SOME FUNDAMENTAL PROPERTIES OF PRESEPARATED SETS

Miguel Caldas

Departamento de Matematica Aplicada
Universidade Federal Fluminense
Rua Mario Santos Braga, s/n 24020-140, Niteroi
RJ Brasil
e-mail: gmamccs@vm.uff.br

Erdal Ekici[1]

Department of Mathematics
Canakkale Onsekiz Mart University
Terzioglu Campus, 17020 Canakkale
Turkey
e-mail: eekici@comu.edu.tr

Saeid Jafari

College of Vestsjaelland South
Herrestraede 11, 4200 Slagelse
Denmark
e-mail: jafari@stofanet.dk

Abstract. In this paper, we offer the new notion of preseparatedness in topological spaces and we study some of its fundamental properties.

Keywords and Phrases: Topological spaces, preopen set, preclosed sets, preseparated sets, pre-symmetric.

2000 Math. Subject Classification: 54C10, 54D10.

1. Introduction

Throughout the paper (X, τ) (or simply X) will always denote a topological space. For a subset A of X, the closure, interior and complement of A in X are denoted by $Cl(A), Int(A)$ and $X - A$, respectively. By $PO(X, \tau)$ and $PC(X, \tau)$ we denote the collection of all preopen sets and the collection of all preclosed sets of (X, τ), respectively. Let A be a subset of a topological space (X, τ). A is preopen [4] (or

[1]Corresponding Author.

locally dense [1]) if $A \subset Int(Cl(A))$. The complement of a preopen set A is called preclosed [4] or equivalently A is preclosed if $Cl(Int(A)) \subset A$. The intersection of all preclosed sets containing A is called the preclosure of A [2] and is denoted by $pCl(A)$. Recall that a function $f : X \to Y$ is said to be precontinuous [4] if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists a preopen set U of X containing x such that $f(U) \subset V$. A topological space X is pre-T_1 [3], if for each pair of distinct points x and y of X, there exists a pair of preopen sets one containing x but not y and the other containing y but not x. A topological space X is pre-T_0 [5] if for any pair of distinct points x and y of X, there exists a preopen set containing x but not y or a preopen set containing y but not x. Observe that every topological space is pre-T_0.

2. Some properties

Definition 2.1. Let X be a topological space and A, $B \subset X$. Then A and B are said to be preseparated if $A \cap pCl(B) = \emptyset$ and $pCl(A) \cap B = \emptyset$.

Remark 2.1.

(1) If A and B are preseparated, then A and B are disjoint.

(2) If $A \neq \emptyset$ is a subset of B and B is preseparated from C, then A and C are preseparated.

(3) If A and B are preseparated and A and C are preseparated, then A and $B \cup C$ are preseparated.

Definition 2.2. Let X be a topological space. Points are called preseparated from preclosed sets in X if for all preclosed sets $C \subset X$ and for each $x \in X - C$, $\{x\}$ and C are preseparated.

Recall that a topological space X is pre-regular [6] if for each preclosed set F and each point $x \in X - F$, there exist disjoint preopen sets U and V such that $x \in U$ and $F \subset V$.

Remark 2.2.

(1) If X is pre-T_1, then points are preseparated from preclosed sets. Hence, this is a weaker separation axiom than pre-T_1.

(2) If X is pre-regular, then points are preseparated from preclosed sets. Hence, pre-regularity is a stronger condition.

This axiom enables us to determine the family of preopen sets of the topology of a space from its preseparated sets.

Theorem 2.1. *Let X be a topological space in which points are preseparated from preclosed sets and let S be the pairs of preseparated sets in X. Then, for each subset A of X, the preclosure of A is*

$$pCl(A) = \{x \in X : \{\{x\}, A\} \notin S\}.$$

Proof. Let $x \notin \{x \in X : \{\{x\}, A\} \notin S\}$. Then $\{\{x\}, A\} \in S$. We have $\{x\} \cap pCl(A) = \emptyset$. Thus, $x \notin pCl(A)$ and hence $pCl(A) \subset \{x \in X : \{\{x\}, A\} \notin S\}$.

Suppose that $x \notin pCl(A)$. Then $pCl(A)$ is a preclosed set disjoint from $\{x\}$ and thus, by hypothesis, $\{\{x\}, A\} \in S$. Hence, $x \notin \{x \in X : \{\{x\}, A\} \notin S\}$. ∎

In general, if $x \in pCl(\{y\})$ in a topological space, then it need not be the case that $y \in pCl(\{x\})$. However, when points are preseparated from preclosed sets, this is the case; in fact, this provides us with alternate characterizations of the axiom.

Definition 2.3. A topological space X is called pre-symmetric if topologically distinct points in X are preseparated.

Observe that a topological space X is pre-symmetric if and only if X is T_1. Since every topological space is pre-T_0, this makes sense.

Theorem 2.2. *Let X be a topological space. Then the following are equivalent:*

(1) *Points are preseparated from preclosed sets in X.*

(2) *For all $x, y \in X$, $x \in pCl(\{y\})$ if and only if $y \in pCl(\{x\})$.*

(3) *X is pre-symmetric.*

Proof. Suppose that (1) holds. If $x \in pCl(\{y\})$, then $\{x\}$ and $\{y\}$ are not preseparated and hence, $y \in pCl(\{x\})$. If $\{x\}$ and $\{y\}$ are topologically distinct, then one of them, say x, has a preopen neighborhood U which does not contain y. We have $pCl(\{y\}) \subset X - U$. This implies that $\{x\}$ and $pCl(\{y\})$ are preseparated and $\{x\}$ and $\{y\}$ are preseparated. Hence, (1) implies (2) and (1) implies (3).

Suppose that (2) is true. Let $C \subset X$ be preclosed and let $x \in X - C$. For each $y \in C$, $x \notin pCl(\{y\})$ and hence $y \notin pCl(\{x\})$. Thus, $pCl(\{x\}) \cap C = \emptyset$. Hence, (2) implies (1).

Finally, suppose that (3) is true and suppose that $x \notin pCl(\{y\})$. Then $\{x\}$ and $\{y\}$ are topologically distinct and hence preseparated. Thus, $pCl(\{x\}) \cap \{y\} = \emptyset$, that is, $y \notin pCl(\{x\})$. Hence, (3) implies (2). ∎

Theorem 2.1 tells us that when X is pre-symmetric, the collection of preseparated sets uniquely determines the family of preopen sets of the topology of X. Note that presymmetry really is necessary.

Example 2.1. Let $X = \{a, b\}$ and suppose that no pair of nonempty subsets are preseparated in X. Then the family of preopen sets of the pre-symmetric topology on X must be the power set of X, but the nonpre-symmetric topology $\{\emptyset, \{a\}, X\}$ also preseparates no pair of nonempty subsets of X.

In fact, pre-symmetric spaces can be presented without reference to preopen sets at all.

Theorem 2.3. *Let S be a set of unordered pairs of subsets of a set X such that*

(1) *if $\{A, B\} \in S$, then A and B are disjoint,*

(2) *if $A \subset B$ and $\{B, C\} \in S$, then $\{A, C\} \in S$,*

(3) *if $\{A, B\} \in S$ and $\{A, C\} \in S$, then $\{A, B \cup C\} \in S$,*

(4) *if $\{\{x\}, B\} \in S$ for each $x \in A$ and $\{\{y\}, A\} \in S$ for each $y \in B$, then $\{A, B\} \in S$ and*

(5) *if $\{\{x\}, B\} \notin S$ and for each $y \in B$, $\{\{y\}, A\} \notin S$, then $\{\{x\}, A\} \notin S$.*

Then there exists a unique pre-symmetric family of the preopen sets of the topology on X for which S is the collection of preseparated sets.

Proof. Let $pCl(A) = \{x \in X : \{\{x\}, A\} \notin S\}$ for every subset A of X. If $x \notin pCl(A)$, then $\{\{x\}, A\} \in S$ and hence $x \notin A$. Thus, $A \subset pCl(A)$ for each subset A.

If $x \in pCl(A)$, then $\{\{x\}, A\} \notin S$ and hence, $\{\{x\}, B\} \notin S$, that is, $x \in pCl(B)$. Thus, $pCl(A) \subset pCl(B)$ whenever $A \subset B$. In particular, since $A \subset pCl(A)$, $pCl(A) \subset pCl(pCl(A))$.

If $x \in pCl(pCl(A))$, then $\{\{x\}, pCl(A)\} \notin S$ and hence, $\{\{x\}, A\} \notin S$. Furthermore, by the final condition, $pCl(pCl(A)) \subset pCl(A)$ and thus, $pCl(pCl(A)) = pCl(A)$ for each $A \subset X$.

(1) Since $X \subset pCl(X) \subset X$, then $pCl(X) = X$.

(2) If $pCl(A_\alpha) = A_\alpha$, $\forall \alpha \in \Phi$, then $pCl(\cap_{\alpha\in\Phi} A_\alpha) \subset pCl(A_\alpha) = A_\alpha$ for each $\alpha \in \Phi$, since $\cap_{\alpha\in\Phi} A_\alpha \subset A_\alpha$ for each $\alpha \in \Phi$. Hence, $pCl(\cap_{\alpha\in\Phi} A_\alpha) \subset \cap A_\alpha$. Also, since $\cap_{\alpha\in\Phi} A_\alpha \subset pCl(\cap_{\alpha\in\Phi} A_\alpha)$, then $pCl(\cap_{\alpha\in\Phi} A_\alpha) = \cap_{\alpha\in\Phi} A_\alpha$.

(3) If $pCl(A) = A$ and $pCl(B) = B$, then

$$\begin{aligned} pCl(A \cup B) &= \{x \in X : \{\{x\}, A \cup B\} \notin S\} \\ &= \{x \in X : \{\{x\}, A\} \notin S \text{ or } \{\{x\}, B\} \notin S\} \\ &= \{x \in X : \{\{x\}, A\} \notin S\} \cup \{x \in X : \{\{x\}, B\} \notin S\} \\ &= pCl(A) \cup pCl(B) \\ &= A \cup B. \end{aligned}$$

Hence, pCl is the preclosure operator of a topology τ on X. If $y \notin C = \{x \in X : \{\{x\}, C\} \notin S\}$, then $\{\{y\}, C\} \in S$. Thus, points are preseparated from preclosed sets in X.

Suppose that $\{A, B\} \in S$. Then

$$\begin{aligned} A \cap pCl(B) &= A \cap \{x \in X : \{\{x\}, B\} \notin S\} \\ &= \{x \in A : \{\{x\}, B\} \notin S\} \\ &= \emptyset. \end{aligned}$$

Similarly, $pCl(A) \cap B = \emptyset$. Hence, if $\{A, B\} \in S$, then A and B are preseparated in τ.

Now suppose that A and B are preseparated. Then $\{x \in A : \{\{x\}, B\} \notin S\} = A \cap pCl(B) = \emptyset$ and $\{x \in A : \{\{x\}, B\} \notin S\} = pCl(A) \cap B = \emptyset$. Hence, $\{\{x\}, B\} \in S$ for each $x \in A$ and $\{\{y\}, A\} \in S$ for each $y \in B$. Thus, $\{A, B\} \in S$. Hence, S is the collection of pairs of sets preseparated by $PO(X)$ and by Theorem 2.1, S determines $PO(X)$ uniquely.

Example 2.2. Let $X = R$ be the set of real numbers with $S = \{\{A, B\} : \forall x \in A, \exists \varepsilon > 0$ such that $|x - y| \geq \varepsilon, \forall y \in B\}$. Since $x \in pCl(A)$ if and only if for each $\varepsilon > 0$, $|x - a| < \varepsilon$, for some $a \in A$.

pCl_X will denote the preclosure operator of X and S_X will denote the collection of preseparated sets of X.

We have seen that pre-symmetric topological spaces can be viewed entirely in terms of their preseparated sets and it would be nice if we could treat precontinuous functions similarly.

Definition 2.4. Let X and Y be topological spaces and let $f : X \to Y$ be a function. If for all A, B not preseparated in X, $f(A)$ and $f(B)$ are not preseparated in Y, then we say that f is nonpreseparating.

Theorem 2.4. *Let X and Y be topological spaces and let $f : X \to Y$ be a function. Then the following are equivalent:*

(1) *f is nonpreseparating.*

(2) *for all A, $B \subset X$, if $f(A)$ and $f(B)$ are preseparated in Y, then A and B are preseparated.*

(3) *for all preseparated C, $D \subset Y$, $f^{-1}(C)$ and $f^{-1}(D)$ are preseparated in X.*

Proof. (2)⇔(1): Obvious.

(2)⇒(3): Let C and D be preseparated subsets of Y. Let $A = f^{-1}(C)$ and $B = f^{-1}(D)$. Then $f(A) \subset C$ and $f(B) \subset D$ and hence, $f(A)$ and $f(B)$ are preseparated in Y. By (2), A and B are preseparated in X.

(3)⇒(2): Let A, $B \subset X$ such that $C = f(A)$ and $D = f(B)$ are preseparated. Then by (3), $f^{-1}(C)$ and $f^{-1}(D)$ are preseparated and hence, $A \subset f^{-1}(f(A)) = f^{-1}(C)$ and $B \subset f^{-1}(f(B)) = f^{-1}(D)$ are preseparated. ∎

Note that if f is precontinuous, then f is nonpreseparating. Really, for preseparated sets C, D in Y,

$$\begin{aligned} f^{-1}(C) \cap pCl_X(f^{-1}(D)) &\subset f^{-1}(C) \cap f^{-1}(pCl_Y(D)) \\ &= f^{-1}(C \cap pCl_Y(D)) \\ &= \emptyset. \end{aligned}$$

Similarly, $pCl_X(f^{-1}(C)) \cap f^{-1}(D) = \emptyset$.

Theorem 2.5. *Let X and Y be topological spaces with Y pre-symmetric and let $f : X \to Y$ be a function. Then f is precontinuous if and only if f is nonpreseparating.*

Proof. If f is precontinuous, then f is nonpreseparating.

Suppose that f is nonpreseparating. Let $A \subset X$ be nonempty and $y \in f(pCl_X(A))$ and let $x \in pCl_X(A) \cap f^{-1}(\{y\})$. Since $x \in pCl_X(A)$, then $\{\{x\}, A\} \notin S_X$. Since f is nonpreseparating, then $\{\{y\}, f(A)\} \notin S_Y$. Since points are preseparated from disjoint preclosed sets in Y, then $y \in pCl_Y(f(A))$. Thus, $f(pCl_X(A)) \subset pCl_Y(f(A))$ for each $A \subset X$. ■

References

[1] Corson, H. and Michael, E., *Metrizability of certain countable unions*, Illinois J. Math., 8 (1964), 351-360.

[2] EL-Deeb, N., Hasanein, I.A., Mashhour, A.S. and Noiri, T., *On p-regular spaces*, Bull. Math. Soc. Sci. Math. R.S. Roumanie, 27 (1983), 311-315.

[3] Kar, A. and Bhattacharyya, P., *Some weak separation axioms*, Bull. Calcutta Math. Soc., 82(1990), 415-422.

[4] Mashhour, A.S., Abd El-Monsef, M.E. and El-Deeb, S.N., *On precontinuous and weak precontinuous mapping*, Proc. Math. Phys. Soc. Egypt, 53 (1982), 47-53.

[5] Nour, T.M., *Contributions to the Theory of Bitopological Spaces*, Ph.D. Thesis, Univ. of Delhi, 1989.

[6] Pal, M.C. and Bhattacharyya, P., *Feeble and strong forms of preirresolute functions*, Bull. Malaysian Math. Soc., (2) 19 (1996), no. 2, 63-75.

Accepted: 17.01.2007