# Fuzzy Minimal Separation Axioms

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**Abstract**
In this paper, we deal with some separation axioms in the context of fuzzy minimal structures.

# 1 Introduction

Zadeh introduced the concept of a fuzzy set in [10]. Subsequently, many attempts have been made to extend many science notions to the fuzzy setting. Fuzzy minimal structure and fuzzy minimal space introduced in [3]. Further results about fuzzy minimal spaces can be found in [1, 3, 4, 5, 7]. In this paper, we introduce and investigate some new fuzzy minimal separation axioms.

For easy understanding of the material incorporated in this paper, we recall some basic definitions and results. For details on the following notions we refer to [1, 3, 4, 5, 7].

A family $\mathcal{M}$ of fuzzy sets in $X$ is said to be a *fuzzy minimal structure* on $X$ if $\alpha 1_X \in M$ for all $\alpha \in I$, where $I = [0, 1]$. In this case $(X, M)$ is called a *fuzzy minimal space* [1]. A fuzzy set $A \in I^X$ is said to be fuzzy $m$-open if $A \in \mathcal{M}$. $B \in I_X$ is called a fuzzy $m$-closed set if $B^c \in \mathcal{M}$. Let

$$m - Int(A) = \bigvee\{U : U \leq A, U \in \mathcal{M}\} \quad \text{and} \tag{1.1}$$

$$m - Cl(A) = \bigwedge\{F : A \leq F, F^c \in \mathcal{M}\}. \tag{1.2}$$

**Proposition 1.1** [*1*] *For any two fuzzy sets $A$ and $B$,*

*(a) $m - Int(A) \leq A$ and $m - Int(A) = A$ if $A$ is a fuzzy $m$-open set.*

*(b) $A \leq m - Cl(A)$ and $A = m - Cl(A)$ if $A$ is a fuzzy $m$-closed set.*

*(c) $m - Int(A) \leq m - Int(B)$ and $m - Cl(A) \leq m - Cl(B)$ if $A \leq B$.*

*(d) $m - Int(A \wedge B) = (m - Int(A)) \wedge (m - Int(B))$ and $(m - Int(A)) \vee (m - Int(B)) \leq m - Int(A \vee B)$.*

(*e*) $m-Cl(A\vee B)=(m-Cl(A))\vee(m-Cl(B))$ *and* $m-Cl(A\wedge B)=(m-Cl(A))\wedge(m-Cl(B))$.

(*f*) $m-Int(m-Int(A))=m-Int(A)$ *and* $m-Cl(m-Cl(B))=m-Cl(B)$.

(*g*) $(m-Cl(A))^c=m-Int(A^c)$ *and* $(m-Int(A))^c=m-Cl(A^c)$.

**Definition 1.1** [1] *A fuzzy minimal space* $(X,\mathcal{M})$ *enjoys the property* $U$ *if arbitrary union of fuzzy* $m$*-open sets is fuzzy* $m$*-open.*

**Proposition 1.2** [2] *For a fuzzy minimal structure* $\mathcal{M}$ *on a set* $X$*, the following are equivalent.*

(*a*) $(X,\mathcal{M})$ *has the property* $U$.

(*b*) *If* $m-Int(A)=A$*, then* $A\in\mathcal{M}$.

(*c*) *If* $m-Cl(B)=B$*, then* $B^c\in\mathcal{M}$.

**Proof**. Suppose (a) satisfies. If $m-Int(A)=A$, then it follows that $A$ is fuzzy $m$-open; i.e., $A\in\mathcal{M}$ which it proves (a)$\Longrightarrow$(b) and also (b)$\Longrightarrow$(c) is straightforward. Finally, assume (c) holds. Consider arbitrary fuzzy $m$-open set $A_\alpha$, set $A=\bigcup_{\alpha\in\mathcal{A}}A_\alpha$. It follows from part (a) of Proposition 1.1 that $m-int(A)\subseteq A$. On the other hand, since $A_\alpha$ is fuzzy $m$-open set for all $\alpha\in\mathcal{A}$, so part (c) of Proposition 1.2 implies that

$$A_\alpha=m-Int(A_\alpha)\subseteq m-Int(\bigcup_{\alpha\in\mathcal{A}}A_\alpha)=m-Int(A)\quad rm\ for\ all\quad \alpha\in\mathcal{A}.$$

Therefore, $A=\bigcup_{\alpha\in\mathcal{A}}A_\alpha\subseteq m-Int(A)$; i.e., $A=m-Int(A)$. Hence, $m-Int((A^c)^c)=A$ and since part (g) of Proposition 1.2 implies that $(m-Cl(A^c))^c=A$; i.e., $m-Cl(A^c)=A^c$. Thus $A\in\mathcal{M}$ follows from the assumption.

Fuzzy minimal continuous functions was introduced and studied in [3].

**Definition 1.2** [3] *Let* $(X,\mathcal{M})$ *and* $(Y,\mathcal{N})$ *be two fuzzy minimal spaces. We say that a fuzzy function* $f:(X,\mathcal{M})\to(Y,\mathcal{N})$ *is fuzzy minimal continuous (briefly fuzzy* $m$*-continuous) if* $f^{-1}(B)\in\mathcal{M}$*, for any* $B\in\mathcal{N}$.

**Theorem 1.1** *[3] Consider the following properties for a fuzzy function* $f:(X,\mathcal{M})\to(Y,\mathcal{N})$ *between two fuzzy minimal spaces.*

(*a*) $f$ *is a fuzzy* $m$*-continuous function.*

(*b*) $f^{-1}(B)$ *is a fuzzy* $m$*-closed set for each fuzzy* $m$*-closed set* $B\in I^Y$.

(*c*) $m-Cl(f^{-1}(B))\leq f^{-1}(m-Cl(B))$ *for each* $B\in I^Y$.

(*d*) $f(m-Cl(A))\leq m-Cl(f(A))$ *for any* $A\in I^X$.

(*e*) $f^{-1}(m-Int(B))\leq m-Int(f^{-1}(B))$ *for each* $B\in I^Y$.

*Then* (*a*) $\Leftrightarrow$ (*b*) $\Rightarrow$ (*c*) $\Leftrightarrow$ (*d*) $\Leftrightarrow$ (*e*)*. Moreover, if* $(X,\mathcal{M})$ *satisfies in the property* $U$*, then all of the above statements are equivalent.*

## 2 Fuzzy minimal separation axioms

**Definition 2.1** *Suppose $(X, \mathcal{M})$ is a fuzzy minimal space. A fuzzy set $A$ in $X$ is said to be a fuzzy minimal q-neighborhood of a fuzzy point $x_\alpha$ if there is a fuzzy $m$-open set $\mu$ in $X$ with $x_\alpha q \mu$ and $\mu \leq A$.*

**Definition 2.2** *Suppose $(X, \mathcal{M})$ is a fuzzy minimal space. A fuzzy point $x_\alpha$ in $X$ is said to be fuzzy minimal cluster point of a fuzzy set $A$ if every fuzzy minimal q-neighborhood of $x_\alpha$ is q-coincident with $A$.*

**Theorem 2.1** *Suppose $(X, \mathcal{M})$ is a fuzzy minimal space. A fuzzy point $x_\alpha$ is a fuzzy minimal cluster point of a fuzzy set $A$ if and only if $x_\alpha \widetilde{\in}\ m - Cl(A)$.*

**Proof**. Suppose $x_\alpha \not\widetilde{\in}\ m - Cl(A)$. Then, one can easily see that there exists $m$-closed set $F$ in $X$ with $A \leq F$ and $F(x) < \alpha$. Therefore, $x_\alpha q F^c$ and $A \not{q} F^c$; i.e., $x_\alpha$ is not a fuzzy minimal cluster point of $A$. Conversely, suppose $x_\alpha$ is not a fuzzy minimal cluster point of $A$. There exists a fuzzy minimal $q$-neighborhood $N$ of $x_\alpha$ for which $N \not{q} A$. Then there exists a fuzzy $m$-open set $\mu$ in $X$ with $x_\alpha q \mu$ and $\mu \leq N$. Therefore, $\mu \not{q} A$ which implies that $A \leq \mu^c$. Since $\mu^c$ is $m$-closed, so (1.2) implies that $m - Cl(A) \leq \mu^c$. That $x_\alpha \not\widetilde{\in}\ m - Cl(A)$ follows from the fact that $x_\alpha \not\widetilde{\in} \mu^c$.

**Definition 2.3** *A fuzzy minimal space $(X, \mathcal{M})$ is said to be*

*$(a)$ fuzzy minimal $T_0$ if for every pair of distinct fuzzy points $x_\alpha$ and $x_\beta$,*

*when $x \neq y$ either $x_\alpha$ has a fuzzy minimal neighborhood which is not q-coincident with $y_\beta$ or $y_\beta$ has a fuzzy minimal neighborhood which is not q-coincident with $x_\alpha$,*

*when $x = y$ and $\alpha \leq \beta$ (say), there is a fuzzy minimal q-neighborhood of $y_\beta$ which is not q-coincident with $x_\alpha$,*

*$(b)$ fuzzy minimal $T_1$ if for every pair of distinct fuzzy points $x_\alpha$ and $x_\beta$,*

*when $x \neq y$ there is a fuzzy minimal neighborhood $\mu$ of $x_\alpha$ and a fuzzy minimal neighborhood $\nu$ of $y_\beta$ with $\mu \not{q} y_\beta$ and $x_\alpha \not{q} \nu$,*

*when $x = y$ and $\alpha \leq \beta$ (say), $y_\beta$ has a fuzzy minimal q-neighborhood which is not q-coincident with $x_\alpha$,*

*$(c)$ fuzzy minimal $T_2$ if for every pair of distinct fuzzy points $x_\alpha$ and $x_\beta$,*

*when $x \neq y$, $x_\alpha$ and $y_\beta$ have fuzzy minimal q-neighborhoods which are not q-coincident,*

*when $x = y$ and $\alpha \leq \beta$ (say), $x_\alpha$ has a fuzzy minimal neighborhood $\mu$ and $y_\beta$ has a fuzzy minimal q-neighborhood $\nu$ in which $\mu \not{q} \nu$.*

*In short fuzzy $m - T_i$(i=0,1,2) space are used for fuzzy minimal $T_i$ space.*

**Theorem 2.2** *Every fuzzy $m - T_2$ space is a fuzzy $m - T_1$ space and also every fuzzy $m - T_1$ space is a fuzzy $m - T_0$ space.*

**Proof**. Obvious.

**Theorem 2.3** *If a fuzzy minimal space $(X, \mathcal{M})$ is fuzzy $m-T_0$, then for any pair of distinct fuzzy points $x_\alpha$ and $y_\beta$ we have $x_\alpha$ is not a fuzzy minimal cluster point of $y_\beta$ or $y_\beta$ is not a fuzzy minimal cluster point of $x_\alpha$.*

**Proof**. Suppose $(X, \mathcal{M})$ is a fuzzy $m - T_0$ space. For any distinct fuzzy points $x_\alpha$ and $y_\beta$ in $X$, there are two cases

(i) $x \neq y$

(ii) $x = y$ and $\alpha \leq \beta$ (say).

When $x \neq y$, then the fuzzy point $x_1$ has a fuzzy minimal neighborhood $\mu$ for which $\mu \not{q} y_\beta$ or $y_1$ has a fuzzy minimal neighborhood $\nu$ for which $x_\alpha \not{q} \nu$. Therefore, $\mu$ is a fuzzy minimal $q$-neighborhood of $x_\alpha$ with $\mu \not{q} y_\beta$ or $\nu$ is a fuzzy minimal $q$-neighborhood of $y_\beta$ such that $x_\alpha \not{q} \nu$. It follows from Definition 2.2 that $x_\alpha$ is not a fuzzy minimal cluster point of $y_\beta$ or $y_\beta$ is not a fuzzy minimal cluster point of $x_\alpha$. In case that $x = y$ and $\alpha \leq \beta$ there is a fuzzy minimal $q$-neighborhood of $y_\beta$ which is not $q$-coincident with $x_\alpha$. Then $y_\beta$ is not a fuzzy minimal cluster point of $x_\alpha$.

**Theorem 2.4** *A fuzzy minimal space $(X, \mathcal{M})$ is fuzzy $m-T_1$ if every fuzzy point $x_\alpha$ is fuzzy $m$-closed in $X$.*

**Proof**. Suppose $x_\alpha$ and $y_\beta$ are distinct fuzzy points in $X$, there are two cases

(i) $x \neq y$

(ii) $x = y$ and $\alpha < \beta$ (say).

Assume that $x \neq y$. By hypothesis $x_\alpha^c$ and $y_\beta^c$ are fuzzy $m$-open sets. It is easy to see that $x_\alpha \widetilde{\in} y_\beta^c$, $y_\beta \widetilde{\in} x_\alpha^c$ and $x_\alpha \not{q} y_\beta$. In case that $x = y$ and $\alpha < \beta$, one can deduce that $x_\alpha^c$ is a fuzzy $m$-open set with $y_\beta q x_\alpha^c$ and $x_\alpha \not{q} x_\alpha^c$ which implies that $(X, \mathcal{M})$ is fuzzy $m - T_1$.

**Theorem 2.5** *Let $(X, \mathcal{M})$ be a fuzzy minimal space. Suppose that $(X, \mathcal{M})$ enjoys the property $U$. Then $(X, \mathcal{M})$ is fuzzy minimal $T_1$ if for each $x \in X$ and each $\alpha \in [0, 1]$ there exists a fuzzy minimal open set $\mu$ such that $\mu(x) = 1 - \alpha$ and $\mu(y) = 1$ for $y \neq x$.*

**Proof**. Let $x_\alpha$ be an arbitrary fuzzy point of $X$. We shall show that the fuzzy point $x_\alpha$ is fuzzy *minimal* closed. By hypothesis, there exists a fuzzy *minimal* open set $\mu$ such that $\mu(x) = 1 - \alpha$ and $\mu(y) = 1$ for $y \neq x$. We have $\mu^c = x_\alpha$. Thus, the fuzzy point $x_\alpha$ is fuzzy *minimal* closed and hence the fuzzy *minimal* space $X$ is *fuzzy minimal* $T_1$.

**Theorem 2.6** *Let $(X, \mathcal{M})$ be a fuzzy minimal space such that $1 \in \mathcal{M}$. Suppose that $(X, \mathcal{M})$ enjoys the property $U$. The following are equivalent:*

*(1) $(X, \mathcal{M})$ is fuzzy minimal $T_1$,*

*(2) for each $x \in X$ and each $\alpha \in [0, 1]$ there exists a fuzzy minimal open set $\mu$ such that $\mu(x) = 1 - \alpha$ and $\mu(y) = 1$ for $y \neq x$.*

**Proof**. (1) $\Rightarrow$ (2) : Let $\alpha = 0$. Take $\mu = 1$. Then $\mu$ is a fuzzy *minimal* open set such that $\mu(x) = 1 - 0$ and $\mu(y) = 1$ for $y \neq x$. Let $\alpha \in (0, 1]$ and $x \in X$. Take $\mu = (x_\alpha)^c$. The set $\mu$ is fuzzy *minimal* open such that $\mu(x) = 1 - \alpha$ and $\mu(y) = 1$ for $y \neq x$.

(2) $\Rightarrow$ (1) : It follows from Theorem 2.5.

**Theorem 2.7** *Let $(X, \mathcal{M})$ be a fuzzy minimal space. Suppose that $(X, \mathcal{M})$ enjoys the property $U$. If $(X, \mathcal{M})$ is fuzzy minimal $T_1$, then every fuzzy point $x_\alpha$ is fuzzy $m$-closed in $X$.*

**Proof**. Let $(X, \mathcal{M})$ be fuzzy *minimal* $T_1$ and $x_\alpha$ and $y_\beta$ be any pair of distinct fuzzy points of $X$.

Let $x \neq y$. Then there exists a fuzzy minimal neighborhood $\mu$ of $x_\alpha$ and a fuzzy minimal neighborhood $\nu$ of $y_\beta$ with $\mu \not{q} y_\beta$ and $x_\alpha \not{q}\nu$. Then $x_\alpha \in \nu^c$. Since $\nu^c$ is fuzzy minimal closed, then $m - Cl(x_\alpha) \leq \nu^c$ and hence $m - Cl(x_\alpha) \not{q}\nu$. Thus, $m - Cl(x_\alpha) \leq x_\alpha$ and hence $x_\alpha = m - Cl(x_\alpha)$. Consequently, every fuzzy point $x_\alpha$ is fuzzy $m$-closed in $X$.

**Corollary 2.1** *Let $(X, \mathcal{M})$ be a fuzzy minimal space. Suppose that $(X, \mathcal{M})$ enjoys the property $U$. The following properties are equivalent:*

*(1) $(X, \mathcal{M})$ is fuzzy minimal $T_1$,*

*(2) every fuzzy point $x_\alpha$ is fuzzy $m$-closed in $X$.*

**Proof**. It is an immediate consequence of Theorem 2.4 and Theorem 2.7.

**Theorem 2.8** *Suppose $i = 0, 1, 2$. A fuzzy minimal space $(X, \mathcal{M})$ is fuzzy $m - T_i$ if and only if for any pair of distinct fuzzy points $x_\alpha$ and $y_\beta$ with distinct supports, there exists a fuzzy $m$-continuous mapping $f$ from $X$ into a fuzzy $m - T_i$ space $(Y, \mathcal{N})$ such that $f(x) \neq f(y)$.*

**Proof**. We only prove the case that $i = 2$ and others are similar. Suppose $(X, \mathcal{M})$ is fuzzy $m - T_2$ space. Let $(Y, \mathcal{N}) := (X, \mathcal{M})$ and $f := id_X$. Clearly, $(Y, \mathcal{N})$ and $f$ have the required properties. Conversely, suppose $x_\alpha$ and $y_\beta$ are distinct fuzzy points in $X$. There are two cases

(i) $x \neq y$

(ii) $x = y$ and $\alpha < \beta$ (say).

When $x \neq y$, by assumption there is fuzzy $m$-continuous mapping $f$ from $(X, \mathcal{M})$ into a fuzzy $m - T_2$ space $(Y, \mathcal{N})$ with $f(x) \neq f(y)$. Since $(Y, \mathcal{N})$ is fuzzy $m - T_2$ space and $(f(x))_\alpha$ and $(f(y))_\beta$ are distinct fuzzy points in $Y$, so there are fuzzy minimal neighborhoods $\mu$ and $\nu$ of $(f(x))_\alpha$ and $(f(y))_\beta$ respectively for which $\mu \not{q}\nu$. It follows from $m$-continuity of $f$ that $f^{-1}(\mu)$ and $f^{-1}(\nu)$ are fuzzy minimal neighborhoods of $x_\alpha$ and $y_\beta$ respectively. Since $\mu \not{q}\nu$, so $f^{-1}(\mu) \not{q} f^{-1}(\nu)$. In case that $x = y$ and $\alpha < \beta$ (say), $(f(x))_\alpha$ and $(f(y))_\beta$ are fuzzy points in $Y$ with $f(x) = f(y)$. Since $(Y, \mathcal{N})$ is fuzzy $m - T_2$ space, so $(f(x))_\alpha$ has a fuzzy minimal neighborhood $\mu$ and $(f(y))_\beta$ has a fuzzy minimal $q$-neighborhoods $\nu$ for which $\mu \not{q}\ \nu$. Then $f^{-1}(\mu)$ is a fuzzy minimal $q$-neighborhood of $x_\alpha$ and $f^{-1}(\nu)$ is a fuzzy minimal $q$-neighborhood of $y_\beta$ with $f^{-1}(\mu) \not{q} f^{-1}(\nu)$. Therefore, $(X, \mathcal{M})$ is fuzzy $m - T_2$ space.

**Theorem 2.9** *Let $(X, \mathcal{M})$ be a fuzzy minimal space. If $(X, \mathcal{M})$ is fuzzy minimal $T_2$, then for any two distinct fuzzy points $x_\alpha$ and $y_\beta$, the following properties hold:*

*(1) If $x \neq y$, then there exist fuzzy open neighborhoods $\mu$ and $\nu$ of $x_\alpha$ and $y_\beta$, respectively, such that $m - Cl(\nu) \leq 1_X - \mu$ and $m - Cl(\mu) \leq 1_X - \nu$,*

*(2) If $x = y$ and $\alpha < \beta$ (say), then there exists a fuzzy open neighborhood $\mu$ of $x_\alpha$ such that $y_\beta \notin m - Cl(\mu)$.*

**Proof**. (1) : Let $x \neq y$. Then there exist fuzzy $m$-open *neighborhoods* $\mu$ and $\nu$ of $x_\alpha$ and $y_\beta$, respectively, such that $\mu \not{q}\nu$. Since $\mu \not{q}\nu$, then $\mu(z) \leq 1 - \nu(z)$ and $\nu(z) \leq 1 - \mu(z)$ for all $z \in X$. Since $1_X - \mu$ and $1_X - \nu$ are fuzzy $m$-closed, then $m - Cl(\nu) \leq 1_X - \mu$ and $m - Cl(\mu) \leq 1_X - \nu$.

(2) : Let $x = y$. Then there exist a fuzzy q-*neighborhood* $\lambda$ of $y_\beta$ and a fuzzy open *neighborhood* $\mu$ of $x_\alpha$ such that $\lambda \not{q}\mu$.

Let $\nu$ be a fuzzy $m$-open set in $X$ such that $y_\beta q\nu$ and $\nu(y) \leq \lambda(y)$. Since $\beta > 1 - \nu(y) = (m - Cl(1 - \nu))(y)$, $\nu(y) \leq \lambda(y)$ and $\mu(y) \leq 1 - \lambda(y)$, then $\beta > m - Cl(\mu)(y)$. Thus, $y_\beta \notin m - Cl(\mu)$.

**Theorem 2.10** *Let $(X, \mathcal{M})$ be a fuzzy minimal space. Suppose that $(X, \mathcal{M})$ enjoys the property $U$. Then $(X, \mathcal{M})$ is fuzzy minimal $T_2$ if and only if for any two distinct fuzzy points $x_\alpha$ and $y_\beta$, the following properties hold:*

*(1) If $x \neq y$, then there exist fuzzy $m$-open neighborhoods $\mu$ and $\nu$ of $x_\alpha$ and $y_\beta$, respectively, such that $m - Cl(\nu) \leq 1_X - \mu$ and $m - Cl(\mu) \leq 1_X - \nu$,*

*(2) If $x = y$ and $\alpha < \beta$ (say), then there exists a fuzzy $m$-open neighborhood $\mu$ of $x_\alpha$ such that $y_\beta \notin m - Cl(\mu)$.*

**Proof.** $(\Rightarrow)$ : It follows from Theorem 2.9.

$(\Leftarrow)$ : Let $x_\alpha$ and $y_\beta$ be distinct fuzzy points in $X$.

Let $x \neq y$. Then there exist fuzzy $m$-open *neighborhoods* $\mu$ and $\nu$ of $x_\alpha$ and $y_\beta$, respectively, such that $m - Cl(\nu) \leq 1_X - \mu$. This implies that for all $z \in X$, $\mu(z) + \nu(z) \leq (m - Cl(\nu))(z) + \mu(z) \leq 1$. Hence, $\mu \not{q}\nu$.

Let $x = y$ and $\alpha < \beta$. Then there exists a fuzzy $m$-open *neighborhood* $\mu$ of $x_\alpha$ such that $y_\beta \notin m - Cl(\mu)$. Let $\lambda = 1_X - m - Cl(\mu)$. Since for all $z \in X$, $\lambda(z) + \mu(z) \leq 1$, then $\lambda \not{q}\mu$. On the other hand, $\lambda$ is a fuzzy open set and $\beta + \lambda(y) > \alpha + \lambda(y) \geq 1$. Hence, $\lambda$ is a fuzzy minimal q-*neighborhood* of $y_\beta$ such that $\lambda \not{q}\mu$.

**Theorem 2.11** *Let $(X, \mathcal{M})$ be a fuzzy minimal space. If $(X, \mathcal{M})$ is fuzzy minimal $T_2$, the the following hold:*

*(1) for every fuzzy point $x_\alpha$ in $X$, $\{x_\alpha\} = \cap\{m{-}Cl(\nu) : \nu$ is a fuzzy minimal neighborhood of $x_\alpha\}$.*

*(2) for every $x$, $y \in X$ with $x \neq y$, there exist a fuzzy minimal neighborhood $\mu$ of $x_1$ such that $y \notin supp(m - Cl(\mu))$.*

**Proof.** (1) : Let $y_\beta \notin \{x_\alpha\}$. We shall show the existence of a fuzzy minimal *neighborhood* of $x_\alpha$ such that $y_\beta \notin m - Cl(\nu)$.

Let $x \neq y$. Then there exist fuzzy minimal open sets $\mu$ and $\nu$ containing $y_1$ and $x_\alpha$, respectively such that $\mu \not{q}\nu$. Then $\nu$ is fuzzy minimal *neighborhood* of $x_\alpha$ and $\mu$ is a fuzzy minimal q-*neighborhood* of $y_\beta$ such that $\mu \not{q}\nu$. Hence, $y_\beta \notin m - Cl(\nu)$.

Let $x = y$. Then $\alpha < \beta$ and there exist a fuzzy minimal q-*neighborhood* $\mu$ of $y_\beta$ and fuzzy minimal *neighborhood* $\nu$ of $x_\alpha$ such that $\mu \not{q}\nu$. Thus, $y_\beta \notin m - Cl(\nu)$.

(2) : For every $x$, $y \in X$ with $x \neq y$, since $(X, \mathcal{M})$ is *fuzzy minimal $T_2$*, then there exist fuzzy minimal open sets $\mu$ and $\nu$ such that $x_1 \in \mu$, $y_1 \in \nu$ and $\mu \not{q}\nu$. Then $\nu^c(y) = 0$ and $\mu \leq \nu^c$. Since $\nu^c$ is fuzzy minimal closed, $m - Cl(\mu) \leq \nu^c$. Thus, $m - Cl(\mu)(y) = 0$ and hence, $y \notin$ *supp*$(m - Cl(\mu))$.

**Theorem 2.12** *Let $(X, \mathcal{M})$ be a fuzzy minimal space. Suppose that $(X, \mathcal{M})$ enjoys the property $U$. Then $(X, \mathcal{M})$ is fuzzy minimal $T_2$ if and only if*

*(1) for every fuzzy point $x_\alpha$ in $X$, $\{x_\alpha\} = \cap\{m{-}Cl(\nu) : \nu$ is a fuzzy minimal neighborhood of $x_\alpha\}$.*

*(2) for every $x$, $y \in X$ with $x \neq y$, there exist a fuzzy minimal neighborhood $\mu$ of $x_1$ such that $y \notin supp(m - Cl(\mu))$.*

**Proof.** $(\Rightarrow)$ : It follows from Theorem 2.11.

$(\Leftarrow)$ : Let $x_\alpha$ and $y_\beta$ be two distinct fuzzy point in $X$.

Let $x \neq y$. Suppose that $0 < \alpha < 1$. There exists a real number $\delta$ such that $0 < \alpha + \delta < 1$. By hypothesis, there exists a fuzzy minimal *neighborhood* $\mu$ of $y_\beta$ such that $x_\delta \notin m - Cl(\mu)$. Then $x_\delta$ has a fuzzy minimal q-*neighborhood* $\nu$ such that $\mu \not{q} \nu$. On the other hand, $\delta + \nu(x) > 1$ and $\nu(x) > 1 - \delta > \alpha$ and hence $\nu$ is a fuzzy minimal *neighborhood* of $x_\alpha$ such that $\mu \not{q} \nu$, where $\mu$ is a fuzzy minimal *neighborhood* of $y_\beta$. If $\alpha = \beta = 1$, by hypothesis there exists a fuzzy minimal *neighborhood* $\mu$ of $x_1$ such that $m - Cl(\mu)(y) = 0$. Thus, $\nu = (m - Cl(\mu))^c$ is a fuzzy minimal *neighborhood* of $y_1$ such that $\mu \not{q} \nu$.

Let $x = y$ and $\alpha < \beta$. Then there exists a fuzzy minimal *neighborhood* of $x_\alpha$ such that $y_\beta \notin m - Cl(\mu)$. Thus, there exists a fuzzy minimal q-*neighborhood* $\nu$ of $y_\beta$ such that $\mu \not{q} \nu$.

Hence, $(X, \mathcal{M})$ is *fuzzy minimal* $T_2$.

**Corollary 2.2** *Suppose $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are fuzzy minimal spaces and $f : X \longrightarrow Y$ is injective and fuzzy m-continuous. $(X, \mathcal{M})$ is fuzzy $m - T_i$ space if $(Y, \mathcal{N})$ is fuzzy $m - T_i$ space.*

**Proof.** It is an immediate consequence of Theorem 2.8.

# 3 Fuzzy $\Lambda_m$-sets and fuzzy minimal separation axioms

The concept of fuzzy $\Lambda_m$-set was introduced and studied in [6]. Investigating the relation of fuzzy $\Lambda_m$-sets and some brand of fuzzy minimal separation is our main task in this section.

For a fuzzy set $A$ in the fuzzy minimal space $(X, \mathcal{M})$, set

$$\Lambda_m(A) = \bigwedge \{U : A \leq U, U \in \mathcal{M}\}.$$

**Proposition 3.1** [6] *For fuzzy sets $A, B$ and $A_\alpha (\forall\ \alpha \in \mathcal{A})$ in a fuzzy minimal space $(X, \mathcal{M})$, the following properties hold :*

(a) *$A \leq \Lambda_m(A)$ and $\Lambda_m(A) = A$ if $A$ is fuzzy m-open set.*

(b) *If $A \leq B$, then $\Lambda_m(A) \leq \Lambda_m(B)$.*

(c) *$\Lambda_m(\Lambda_m(A)) = \Lambda_m(A)$.*

(d) *$\bigvee_\alpha \Lambda_m(A_\alpha) \leq \Lambda_m(\bigvee_\alpha A_\alpha)$.*

(e) *$\bigwedge_\alpha \Lambda_m(A_\alpha) \geq \Lambda_m(\bigwedge_\alpha A_\alpha)$.*

**Definition 3.1** *A fuzzy set $A$ in a fuzzy minimal space $(X, \mathcal{M})$ is called a fuzzy $\Lambda_m$-set if $\Lambda_m(A) = A$. The family of all fuzzy $\Lambda_m$-sets in $X$ is denoted by $F\Lambda_m$.*

**Proposition 3.2** [6] *For fuzzy sets $A, B$ and $A_\alpha (\forall\ \alpha \in \mathcal{A})$ in a fuzzy minimal space $(X, \mathcal{M})$, the following assertions hold :*

(a) *If $A$ is a fuzzy m-open set, then $A$ is fuzzy $\Lambda_m$-set.*

(b) *$\Lambda_m(A)$ is a fuzzy $\Lambda_m$-set.*

(c) *$\bigwedge_\alpha A_\alpha$ is fuzzy $\Lambda_m$-set if $A_\alpha$ is fuzzy $\Lambda_m$-set for each $\alpha \in \mathcal{A}$.*

(d) *If in addition $(X, \mathcal{M})$ has the property $U$, then $\bigvee_\alpha A_\alpha$ is fuzzy $\Lambda_m$-set, whenever $A_\alpha$ is fuzzy $\Lambda_m$-set for each $\alpha \in \mathcal{A}$.*

**Definition 3.2** *A fuzzy set $A$ in a fuzzy minimal space $(X, \mathcal{M})$ is said to be fuzzy $(\Lambda, m)$-closed if $A = B \wedge F$, where $B$ is a fuzzy $\Lambda_m$-set and $F$ is a fuzzy $m$-closed set. The family of all fuzzy $(\Lambda, m)$-closed sets in $X$ is denoted by $F\Lambda_{mc}$.*

**Proposition 3.3** [6] *In a fuzzy minimal space $(X, \mathcal{M})$,*
$(a)$ *every fuzzy $m$-closed set is a fuzzy $(\Lambda, m)$-closed set,*
$(b)$ *every fuzzy $\Lambda_m$-set is a fuzzy $(\Lambda, m)$-closed set.*

**Theorem 3.1** [6] *Consider following conditions for a fuzzy set $A$ in a fuzzy minimal space $(X, \mathcal{M})$ :*
$(a)$ *$A$ is fuzzy $(\Lambda, m)$-closed.*
$(b)$ *$A = B \wedge m - Cl(A)$, where $B$ is fuzzy $\Lambda_m$-set.*
$(c)$ *$A = \Lambda_m(A) \wedge m - Cl(A)$.*
*Then $(a) \Rightarrow (b) \Leftrightarrow (c)$. Moreover, if $(X, \mathcal{M})$ has the property $U$, then the above conditions are equivalent.*

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