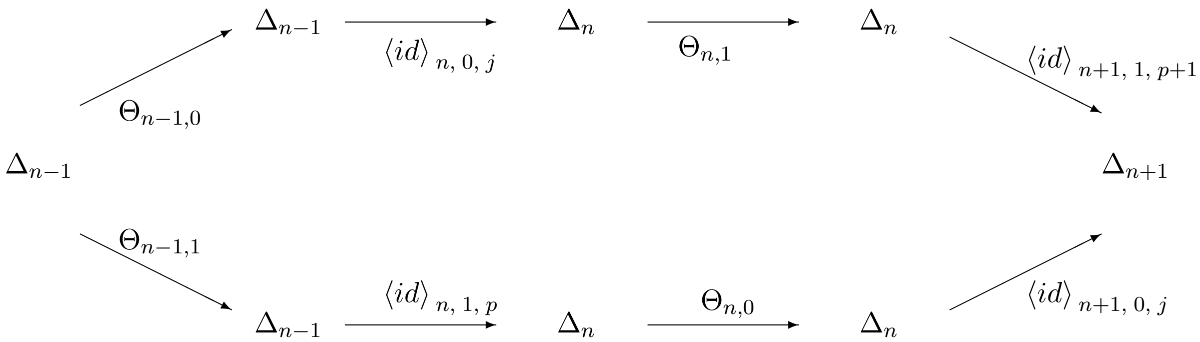

Figure 18:

Proof. (of Theorem (2)). We repeat the beginning of the induction from the third section.

<u>Start of the induction:</u> We had defined the $\frac{1}{4}$-cross of Δ_1, $\clubsuit_{1,\frac{1}{4}} = \left\{\left(\frac{1}{4}, \frac{3}{4}\right), \left(\frac{3}{4}, \frac{1}{4}\right)\right\}$. The constructed two homeomorphisms $\Theta_{1,0}, \Theta_{1,1}$ are elements of $\mathcal{COMFORT}(\Delta_1)$, and together with $\Theta_{0,0}, \Theta_{0,1}$ they fulfil the equations $\texttt{EQUATION}_{n=1,j\leq p,i,k}$ for $j, p \in \{0, 1\}$ with $j \leq p$, and $i, k \in \{0, 1\}$. Moreover, if we restrict $\Theta_{1,0}$ and $\Theta_{1,1}$ to $\clubsuit_{1,\frac{1}{4}}$, then $\Theta_{1,0}$ maps $\clubsuit_{1,\frac{1}{4}}$ onto the $\frac{1}{6}$-cross $\clubsuit_{1,\frac{1}{6}}$, and $\Theta_{1,1}$ maps $\clubsuit_{1,\frac{1}{4}}$ onto $\clubsuit_{1,\frac{1}{5}}$. (E.g. we have $\Theta_{1,1}\left(\frac{1}{4}, \frac{3}{4}\right) = \left(\frac{1}{5}, \frac{4}{5}\right)$). In short, we have maps

$$\Theta_{1,0}|\clubsuit_{1,\frac{1}{4}} : \clubsuit_{1,\frac{1}{4}} \xrightarrow{\cong} \clubsuit_{1,\frac{1}{6}}\ , \qquad \text{and} \qquad \Theta_{1,1}|\clubsuit_{1,\frac{1}{4}} :\ \clubsuit_{1,\frac{1}{4}} \xrightarrow{\cong} \clubsuit_{1,\frac{1}{5}}\ ,$$

which trivially are homeomorphisms.

<u>The induction step from n to $n + 1$:</u> Let for an $n \in \mathbb{N}$ for all $q \in \{0, 1, 2, \ldots, n\}$ the homeomorphisms $\Theta_{q,1}$ on Δ_q are constructed, $\Theta_{q,1}$ and $\Theta_{q,0}$ are elements of $\mathcal{COMFORT}(\Delta_q)$, and let four at a time are used to satisfy the equations $\texttt{EQUATION}_{q,j\leq p,i,k}$, for all $j, p \in \{0, 1, \ldots, q\}$, with $j \leq p$, and $i, k \in \{0, 1\}$. Furthermore, if we restrict $\Theta_{q,0}$ and $\Theta_{q,1}$ to the $\frac{1}{2\cdot(q+1)}$-cross of Δ_q, we get a homeomorphism $\Theta_{q,1} : \clubsuit_{q,\frac{1}{2\cdot(q+1)}} \xrightarrow{\cong} \clubsuit_{q,\frac{1}{2\cdot(q+1)+1}}$, (by this assumption of the induction), and also a homeomorphism $\Theta_{q,0} : \clubsuit_{q,\frac{1}{2\cdot(q+1)}} \xrightarrow{\cong} \clubsuit_{q,\frac{1}{2\cdot(q+2)}}$ (by the construction of $\Theta_{q,0}$, see Lemma (14)).

That means for $q = n$ and for positions $j = p = 0$, and $i = 0$, $k = 1$, we assume by the induction hypothesis that the following equation $\texttt{EQUATION}_{n,j=0\leq p=0,i=0,k=1}$ holds, i.e. we assume that the diagram (Figure 19) commutes,

$$\langle id \rangle_{n+1,\,0,\,0} \ \circ\ \Theta_{n,0} \ \circ\ \langle id \rangle_{n,\,1,\,0} \ \circ\ \Theta_{n-1,1} \quad = \quad \langle id \rangle_{n+1,\,1,\,1} \ \circ\ \Theta_{n,1} \ \circ\ \langle id \rangle_{n,\,0,\,0} \ \circ\ \Theta_{n-1,0}\ .$$

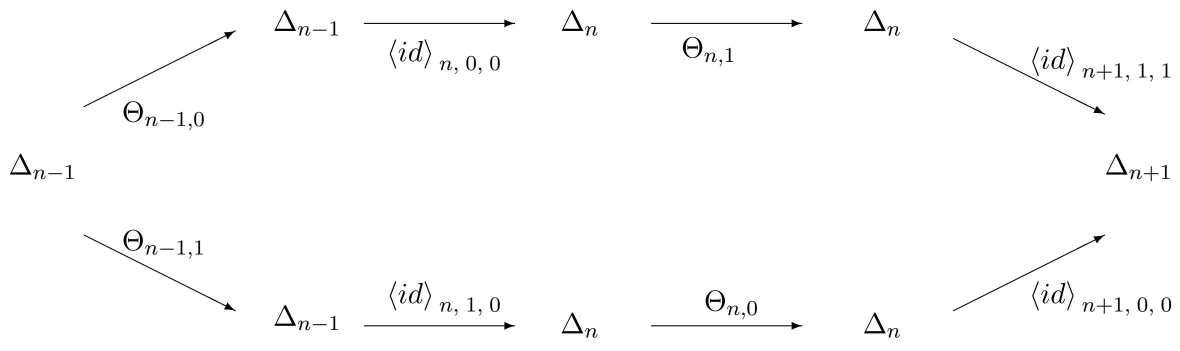

Figure 19:

We are going to make the induction step $n \mapsto n+1$. We fix positions $j = p = 0$, and let $i = 0$, $k = 1$. We want to show $\texttt{EQUATION}_{n+1,j=0\leq p=0,i=0,k=1}$. This means that we will show the commutativity of Figure 20.

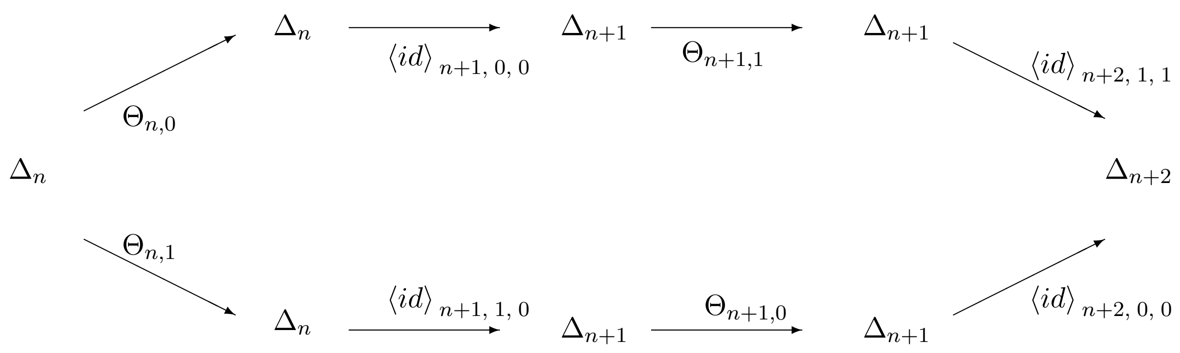

Figure 20:

The homeomorphism $\Theta_{n,1} \in \mathcal{COMFORT}(\Delta_n)$ already exists by the assumption of the induction. And we already had defined $\Theta_{n,0}$ and $\Theta_{n+1,0} = \mathbf{\Psi_{n+1}}(\phi_{n+1,0}) \in \mathcal{COMFORT}(\Delta_{n+1})$. Hence it lacks $\Theta_{n+1,1}$. Note that $\Theta_{n+1,0}$ yields a homeomorphism (see Definition (15)),

$$\Theta_{n+1,0|\clubsuit_{n+1,\frac{1}{2\cdot(n+2)}}} \;:\; \clubsuit_{n+1,\frac{1}{2\cdot(n+2)}} \xrightarrow{\cong} \clubsuit_{n+1,\frac{1}{2\cdot(n+3)}} \;,$$

as it was mentioned in Lemma (14).
We show the construction of $\Theta_{n+1,1}$ similarly as we just have described it for $n = 2$. Let $\vec{x}$ be an arbitrary element of Δ_n. We name the images of $\vec{x}$ by $\Theta_{n,0}(\vec{x}) =: \vec{y}$, and $\Theta_{n,1}(\vec{x}) =: \vec{z}$. Hence $\langle id \rangle_{n+1,\,0,\,0}\,(\vec{y}) = (0, \vec{y})$ and $\langle id \rangle_{n+1,\,1,\,0}\,(\vec{z}) = \left(\frac{1}{2\cdot(n+2)}, \left(1 - \frac{1}{2\cdot(n+2)}\right) \cdot \vec{z}\right) \in \Delta_{n+1}$. We call $\Theta_{n+1,1}(0, \vec{y}) =: (0, \vec{v})$ and $\Theta_{n+1,0}\left(\frac{1}{2\cdot(n+2)}, \left(1 - \frac{1}{2\cdot(n+2)}\right) \cdot \vec{z}\right) =: \left(\frac{1}{2\cdot(n+3)}, \vec{w}\right)$. Hence the bottom path of the above diagram (Figure 20) is

$$\langle id \rangle_{n+2,\,0,\,0} \circ \Theta_{n+1,0} \circ \langle id \rangle_{n+1,\,1,\,0} \circ \Theta_{n,1}\,(\vec{x}) = \left(0, \frac{1}{2\cdot(n+3)}, \vec{w}\right) \in \Delta_{n+2}\;.$$

The upper path of the same diagram (Figure 20) is

$$\langle id \rangle_{n+2,\,1,\,1} \circ \Theta_{n+1,1} \circ \langle id \rangle_{n+1,\,0,\,0} \circ \Theta_{n,0}\ (\vec{x}) = \langle id \rangle_{n+2,\,1,\,1}\ (0, \vec{v})$$
$$= \left(0, \tfrac{1}{2\cdot(n+3)}, \left(1 - \tfrac{1}{2\cdot(n+3)}\right) \cdot \vec{v}\right) \in \Delta_{n+2}\ .$$

Because we need the commutativity of the above diagram (Figure 20), we must define $\Theta_{n+1,1}$ such that $\vec{w} = \left(1 - \frac{1}{2\cdot(n+3)}\right) \cdot \vec{v}$. We need to enforce the commutativity of Figure 21.

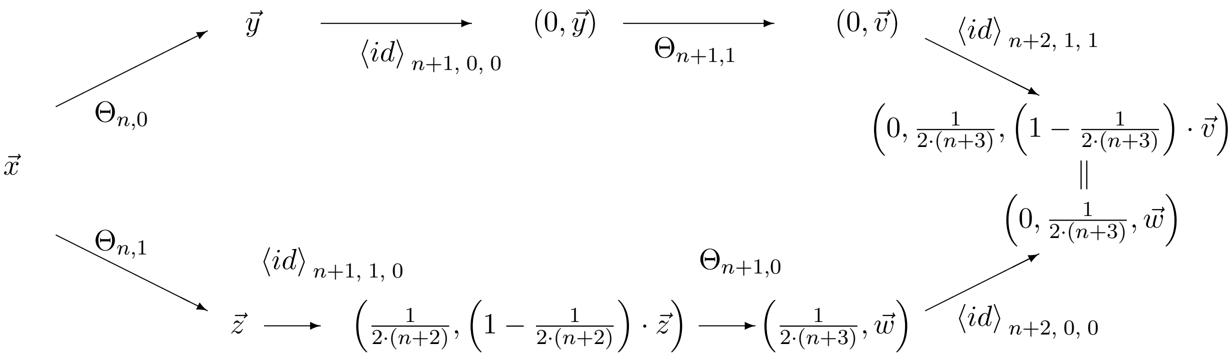

Figure 21:

We have to define the $(n+1)$-tuple $\vec{v} \in \Delta_n$. Let $\vec{v} := \left(1 - \frac{1}{2\cdot(n+3)}\right)^{-1} \cdot \vec{w}$, but we describe details.

To force the commutativity we reverse the directions of $\Theta_{n,0}$, $\langle id \rangle_{n+1,\,0,\,0}$, and $\langle id \rangle_{n+2,\,1,\,1}$, respectively, in a way we have described it after Figure 15 in the case $n = 2$.

Since $\Theta_{n,0}$ is a homeomorphism the meaning of $\Theta_{n,0}^{-1}$ is clear. It holds $\Theta_{n,0}^{-1}(\vec{y}) = (\vec{x})$. We define the left inverse maps $\langle id \rangle^{-1}_{n+1,\,0,\,0}$ and $\langle id \rangle^{-1}_{n+2,\,1,\,1}$. The map $\langle id \rangle^{-1}_{n+1,\,0,\,0}$ will map certain subsets of Δ_{n+1} homeomorphicly onto Δ_n, and $\langle id \rangle^{-1}_{n+2,\,1,\,1}$ will map some subsets of Δ_{n+2} homeomorphicly onto Δ_{n+1}. Let $\langle id \rangle^{-1}_{n+1,\,0,\,0}\,(0, \vec{y}) := (\vec{y}) \in \Delta_n$.

$\langle id \rangle^{-1}_{n+2,\,1,\,1}$ acts on any element of the form $\left(a, \frac{1}{2\cdot(n+3)}, b, c\right) \in \Delta_{n+2}$ by deleting the fraction $\frac{1}{2\cdot(n+3)}$ at the second place and multiplying the other components with $\left(1 - \frac{1}{2\cdot(n+3)}\right)^{-1}$. For any $(n+1)$-tuple $\vec{w}$ such that $\left(0, \frac{1}{2\cdot(n+3)}, \vec{w}\right) \in \Delta_{n+2}$ we define

$$\langle id \rangle^{-1}_{n+2,\,1,\,1}\ \left(0, \frac{1}{2\cdot(n+3)}, \vec{w}\right) := \left(0, \left(1 - \frac{1}{2\cdot(n+3)}\right)^{-1} \cdot \vec{w}\right) \in \Delta_{n+1}\ .$$

We get the identities
$\langle id \rangle^{-1}_{n+1,\,0,\,0} \circ \langle id \rangle_{n+1,\,0,\,0} = id(\Delta_n)$, and $\langle id \rangle^{-1}_{n+2,\,1,\,1} \circ \langle id \rangle_{n+2,\,1,\,1} = id(\Delta_{n+1})$, and

$$\langle id \rangle_{n+2,\,1,\,1} \circ \langle id \rangle^{-1}_{n+2,\,1,\,1}\ \left(0, \frac{1}{2\cdot(n+3)}, \vec{w}\right) = \left(0, \frac{1}{2\cdot(n+3)}, \vec{w}\right)\ .$$

Hence, at first we define $\Theta_{n+1,1}$ on the face $\mathbf{Bound}_{n+1} \cap \mathbf{Section}_{n+1,0}$:

$$\Theta_{n+1,1} := \langle id \rangle^{-1}_{n+2,\,1,\,1} \circ \langle id \rangle_{n+2,\,0,\,0} \circ \Theta_{n+1,0} \circ \langle id \rangle_{n+1,\,1,\,0} \circ \Theta_{n,1} \circ \Theta_{n,0}^{-1} \circ \langle id \rangle^{-1}_{n+1,\,0,\,0}\ ,$$

$$i.e. \qquad \Theta_{n+1,1}(0,\vec{y}) \ := \ \left(0, \left(1 - \frac{1}{2\cdot(n+3)}\right)^{-1} \cdot \vec{w}\right) .$$

We obtain the commutativity of Figure 22, see the following diagram. There we start at $(0,\vec{y})$.

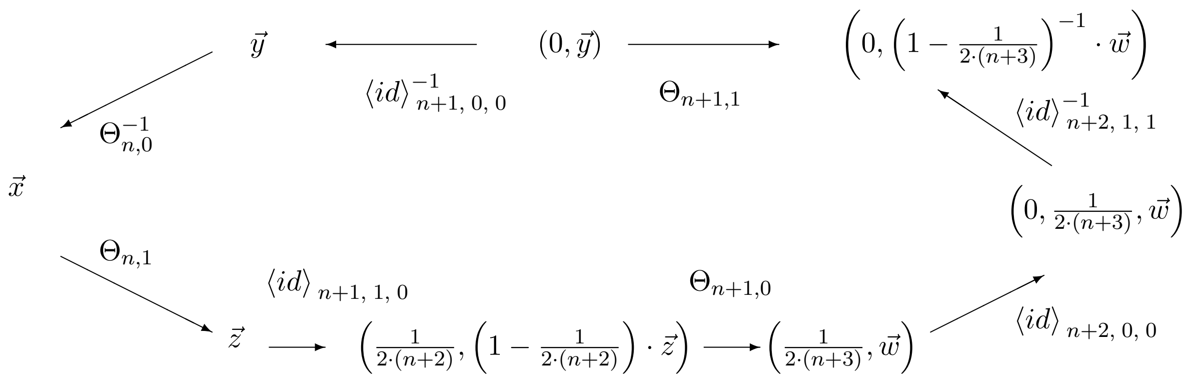

Figure 22:

Lemma 16. *The just constructed map* $\Theta_{n+1,1}|_{\mathbf{Bound}_{n+1}\cap\mathbf{Section}_{n+1,0}}$ *is a homeomorphism on* $\mathbf{Bound}_{n+1} \cap \mathbf{Section}_{n+1,0}$. *Further, it satisfies the conditions* $\widehat{[\mathbf{2}]}$ *(respecting permutations) and* $\widehat{[\mathbf{3}]}$ *(keeping the order) of Definition* (14)*.*

Proof.

- Continuity: The map $\Theta_{n+1,1}|_{\mathbf{Bound}_{n+1}\cap\mathbf{Section}_{n+1,0}}$ is a product of seven continuous maps.
- $\Theta_{n+1,1}|_{\mathbf{Bound}_{n+1}\cap\mathbf{Section}_{n+1,0}}$ is injective, because all seven maps are injective.
- $\Theta_{n+1,1}|_{\mathbf{Bound}_{n+1}\cap\mathbf{Section}_{n+1,0}}$ is surjective on $\mathbf{Bound}_{n+1} \cap \mathbf{Section}_{n+1,0}$.
- $\widehat{[\mathbf{2}]}$ and $\widehat{[\mathbf{3}]}$ are easy to verify. See Figure 22, and note that all maps $\Theta_{n,0}, \Theta_{n,1}, \Theta_{n+1,0}$ are from $\mathcal{COMFORT}(\Delta_n)$ or $\mathcal{COMFORT}(\Delta_{n+1})$, respectively. Note Lemma (1).

□

So far the map $\Theta_{n+1,1}$ is defined on $\mathbf{Bound}_{n+1} \cap \mathbf{Section}_{n+1,0}$. We had fixed the positions $j = p = 0$. If we vary j, p and take them from the set $\{0, 1, \ldots, n, n+1\}$ with $j \le p$, we get $\Theta_{n+1,1}$ defined on the other faces $\mathbf{Bound}_{n+1} \cap \mathbf{Section}_{n+1,k}$, for $1 \le k \le n+1$. All maps $\Theta_{n,0}$, $\Theta_{n,1}$, $\langle id\rangle_{n+1,\,0,\,j}$, $\langle id\rangle_{n+1,\,1,\,p}$, $\Theta_{n+1,0}$, $\langle id\rangle_{n+2,\,0,\,j}$, $\langle id\rangle_{n+2,\,1,\,p+1}$ respect permutations, see Lemma (1), hence there are no contradictions in the definition of $\Theta_{n+1,1}$ on $\mathbf{Bound}_{n+1}$, the boundary of Δ_{n+1}.

As in the case $n = 2$ we use Proposition (2). Before we can use Proposition (2) we have to check whether all the conditions of this proposition are fulfilled.

Lemma 17. *The constructed map* $\Theta_{n+1,1}|_{\mathbf{Bound}_{n+1}}$ *is a homeomorphism on* $\mathbf{Bound}_{n+1}$*. Further, it satisfies the conditions* $\widehat{[\mathbf{2}]}$ *(respecting permutations) and* $\widehat{[\mathbf{3}]}$ *(keeping the order). If we restrict* $\Theta_{n+1,1}|_{\mathbf{Bound}_{n+1}}$ *to the* $\frac{1}{2\cdot(n+2)}$*-cross* $\clubsuit_{n+1,\frac{1}{2\cdot(n+2)}}$*, we get a homeomorphism*

$$\Theta_{n+1,1}|_{\mathbf{Bound}_{n+1}} : \ \mathbf{Bound}_{n+1} \cap \clubsuit_{n+1,\frac{1}{2\cdot(n+2)}} \ \stackrel{\cong}{\longrightarrow} \ \mathbf{Bound}_{n+1} \cap \clubsuit_{n+1,\frac{1}{2\cdot(n+2)+1}} \ .$$

Proof. The fact that we have a homeomorphism $\Theta_{n+1,1}|_{\mathbf{Bound}_{n+1}} : \mathbf{Bound}_{n+1} \xrightarrow{\cong} \mathbf{Bound}_{n+1}$, and the properties $\widehat{[\mathbf{2}]}$ and $\widehat{[\mathbf{3}]}$ just have been discussed in and after Lemma (16).

The last claim means that if $\Theta_{n+1,1}(y_0, y_1, \ldots, y_n, y_{n+1}) = (z_0, z_1, \ldots, z_n, z_{n+1})$, for $(y_0, y_1, \ldots, y_{n+1}) \in \mathbf{Bound}_{n+1}$, and if $y_j = \frac{1}{2\cdot(n+2)}$ for any $j \in \{0, 1, \ldots, n, n+1\}$, it follows $z_j = \frac{1}{2\cdot(n+2)+1}$. This can be shown by using the definition of $\Theta_{n+1,1}$ on $\mathbf{Bound}_{n+1}$. Please look at Figure 22. We have to note that, if we restrict the following three maps to the corresponding 'crosses', we have three homeomorphisms

$$\Theta_{n,0}^{-1} : \clubsuit_{n,\frac{1}{2\cdot(n+2)}} \xrightarrow{\cong} \clubsuit_{n,\frac{1}{2\cdot(n+1)}}, \qquad \Theta_{n,1} : \clubsuit_{n,\frac{1}{2\cdot(n+1)}} \xrightarrow{\cong} \clubsuit_{n,\frac{1}{2\cdot(n+1)+1}}, \qquad \text{and}$$
$$\Theta_{n+1,0} : \clubsuit_{n+1,\frac{1}{2\cdot(n+2)}} \xrightarrow{\cong} \clubsuit_{n+1,\frac{1}{2\cdot(n+3)}} .$$

□

Now we are prepared to use Proposition (2). Let $\alpha := \frac{1}{2\cdot(n+2)}$ and $\beta := \frac{1}{2\cdot(n+2)+1}$. Hence by Proposition (2), the constructed map $\Theta_{n+1,1}|_{\mathbf{Bound}_{n+1}}$ can be extended to a homeomorphism $\Theta_{n+1,1}$ on Δ_{n+1}, even $\Theta_{n+1,1} \in \mathcal{COMFORT}(\Delta_{n+1})$, with the property

$$\Theta_{n+1,1}|_{\clubsuit_{n+1,\frac{1}{2\cdot(n+2)}}} : \clubsuit_{n+1,\frac{1}{2\cdot(n+2)}} \xrightarrow{\cong} \clubsuit_{n+1,\frac{1}{2\cdot(n+2)+1}} .$$

By the construction of $\Theta_{n+1,1}|_{\mathbf{Bound}_{n+1}}$, the equations $\texttt{EQUATION}_{n+1,j\le p,i,k}$ are satisfied for $(i,k) \in \{(0,1),(1,0)\}$. For $i = k = 0$, $\texttt{EQUATION}_{n+1,j\le p,0,0}$ is trivial. It remains to consider the case $i = k = 1$, i.e. we want to show $\texttt{EQUATION}_{n+1,j\le p,1,1}$. With the homeomorphism $\Theta_{n+1,1} : \clubsuit_{n+1,\frac{1}{2\cdot(n+2)}} \xrightarrow{\cong} \clubsuit_{n+1,\frac{1}{2\cdot(n+2)+1}}$ it is easy to see that the following Figure 23

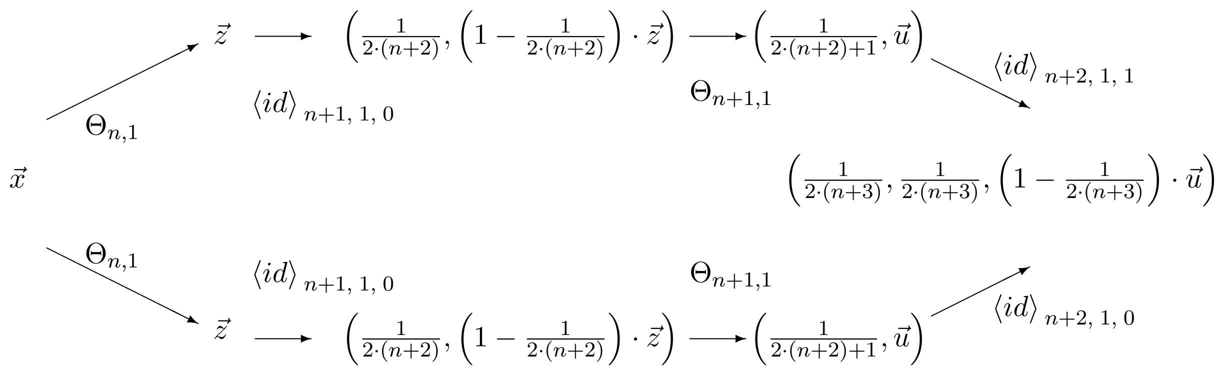

Figure 23:

commutes, with a suitable $(n+1)$-tuple $\vec{u}$. Hence we have confirmed that also the identity $\texttt{EQUATION}_{n+1,j=0\le p=0,i=1,k=1}$ holds. The other cases of $\texttt{EQUATION}_{n+1,j\le p,1,1}$ work correspondingly, for positions $j, p \in \{0, 1, 2, \ldots, n, n+1\}$ with $j \le p$.

Now all cases of the equations $\texttt{EQUATION}_{n+1,j\le p,i,k}$ are proven, for $i, k \in \{0, 1\}$, and $j, p \in \{0, 1, 2, \ldots, n, n+1\}$ with $j \le p$.

As a brief summary, we have fixed $i = 0, k = 1$, and we have varied $j, p \in \{0, 1, \ldots, n, n+1\}$ with $j \le p$. Thereby we were able to define the homeomorphism $\Theta_{n+1,1}$ on the boundary $\mathbf{Bound}_{n+1}$ without contradictions. After that we could extend $\Theta_{n+1,1}$ on the entire Δ_{n+1} by Proposition (2). The map $\Theta_{n+1,1}$ has all properties demanded in Theorem (2), and we have confirmed all equations $\texttt{EQUATION}_{n+1,j\le p,i,k}$.

Thus we have done the induction step from n to $n+1$, and the proof of Theorem (2) is finished. □

With Theorem (2) we have completed the proof of Theorem (1), i.e. we have confirmed all equations $\texttt{EQUATION}_{n+1,j\le p,i,k}$ which are assumed in Theorem (1). This means that we have proven the identity ${}_{\vec m}\partial_{n-1} \circ {}_{\vec m}\partial_n(T) = 0$ for an arbitrary $T \in \mathcal{C}_n(X)$ in the case of $L = 1$!

6 The Homology Modules of a Point

As we announced in the introduction, we have developed a sequence $(\mathcal{H}_n)_{n\ge 0}$ of functors $\mathcal{H}_n : \mathsf{TOP} \longrightarrow \mathsf{AB}$ in the case $L = 1$. But for a complete 'extraordinary homology theory' still it lacks the *Excision Axiom* and the *Homotopy Axiom.* Therefore we are only able to compute the homology groups for the one-point space.

The reader should recall the definitions of $\mathcal{C}_n(X)$ and of $\mathcal{F}(\mathcal{R})_n(X)$ from the introduction of this paper. Let us assume that for a fixed $L \in \mathbb{N}_0$, for all $\vec m \in \mathcal{R}^{L+1}$ and $n \in \mathbb{N}$ we have proved ${}_{\vec m}\partial_{n-1} \circ {}_{\vec m}\partial_n(T) = 0$ for all basis elements $T \in \mathcal{C}_n(X)$, as we just have done it in the case $L = 1$. Let us take an arbitrary chain $\mathtt{u} \in \mathcal{F}(\mathcal{R})_n(X)$, i.e. $\mathtt{u}$ is a $\mathcal{R}$-linear combination of some T's from the set $\mathcal{C}_n(X)$. We can extend the boundary operators ${}_{\vec m}\partial_n$ from $\mathcal{C}_n(X)$ to $\mathcal{F}(\mathcal{R})_n(X)$ by linearity, hence we obtain for each chain $\mathtt{u}$ the equation ${}_{\vec m}\partial_{n-1} \circ {}_{\vec m}\partial_n(\mathtt{u}) = 0$. This holds for all $n \in \mathbb{N}$, and for the chain complex

$$\cdots \xrightarrow{{}_{\vec m}\partial_{n+1}} \mathcal{F}(\mathcal{R})_n(X) \xrightarrow{{}_{\vec m}\partial_n} \mathcal{F}(\mathcal{R})_{n-1}(X) \xrightarrow{{}_{\vec m}\partial_{n-1}} \cdots \xrightarrow{{}_{\vec m}\partial_2} \mathcal{F}(\mathcal{R})_1(X) \xrightarrow{{}_{\vec m}\partial_1} \mathcal{F}(\mathcal{R})_0(X) \xrightarrow{{}_{\vec m}\partial_0} \{0\}$$

we can deduce that $\mathrm{image}({}_{\vec m}\partial_n)$ is a submodule of $\mathrm{kernel}({}_{\vec m}\partial_{n-1})$, hence the $\mathcal{R}$-module

$${}_{\vec m}\mathcal{H}_n(X) := \frac{kernel({}_{\vec m}\partial_n)}{image({}_{\vec m}\partial_{n+1})}$$

is well defined for this L and all tuples $\vec m = (m_0, m_1, \ldots, m_L) \in \mathcal{R}^{L+1}$, for every topological space X and all $n \in \mathbb{N}_0$.

Let $\mathcal{R} := \mathbb{Z}$. For the one-point space $\{p\}$ and for $n \in \mathbb{N}_0$ there is only one $T : \Delta_n \to \{p\}$, thus it holds $\mathcal{F}(\mathbb{Z})_n(p) \cong \mathbb{Z}$. And for the generated chain complex

$$\cdots \xrightarrow{{}_{\vec m}\partial_{n+1}} \mathcal{F}(\mathbb{Z})_n(p) \xrightarrow{{}_{\vec m}\partial_n} \mathcal{F}(\mathbb{Z})_{n-1}(p) \xrightarrow{{}_{\vec m}\partial_{n-1}} \cdots \xrightarrow{{}_{\vec m}\partial_2} \mathcal{F}(\mathbb{Z})_1(p) \xrightarrow{{}_{\vec m}\partial_1} \mathcal{F}(\mathbb{Z})_0(p) \xrightarrow{{}_{\vec m}\partial_0} \{0\}$$

we get

$$\cdots\cdots \xrightarrow{{}_{\vec m}\partial_{n+1}} \mathbb{Z} \xrightarrow{{}_{\vec m}\partial_n} \mathbb{Z} \xrightarrow{{}_{\vec m}\partial_{n-1}} \cdots\cdots \xrightarrow{{}_{\vec m}\partial_3} \mathbb{Z} \xrightarrow{{}_{\vec m}\partial_2} \mathbb{Z} \xrightarrow{{}_{\vec m}\partial_1} \mathbb{Z} \xrightarrow{{}_{\vec m}\partial_0} \{0\}.$$

We abbreviate the integer $\sigma := \sum_{i=0}^{L} m_i$. If we define the map $\times\sigma : \mathbb{Z} \to \mathbb{Z}, x \mapsto \sigma \cdot x$, we can describe the boundary operators by

$${}_{\vec m}\partial_n \cong \begin{cases} 0 & \text{if } n \text{ is odd, or } \; n = 0 \\ \times\sigma & \text{if } n \text{ is even, but } n \neq 0\,. \end{cases}$$

Explanation: We had defined ${}_{\vec m}\partial_n(T) = \sum_{j=0}^{n}(-1)^j \cdot \sum_{i=0}^{L} m_i \cdot \Big(\langle T\rangle_{\,L,\,n,\,i,\,j} \circ \Theta_{L,n-1,i}\Big)$. This means in the special case of a one-point space $\{p\}$ that σ of the unique map from Δ_{n-1} to $\{p\}$ cancel pairwise because of the alternating signs. It follows for $\sigma \neq 0$:

$${}_{\vec m}\mathcal{H}_n(p) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}_{/(\sigma\cdot\mathbb{Z})} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even and } n \neq 0 \;, \end{cases}$$

and for $\sigma = 0$ we get ${}_{\vec{m}}\mathcal{H}_n(p) \cong \mathbb{Z}$ for all $n \in \mathbb{N}_0$.

Remark:
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IXth CONGRESO IBEROAMERICANO de TOPOLOGIA y sus APLICACIONES

which took place in Almeria, Spain, in June 2014.

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