ON FUZZY UPPER AND LOWER CONTRA-CONTINUOUS MULTIFUNCTIONS

M. Alimohammady, E. Ekici*, S. Jafari and M. Roohi

February 18, 2009

Abstract

This paper is devoted to the concepts of fuzzy upper and fuzzy lower contra-continuous multifunctions and also some characterizations of them are considered.
2000 Mathematics Subject Classifications: 03E72, 54A40, 54C60.
Keywords: fuzzy topological space, fuzzy multifunctions, fuzzy lower contra-continuous multifunction, fuzzy upper contra-continuous multifunction.

1 Introduction

In the last three decades, the theory of multifunctions has advanced in a variety of ways and applications of this theory can be found, specially in functional analysis and fixed point theory [5, 23, 24] etc. The initiation of fuzzy multifunctions is due to Papageorgiou [20]. He studied upper and lower semi-continuous multifunctions. Mukherjee and Malakar [15] have studied fuzzy multifunctions with q-coincidence. Recently many authors for example Albrycht and Maltoka, Nouh and El-Shafei [1, 17] and Beg [3, 4] have studied fuzzy multifunctions and have characterized, some properties of fuzzy multifunctions defined on a fuzzy topological space. Several authors have studied different types of fuzzy continuity for fuzzy multifunctions, for example see [2, 9, 20, 21] and also for more details on fuzzy multifunctions one can see [4]. On the other hand, Dontchev [8] introduced the notion of contra-continuous functions. It is shown in [8] that contra-continuous images of strongly S-closed spaces are compact. Joseph and Kwack [14] introduced another form of contra-continuity called (θ, s)-continuous functions in order to investigate S-closed spaces due to Thompson [25]. In recent years, several authors have studied some new forms of contra-continuity for functions and multifunctions, for example see [6, 11, 12, 13, 16]. In the present paper, we study the notions of fuzzy upper and fuzzy lower contra-continuous multifunctions. Also, some characterizations and properties of such notions are discussed.

*Corresponding Author: E. Ekici

2 Preliminaries

The class of all fuzzy sets on a universe Y will be denoted by I^Y and fuzzy sets on Y will be denoted by μ, η, etc. A family τ of fuzzy sets in Y is called a *fuzzy topology* for Y [7] if

(1) $\emptyset, Y \in \tau$,

(2) $\mu \wedge \eta \in \tau$ whenever $\mu, \eta \in \tau$,

(3) If $\mu_i \in \tau$ for each $i \in I$, then $\vee \mu_i \in \tau$.

The pair (Y, σ) is called a *fuzzy topological space.* Every member of σ is called a *fuzzy open set.* A fuzzy set in Y is called a *fuzzy point* if it takes the value 0 for all $y \in Y$ except one, say, $x \in Y$. If its value at x is ε $(0 < \varepsilon \leq 1)$, we denote this fuzzy point by x_ε, where the point x is called its *support* [18, 19]. For any fuzzy point x_ε and any fuzzy set μ, $x_\varepsilon \in \mu$ if and only if $\varepsilon \leq \mu(x)$. A fuzzy point x_ε is called *quasi-coincident* with a fuzzy set η, denoted by $x_\varepsilon q \eta$, if $\varepsilon + \eta(x) > 1$. A fuzzy set μ is called *quasi-coincident* with a fuzzy set η, denoted by $\mu q \eta$, if there exists a $x \in Y$ such that $\mu(x) + \eta(x) > 1$ [18, 19]. When they are not quasi-coincident, it will be denoted by $\mu \overline{q} \eta$.

Throughout this paper, (X, τ) or simply X will stand for ordinary topological space and (Y, σ) or simply Y will be denoted a fuzzy topological space.

Let X and Y be a topological space in the classical sense and a fuzzy topological space, respectively. $F : X \multimap Y$ is called a *fuzzy multifunction* [20] if for each $x \in X$, $F(x)$ is a fuzzy set in Y. Throughout the paper, by $F : X \multimap Y$ we will mean that F is a fuzzy multifunction from a classical topological space X to a fuzzy topological space Y. For a fuzzy multifunction $F : X \multimap Y$, the upper inverse $F^+(\mu)$ and lower inverse $F^-(\mu)$ of a fuzzy set μ in Y are defined as follows: $F^+(\mu) = \{x \in X : F(x) \leq \mu\}$ and $F^-(\mu) = \{x \in X : F(x) q \mu\}$. For any fuzzy set μ in Y, we have $F^-(1 - \mu) = X - F^+(\mu)$ [15]. We denote the interior and the closure of a subset A of a topological space X by $Int(A)$ and $Cl(A)$, respectively.

3 Fuzzy upper and lower contra-continuous multifunctions

Definition 1 *A fuzzy multifunction $F : X \multimap Y$ is called fuzzy lower contra-continuous multifunction if for each fuzzy closed set μ in Y with $x \in F^-(\mu)$, there exists an open set B in X containing x such that $B \subset F^-(\mu)$.*

Definition 2 *A fuzzy multifunction $F : X \multimap Y$ is called fuzzy upper contra-continuous multifunction if for each fuzzy closed set μ in Y with $x \in F^+(\mu)$, there exists an open set B in X containing x such that $B \subset F^+(\mu)$.*

Theorem 3 *The following are equivalent for a fuzzy multifunction $F : X \multimap Y$:*

(1) F is fuzzy upper contra-continuous,

(2) For each fuzzy closed set μ and $x \in X$ such that $F(x) \leq \mu$, there exists an open set B containing x such that if $y \in B$, then $F(y) \leq \mu$,

(3) $F^+(\mu)$ is open for any fuzzy closed set μ in Y,
(4) $F^-(\rho)$ is closed for any fuzzy open set ρ in Y.

Proof. (1) $\Leftrightarrow$ (2) : Obvious.

(1) $\Rightarrow$ (3) : Let μ be any fuzzy closed set in Y and $x \in F^+(\mu)$. By (1), there exists an open set A_x containing x such that $A_x \subset F^+(\mu)$. Thus, $x \in Int(F^+(\mu))$ and hence $F^+(\mu)$ is an open set in X.

(3) $\Rightarrow$ (4) : Let ρ be a fuzzy open set in Y. Then $Y\backslash\rho$ is a fuzzy closed set in Y. By (3), $F^+(Y\backslash\rho)$ is open. Since $F^+(Y\backslash\rho) = X\backslash F^-(\rho)$, then $F^-(\rho)$ is closed in X.

(4) $\Rightarrow$ (3) : It is similar to that of (3) $\Rightarrow$ (4).

(3) $\Rightarrow$ (1) : Let ρ be any fuzzy closed set in Y and $x \in F^+(\rho)$. By (3), $F^+(\rho)$ is an open set in X. Take $A = F^+(\rho)$. Then, $A \subset F^+(\rho)$. Thus, F is fuzzy upper contra-continuous. ■

Definition 4 *The set $\wedge\{\rho \in \tau : \mu \leq \rho\}$ is called the fuzzy kernel of a fuzzy set μ in a fuzzy topological space (X, τ) and is denoted by $Ker(\mu)$.*

Lemma 5 *For fuzzy set μ in a fuzzy topological space (X, τ), if $\mu \in \tau$, then $\mu = Ker(\mu)$.*

Theorem 6 *Let $F : (X, \tau) \multimap (Y, \sigma)$ be a fuzzy multifunction. If $Cl(F^-(\mu)) \leq F^-(Ker(\mu))$ for every fuzzy set μ in Y, then F is fuzzy upper contra-continuous.*

Proof. Suppose that $Cl(F^-(\mu)) \leq F^-(Ker(\mu))$ for every fuzzy set μ in Y. Let $\rho \in \sigma$. By Lemma 5, $Cl(F^-(\rho) \leq F^-(Ker(\rho)) = F^-(\rho)$. This implies that $Cl((F^-(\rho)) = F^-(\rho)$ and hence $F^-(\rho)$ is closed in X. Thus, by Theorem 3, F is fuzzy upper contra-continuous. ■

Definition 7 *A fuzzy multifunction $F : X \multimap Y$ is called*

(1) fuzzy lower semi-continuous [15] if for each fuzzy open set μ in Y with $x \in F^-(\mu)$, there exists an open subset B of X containing x such that $B \subset F^-(\mu)$.

(2) fuzzy upper semi-continuous [15] if for each fuzzy open set μ in Y with $x \in F^+(\mu)$, there exists an open subset B of X containing x such that $B \subset F^+(\mu)$.

Remark 8 *The notions of fuzzy upper contra-continuous multifunctions and fuzzy upper semi-continuous multifunctions are independent as shown in the following examples.*

Example 9 *Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}\}$ and $Y = [0, 1]$, $\sigma = \{Y, \emptyset, \mu, \rho, \eta\}$, where $\mu(y) = 0,5$, $\rho(y) = 0,6$, $\eta(y) = 0,7$ for $y \in Y$. Define a fuzzy multifunction as follows: $F(a) = \mu$, $F(b) = \rho$, $F(c) = \eta$. Then the fuzzy multifunction $F : (X, \tau) \multimap (Y, \sigma)$ is fuzzy upper contra-continuous but it is not fuzzy upper semi-continuous.*

Example 10 *Let* $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{b, c\}\}$ *and* $Y = [0, 1]$, $\sigma = \{Y, \emptyset, \mu, \rho, \eta\}$, *where* $\mu(y) = 0,3$, $\rho(y) = 0,2$, $\eta(y) = 0,6$, $\gamma(y) = 0,4$, $\zeta(y) = 0,5$ *for* $y \in Y$. *Define a fuzzy multifunction as follows:* $F(a) = \gamma$, $F(b) = \zeta$, $F(c) = \eta$. *Then the fuzzy multifunction* $F : (X, \tau) \multimap (Y, \sigma)$ *is fuzzy upper semi-continuous but it is not fuzzy upper contra-continuous.*

Theorem 11 *The following are equivalent for a fuzzy multifunction* $F : X \multimap Y$*:*

(1) F *is fuzzy lower contra-continuous,*

(2) For each fuzzy closed set μ *and* $x \in X$ *such that* $F(x)q\mu$*, there exists an open set* B *containing* x *such that if* $y \in B$*, then* $F(y)q\mu$*,*

(3) $F^-(\mu)$ *is open for any fuzzy closed set* μ *in* Y*,*

(4) $F^+(\rho)$ *is closed for any fuzzy open set* ρ *in* Y*.*

Proof. It is similar to that of Theorem 3. ■

Theorem 12 *For a fuzzy multifunction* $F : (X, \tau) \multimap (Y, \sigma)$*, if* $Cl(F^+(\rho)) \leq F^+(Ker(\rho))$ *for every fuzzy set* ρ *in* Y*, then* F *is fuzzy lower contra-continuous.*

Proof. Suppose that $Cl(F^+(\rho)) \leq F^+(Ker(\rho))$ for every fuzzy set ρ in Y. Let $\rho \in \sigma$. We have $Cl(F^+(\rho) \leq F^+(Ker(\rho)) = F^+(\rho)$. Thus, $Cl((F^+(\rho)) = F^+(\rho)$ and hence $F^+(\rho)$ is closed in X. By Theorem 11, F is fuzzy lower contra-continuous. ■

Theorem 13 *If* $F_i : X \multimap Y$ *are fuzzy upper contra-continuous multifunctions for* $i = 1, 2, ..., n$*, then* $\vee_{i=1}^n F_i$ *is a fuzzy upper contra-continuous multifunction.*

Proof. Let μ be a fuzzy closed set of Y. We will show that $(\vee_{i=1}^n F_i)^+(\mu) = \{x \in X : \vee_{i=1}^n F_i(x) \leq \mu\}$ is open in X. Let $x \in (\vee_{i=1}^n F_i)^+(\mu)$. Then $F_i(x) \leq \mu$ for $i = 1, 2, ..., n$. Since $F_i : X \multimap Y$ is fuzzy upper contra-continuous multifunction for $i = 1, 2, ..., n$, then there exists an open set U_x containing x such that for all $z \in U_x$, $F_i(z) \leq \mu$. Let $U = \wedge_{i=1}^n U_x$. Then $U \subset (\vee_{i=1}^n F_i)^+(\mu)$. Thus, $(\vee_{i=1}^n F_i)^+(\mu)$ is open and hence $\vee_{i=1}^n F_i$ is a fuzzy upper contra-continuous multifunction. ■

Lemma 14 *([4]) Let* $\{\mu_i\}_{i \in I}$ *be a family of fuzzy sets in a fuzzy topological space* X*. Then a fuzzy point* x *is quasi-coincident with* $\vee \mu_i$ *if and only if there exists an* $i_0 \in I$ *such that* $xq\mu_{i_0}$*.*

Theorem 15 *If* $F_i : X \multimap Y$ *are fuzzy lower contra-continuous multifunctions for* $i = 1, 2, ..., n$*, then* $\vee_{i=1}^n F_i$ *is a fuzzy lower contra-continuous multifunction.*

Proof. Let μ be a fuzzy closed set of Y. We will show that $(\vee_{i=1}^n F_i)^-(\mu) = \{x \in X : (\vee_{i=1}^n F_i)(x)q\mu\}$ is open in X. Let $x \in (\vee_{i=1}^n F_i)^-(\mu)$. Then $(\vee_{i=1}^n F_i)(x)q\mu$ and hence $F_{i_0}(x)q\mu$ for an i_0. Since $F_{i_0} : X \multimap Y$ is fuzzy lower contra-continuous multifunction, then there exists an open set U_x containing x such that for all $z \in U$, $F_{i_0}(z)q\mu$. Then $(\vee_{i=1}^n F_i)(z)q\mu$ and hence $U \subset (\vee_{i=1}^n F_i)^-(\mu)$. Thus, $(\vee_{i=1}^n F_i)^-(\mu)$ is open and hence $\vee_{i=1}^n F_i$ is a fuzzy lower contra-continuous multifunction. ■

Theorem 16 *Let $F : X \multimap Y$ be a fuzzy multifunction and $\{U_i : i \in I\}$ be an open cover for X. Then the following are equivalent:*
(1) $F_i = F\mid_{U_i}$ is a fuzzy lower contra-continuous multifunction for all $i \in I$,
(2) F is fuzzy lower contra-continuous.

Proof. (1) $\Rightarrow$ (2) : Let $x \in X$ and μ be a fuzzy closed set in Y with $x \in F^-(\mu)$. Since $\{U_i : i \in I\}$ is an open cover for X, then $x \in U_{i_0}$ for an $i_0 \in I$. We have $F(x) = F_{i_0}(x)$ and hence $x \in F_{i_0}^-(\mu)$. Since $F\mid_{U_{i_0}}$ is fuzzy lower contra-continuous, then there exists an open set $B = G \cap U_{i_0}$ in U_{i_0} such that $x \in B$ and $F^-(\mu) \cap U_{i_0} = F\mid_{U_i}^-(\mu) \supset B = G \cap U_{i_0}$, where G is open in X. We have $x \in B = G \cap U_{i_0} \subset F\mid_{U_{i_0}}^-(\mu) = F^-(\mu) \cap U_{i_0} \subset F^-(\mu)$. Hence, F is fuzzy lower contra-continuous.

(2) $\Rightarrow$ (1) : Let $x \in X$ and $x \in U_i$. Let μ be a fuzzy closed set in Y with $F_i(x)q\mu$. Since F is lower contra-continuous and $F(x) = F_i(x)$, then there exists an open set U containing x such that $U \subset F^-(\mu)$. Take $B = U_i \cap U$. Then B is open in U_i containing x. We have $B \subset F_i^-(\mu)$. Thus, F_i is a fuzzy lower contra-continuous. ■

Theorem 17 *Let $F : X \multimap Y$ be a fuzzy multifunction and $\{U_i : i \in I\}$ be an open cover for X. Then the following are equivalent:*
(1) $F_i = F\mid_{U_i}$ is a fuzzy upper contra-continuous multifunction for all $i \in I$,
(2) F is fuzzy upper contra-continuous.

Proof. It is similar to that of Theorem 16. ■

Recall that for a multifunction $F_1 : X \multimap Y$ and a fuzzy multifunction $F_2 : Y \multimap Z$, the fuzzy multifunction $F_2 \circ F_1 : X \multimap Z$ is defined by $(F_2 \circ F_1)(x) = F_2(F_1(x))$ for $x \in X$.

Definition 18 *Let X and Y be topological spaces. A multifunction $F : X \multimap Y$ is called*

(1) lower semi-continuous [21] if for each open subset $A \subset Y$ with $x \in F^-(A)$, there exists an open set B in X containing x such that $B \subset F^-(A)$.

(2) upper semi-continuous [21] if for each open subset $A \subset Y$ with $x \in F^+(A)$, there exists an open set B in X containing x such that $B \subset F^+(A)$.

Theorem 19 *If $F_1 : X \multimap Y$ is an upper semi-continuous multifunction and $F_2 : Y \multimap Z$ is a fuzzy upper contra-continuous multifunction, then $F_2 \circ F_1$ is fuzzy upper contra-continuous.*

Proof. Let $x \in X$ and μ be a fuzzy closed set in Z. We have $(F_2 \circ F_1)^+(\mu) = F_1^+(F_2^+(\mu))$. Since F_2 is fuzzy upper contra-continuous, then $F_2^+(\mu)$ is open in Y. Since F_1 is upper semi-continuous, then $F_1^+(F_2^+(\mu)) = (F_2 \circ F_1)^+(\mu)$ is open in X. Thus, $F_2 \circ F_1$ is fuzzy upper contra-continuous. ■

Definition 20 *A fuzzy set μ in a fuzzy topological space X is called a fuzzy cl-neighbourhood of a fuzzy point x in X if there exists a fuzzy closed set ρ in X such that $x \in \rho \leq \mu$.*

Theorem 21 *If $F : X \multimap Y$ is a fuzzy upper contra-continuous multifunction, then for each point x of X and each fuzzy cl-neighbourhood μ of $F(x)$, $F^+(\mu)$ is a neighbourhood of x.*

Proof. Let $x \in X$ and μ be a fuzzy cl-neighbourhood of $F(x)$. There exists a fuzzy closed set ρ in Y such that $F(x) \leq \rho \leq \mu$. We have $x \in F^+(\rho) \leq F^+(\mu)$. Since $F^+(\rho)$ is an open set, $F^+(\mu)$ is a neighbourhood of x. ■

Remark 22 *([26]) A subset A of a topological space (X, τ) can be considered as a fuzzy set with characteristic function defined by*

$$A(x) = \begin{cases} 1 & , x \in A \\ 0 & , x \notin A \end{cases}$$

Let (Y, σ) be a fuzzy topological space. The fuzzy sets of the form $A \times \rho$ with $A \in \tau$ and $\rho \in \sigma$ form a basis for the product fuzzy topology $\tau \times \sigma$ on $X \times Y$, where for any $(x, y) \in X \times Y$,

$$(A \times \rho)(x, y) = min\{A(x), \rho(y)\}$$

Definition 23 *([15]) For a fuzzy multifunction $F : X \multimap Y$, the fuzzy graph multifunction $G_F : X \multimap X \times Y$ of F is defined by $G_F(x) = x_1 \times F(x)$ for every $x \in X$.*

Theorem 24 *If the fuzzy graph multifunction G_F of a fuzzy multifunction $F : X \multimap Y$ is fuzzy lower contra-continuous, then F is fuzzy lower contra-continuous.*

Proof. Suppose that G_F is fuzzy lower contra-continuous and $x \in X$. Let μ be a fuzzy closed set in Y such that $F(x)q\mu$. Then there exists $y \in Y$ such that $(F(x))(y)+\mu(y) > 1$. Then $(G_F(x))(x, y)+(X\times\mu)(x, y) = (F(x))(y)+\mu(y) > 1$. Hence, $G_F(x)q(X \times \mu)$. Since G_F is fuzzy lower contra-continuous, there exists an open set B in X such that $x \in B$ and $G_F(b)q(X \times \mu)$ for all $b \in B$.

Let there exists a $b_0 \in B$ such that $F(b_0)\overline{q}\mu$. Then for all $y \in Y$, $(F(b_0))(y)+\mu(y) \leq 1$. For any $(a, c) \in X\times Y$, we have $(G_F(b_0))(a, c) \leq (F(b_0))(c)$ and $(X\times\mu)(a, c) \leq \mu(c)$. Since for all $y \in Y$, $(F(b_0))(y)+\mu(y) \leq 1$, then $(G_F(b_0))(a, c)+$

$(X \times \mu)(a, c) \leq 1$. Thus, $G_F(b_0)\overline{q}(X \times \mu)$, where $b_0 \in B$. This is a contradiction since $G_F(b)q(X \times \mu)$ for all $b \in B$.

Hence, F is fuzzy lower contra-continuous. ■

Theorem 25 *If the fuzzy graph multifunction G_F of a fuzzy multifunction $F : X \multimap Y$ is fuzzy upper contra-continuous, then F is fuzzy upper contra-continuous.*

Proof. Suppose that G_F is fuzzy upper contra-continuous and $x \in X$. Let μ be fuzzy closed in Y with $F(x) \leq \mu$. Then $G_F(x) \leq X \times \mu$. Since G_F is fuzzy upper contra-continuous, there exists an open set B containing x such that $G_F(B) \leq X \times \mu$. For any $b \in B$ and $y \in Y$, we have $(F(b))(y) = (G_F(b))(b, y) \leq (X \times \mu)(b, y) = \mu(y)$. Then $(F(b))(y) \leq \mu(y)$ for all $y \in Y$. Thus, $F(b) \leq \mu$ for any $b \in B$. Hence, F is fuzzy upper contra-continuous. ■

Theorem 26 *Let $F : X \multimap Y$ be a fuzzy multifunction. Then the following are equivalent:*

(1) F is fuzzy lower contra-continuous,

(2) For each $x \in X$ and each net $(x_i)_{i \in I}$ converging to x in X and each fuzzy closed set ρ in Y with $x \in F^-(\rho)$, the net $(x_i)_{i \in I}$ is eventually in $F^-(\rho)$.

Proof. $(1) \Rightarrow (2)$: Let x_i be a net converging to x in X and ρ be any fuzzy closed set in Y with $x \in F^-(\rho)$. Since F is fuzzy lower contra-continuous, then there exists an open set $A \subset X$ containing x such that $A \subset F^-(\rho)$. Since $x_i \to x$, then there exists an index $i_0 \in I$ such that $x_i \in A$ for every $i \geq i_0$. We have $x_i \in A \subset F^-(\rho)$ for all $i \geq i_0$. Hence, $(x_i)_{i \in I}$ is eventually in $F^-(\rho)$.

$(2) \Rightarrow (1)$: Suppose that F is not fuzzy lower contra-continuous. There exists a point x and a fuzzy closed set μ containing x with $x \in F^-(\mu)$ such that $B \nsubseteq F^-(\mu)$ for each open set $B \subset X$ containing x. Let $x_i \in B$ and $x_i \notin F^-(\mu)$ for each open set $B \subset X$ containing x. Then the neighborhood net (x_i) converges to x but $(x_i)_{i \in I}$ is not eventually in $F^-(\mu)$. This is a contradiction. ■

Theorem 27 *Let $F : X \multimap Y$ be a fuzzy multifunction. Then the following are equivalent:*

(1) F is fuzzy upper contra-continuous,

(2) For each $x \in X$ and each net (x_i) converging to x in X and each fuzzy closed set ρ in Y with $x \in F^+(\rho)$, the net (x_i) is eventually in $F^+(\rho)$.

Proof. The proof is similar to that of Theorem 26. ■

Recall that the frontier of a subset A of a topological space X, denoted by $Fr(A)$, is defined by $Fr(A) = Cl(A) \cap Cl(X \backslash A) = Cl(A) \backslash Int(A)$.

Theorem 28 *The set all points of X at which a fuzzy multifunction $F : X \multimap Y$ is not fuzzy upper contra-continuous is identical with the union of the frontier of the upper inverse image of fuzzy closed sets containing $F(x)$.*

Proof. Suppose F is not fuzzy upper contra-continuous at $x \in X$. Then there exists a fuzzy closed set η in Y containing $F(x)$ such that $A \cap (X \backslash F^+(\eta)) \neq \emptyset$ for every open set A containing x. We have $x \in Cl(X \backslash F^+(\eta)) = X \backslash Int(F^+(\eta))$ and $x \in F^+(\eta)$. Thus, $x \in Fr(F^+(\eta))$.

Conversely, let η be a fuzzy closed set in Y containing $F(x)$ with $x \in Fr(F^+(\eta))$. Suppose that F is fuzzy upper contra-continuous at x. There exists an open set A containing x such that $A \subset F^+(\eta)$. We have $x \in Int(F^+(\eta))$. This is a contradiction. Thus, F is not fuzzy upper contra-continuous at x. ■

Theorem 29 *The set all points of X at which a fuzzy multifunction $F : X \multimap Y$ is not fuzzy lower contra-continuous is identical with the union of the frontier of the lower inverse image of fuzzy closed sets which are quasi-coincident with $F(x)$.*

Proof. It is similar to that of Theorem 28. ■

Theorem 30 *If $F : X \multimap Y$ is a fuzzy upper contra-continuous point closed multifunction and $F(x) \wedge F(y) = \emptyset$ for each distinct pair $x, y \in X$, then X is a T_2 space.*

Proof. Let x and y be any two distinct points in X. We have $F(x) \wedge F(y) = \emptyset$. Since F is fuzzy upper contra-continuous and point closed, $F^+(F(x))$ and $F^+(F(y))$ are disjoint fuzzy open sets containing x and y, respectively. Hence, X is T_2. ■

Definition 31 *A fuzzy topological space X is called fuzzy strongly S-closed [2] if every fuzzy closed cover of X has a finite subcover.*

Theorem 32 *Let $F : X \multimap Y$ be a fuzzy upper contra-continuous surjective multifunction. Suppose that $F(x)$ is fuzzy strongly S-closed for each $x \in X$. If X is compact, then Y is fuzzy strongly S-closed.*

Proof. Let $\{\mu_k\}_{k \in I}$ be a fuzzy closed cover of Y. Since $F(x)$ is fuzzy strongly S-closed for each $x \in X$, there exists a finite subset I_x of I such that $F(x) \leq \vee_{k \in I_x} \mu_k$. Take $\mu_x = \vee_{k \in I_x} \mu_k$. Since F is fuzzy upper contra-continuous, there exists a fuzzy open set U_x of X containing x such that $F(U_x) \leq \mu_x$. Then $\{U_x\}_{x \in X}$ is an open cover of X. Since X is compact, there exist x_1, x_2, x_3, ...,x_n in X such that $X = \vee_{i=1}^n U_{x_i}$. We have $Y = F(X) = F(\bigcup_{i=1}^n U_{x_i}) \leq \bigvee_{i=1}^n F(U_{x_i}) \leq \bigvee_{i=1}^n \mu_{x_i} = \bigvee_{i=1}^n \bigvee_{k \in I_{x_i}} \mu_k$. Thus, Y is fuzzy strongly S-closed. ■

Definition 33 *A fuzzy topological space X is said to be disconnected [26] if $X = \mu \vee \eta$, where μ and η are nonempty fuzzy open sets in X such that $\mu \wedge \eta = \varnothing$.*

Theorem 34 *Let $F : X \multimap Y$ be a fuzzy upper contra-continuous punctually fuzzy connected surjective multifunction. If X is connected, then Y is a fuzzy connected space.*

Proof. Suppose that Y is not fuzzy connected. Let $Y = \mu \vee \eta$ be a partition of Y. Then, μ and η are fuzzy open and closed in Y. Since $F(x)$ is fuzzy connected for each $x \in X$, $F(x) \leq \mu$ or $F(x) \leq \eta$. This implies that $x \in F^+(\mu) \cup F^+(\eta)$. We have $F^+(\mu) \cup F^+(\eta) = X$ and $F^+(\mu) \cap F^+(\eta) = \varnothing$. Since F is fuzzy upper contra-continuous, $F^+(\mu)$ and $F^+(\eta)$ are open in X. Thus, $X = F^+(\mu) \cup F^+(\eta)$ is a partition of X. This is a contradiction. ■

Theorem 35 *Let $F : X \multimap Y$ be a fuzzy lower contra-continuous punctually fuzzy connected surjective multifunction. If X is connected, then Y is a fuzzy connected space.*

Proof. Suppose that Y is not fuzzy connected. Let $Y = \mu \vee \eta$ be a partition of Y. Then μ and η are fuzzy open and closed in Y. Since F is fuzzy lower contra-continuous multifunction, $F^+(\mu)$ and $F^+(\eta)$ are closed. Since $X = F^+(\mu) \cup F^+(\eta)$ and $F^+(\mu) \cap F^+(\eta) = \varnothing$, then X is not connected. This is a contradiction. ■

4 Acknowledgment

The first and forth authors of this paper are partially supported by the Research Center in Algebraic Hyperstructures and Fuzzy Mathematics, University of Mazandaran, Babolsar, Iran.

References

[1] J. Albrycht and M. Maltoka, *On fuzzy multivalued function*, Fuzzy Sets and Systems, **12**(1984), 61–69.

[2] M. Alimohammady and M. Roohi, *On fuzzy $\varphi\psi$-continuous multifunction*, J. Appl. Math. Stoch. Anal. **2006**, 1–7.

[3] I. Beg, *Continuity of fuzzy multifunction*, J. Appl. Math. Stochastic Anal. **12**(1)(1999), 17–22.

[4] I. Beg, *Fuzzy multivalued functions*, Bull. Allahabad Math. Soc. **21**(2006), 41–104.

[5] I. Beg, Fixed points of fuzzy multivalued mappings with values in fuzzy ordered sets, J. Fuzzy Math., 6 (1) (1998), 127-131.

[6] M. Caldas, E. Ekici, S. Jafari and T. Noiri, On the class of contra λ-continuous functions, Annales Univ. Sci. Budapest Sec. Math., 49 (2006), 75-86.

[7] C. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl. *24* (1968), 191–201.

[8] J. Dontchev, Contra-continuous functions and strongly S-closed spaces, Internat. J. Math. Math. Sci., 19 (2) (1996), 303-310.

[9] E. Ekici, *On some type of continuous fuzzy functions*, Appl. Math. E-Not. **4**(2004), 21–25.

[10] E. Ekici and E. E. Kerre, *On fuzzy contra-continuities, Advances in Fuzzy Mathematics*, *1*(1)(2006), 35-44.

[11] E. Ekici, S. Jafari and T. Noiri, On upper and lower contra-continuous multifunctions, Analele Stii. Univ. Cuza Iasi Mat., LIV (1) (2008), 75-85.

[12] E. Ekici, New forms of contra-continuity, Carpathian J. Math., 24 (1) (2008), 37-45.

[13] E. Ekici, On (LC,s)-continuous functions, Chaos, Solitons and Fractals, 38 (2) (2008), 430-438.

[14] J. E. Joseph and M. H. Kwack, On S-closed spaces, Proc. Amer. Math. Soc., 80 (1980), 341-348.

[15] M. N. Mukherjee and S. Malakar, *On almost continuous and weakly continuous fuzzy multifunctions*, Fuzzy Sets and Systems, **41**(1991), 113–125.

[16] T. Noiri and S. Jafari, Properties of (θ, s)-continuous functions, Topology and its Appl., 123 (2002), 167-179.

[17] A. A. Nouh and M. E. El-Shafei, Cluster sets of functions and multifunctions in fuzzy topological spaces, J. Fuzzy Math., 15 (4) (2007), 759-768.

[18] P. Pao-Ming and L. Ying-Ming, *Fuzzy Topology I. Neighborhood structure of a fuzzy point and Moore-Smith convergence*, J. Math. Anal. Appl. **76**(1980), 571–599.

[19] P. Pao-Ming and L. Ying-Ming, *Fuzzy Topology II. Product and quotient spaces*, J. Math. Anal. Appl. **77**(1980), 20–37.

[20] N. S. Papageorgiou, *Fuzzy topology and fuzzy multifunctions*, J. Math. Anal. Appl. **109**(1985), 397–425.

[21] V. I. Ponomarev, *Properties of topological spaces preserved under multivalued continuous mappings on compacta*, Amer. Math. Soc. Translations, **38**(2)(1964), 119–140.

[22] K. S. Raja Sethupathy and S. Laksmivarahan, *Connectedness in fuzzy topology*, Kybernetika, **13**(3)(1977), 190–193.

[23] A. Stouti, Fixed points of expanding fuzzy multifunctions, J. Fuzzy Math., 13 (1) (2005), 95-99.

[24] A. Stouti, An α-fuzzy version of Amann's fixed point theorem for α-fuzzy monotone multifunctions, J. Fuzzy Math., 14 (1) (2006), 175-183

[25] T. Thompson, S-closed spaces, Proc. Amer. Math. Soc., 60 (1976), 335-338.

[26] C. K. Wong, *Fuzzy topology: product and quotient theorems*, J. Math. Anal. Appl. **45**(1974), 512–521.

M. Alimohammady
Department of Mathematics, University of Mazandaran,
Babolsar, Iran
E-mail: amohsen@umz.ac.ir

E. Ekici
Department of Mathematics,
Canakkale Onsekiz Mart University,
Terzioglu Campus,
17020 Canakkale/TURKEY
E-mail: eekici@comu.edu.tr

S. Jafari
College of Vestsjaelland South,
Herrestraede 11,
4200 Slagelse, Denmark
E-mail: jafari@stofanet.dk

M. Roohi
Islamic Azad University-Ghaemshahr branch, Iran
E-mail: mehdi.roohi@gmail.com