2014 International Conference on Topology and its Applications,
July 3-7, 2014, Nafpaktos,
Greece

Selected papers of the 2014 International Conference on Topology and its Applications

Editors

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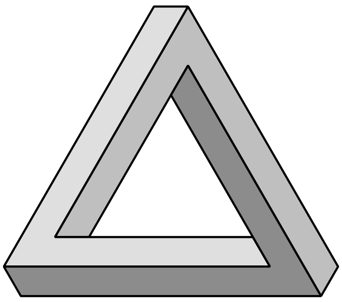

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Preface

The **2014 International Conference on Topology and its Applications** took place from July 3 to 7 in the **3rd High School of Nafpaktos, Greece**. It covered all areas of Topology and its Applications (especially General Topology, Set-Theoretic Topology, Geometric Topology, Topological Groups, Dimension Theory, Dynamical Systems and Continua Theory, Computational Topology, History of Topology). This conference was attended by 235 participants from 44 countries and the program consisted by 147 talks.

The Organizing Committee consisted of S.D. Iliadis (University of Patras), D.N. Georgiou (University of Patras), I.E. Kougias (Technological Educational Institute of Western Greece), A.C. Megaritis (Technological Educational Institute of Western Greece), and I. Boules (Mayor of the city of Nafpaktos).

The Organizing Committee is very much indebted to the City of Nafpaktos for its hospitality and for its excellent support during the conference.

The conference was sponsored by University of Patras, Technological Educational Institute of Western Greece, Municipality of Nafpaktos, New Media Soft – Internet Solutions, Loux Marlafekas A.B.E.E., TAXYTYPO – TAXYEKTYPOSEIS GRAVANIS EPE, Alpha Bank, and Wizard Solutions LTD.

This volume is a special volume under the title: "Selected papers of the 2014 International Conference on Topology and its Applications" which will be edited by the organizers (D.N. Georgiou, S.D. Iliadis, I.E. Kougias, and A.C. Megaritis) and published by the University of Patras. We thank the authors for their submissions.

Editors

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Regularity and normality via $\beta\theta$-open sets

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Abstract

The aim of this paper is to present and study a new type of regularity and normality called $\beta\theta$-regularity and $\beta\theta$-normality, repectively by using $\beta\theta$-open sets.

Key words: Topological spaces, $\beta\theta$-open sets, $\beta\theta$-closed sets, regular spaces, normal spaces, $\beta\theta$-regular spaces, $\beta\theta$-normal spaces.
1991 MSC: 54C10, 54D10.

1. Introduction and preliminaries

It is common viewpoint of many topologists that generalized open sets are important ingredients in General Topology and they are now the research topics of many topologists worldwide of which lots of important and interesting results emerged. Indeed a significant theme in General Topology and Real Analysis concerns the variously modified forms of continuity, separation axioms etc by utilizing generalized open sets. One of the most well-known notions and also an inspiration source is the notion of β-open sets or semipreopen sets introduced by Abd El Monsef et al. [1] and Andrijević [2] respectively. In 2003, Noiri [10] used this notion and the β-closure [1] of a set to introduce the concepts of $\beta\theta$-open and $\beta\theta$-closed sets which provide a formulation of the $\beta\theta$-closure of a set in a topological space. Caldas [4-7] continued the work of Noiri and defined other concepts utilizing $\beta\theta$-closed sets. In this direction we shall study some properties of regularity and normality via $\beta\theta$-open sets and the $\beta\theta$-closure operator.

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Throughout the present paper, spaces (X, τ) and (Y, σ) (or simply, X and Y) denote topological spaces on which no separation axioms are assumed unless explicitly stated. Abd El Monsef et al. [1] and Andrijević [2] introduced the notion of β-open set, which Andrijević called semipreopen, completely independent of each other. In this paper, we adopt the word β-open for the sake of clarity. They characterized the most important properties of β-open sets. A subset A of a topological space (X, τ) is called β-open if $A \subseteq Cl(Int(Cl(A)))$, where $Cl(A)$ and $Int(A)$ denote the closure and the interior of A, respectively. The complement of a β-open set is called β-closed. The intersection of all β-closed sets containing A is called the β-closure of A and is denoted by $\beta Cl(A)$. The family of all β-open (resp. β-closed, open) subsets of X is denoted by $\beta O(X, \tau)$ or $\beta O(X)$ (resp. $\beta C(X, \tau), O(X, \tau)$). We set

$$\beta O(X, x) = \{U : x \in U \in \beta O(X, \tau)\}$$

and

$$\beta C(X, x) = \{U : x \in U \in \beta C(X, \tau)\}.$$

Now we begin to recall some known notions which will be used in the sequel.

Definition 1.1. [10]. Let A be a subset of X. The *$\beta\theta$-closure* of A, denoted by $\beta Cl_\theta(A)$, is the set of all $x \in X$ such that $\beta Cl(O) \cap A \neq \emptyset$ for every $O \in \beta O(X, x)$. A subset A is called *$\beta\theta$-closed* if $A = \beta Cl_\theta(A)$. The set

$$\{x \in X \mid \beta Cl(O) \subset A \text{ for some } O \in \beta O(X, x)\}$$

is called the *$\beta\theta$-interior* of A and is denoted by $\beta Int_\theta(A)$. A subset A is called *$\beta\theta$-open* if $A = \beta Int_\theta(A)$. The family of all $\beta\theta$-open (resp. $\beta\theta$-closed) subsets of X is denoted by $\beta\theta O(X, \tau)$ or $\beta\theta O(X)$ (resp. $\beta\theta C(X, \tau)$). We set $\beta\theta O(X, x) = \{U : x \in U \in \beta\theta O(X, \tau)\}$ and $\beta\theta C(X, x) = \{U : x \in U \in \beta\theta C(X, \tau)\}$.

The following theorem is known and given by Noiri [10].

Theorem 1.2. [10]. For any subset A of X:
(1) $\beta Cl_\theta(\beta Cl_\theta(A)) = \beta Cl_\theta(A)$.
(2) $\beta Cl_\theta(A)$ is $\beta\theta$-closed.
(3) If $A_\alpha \in \beta\theta C(X)$ for each $\alpha \in A$, then $\bigcap\{A_\alpha \mid \alpha \in A\} \in \beta\theta C(X)$.
(4) If $A_\alpha \in \beta\theta O(X)$ for each $\alpha \in A$, then $\bigcup\{A_\alpha \mid \alpha \in A\} \in \beta\theta O(X)$.
(5) $\beta Cl_\theta(A)$ is the intersection of all $\beta\theta$-closed sets each containing A.
(6) $A \subset \beta Cl(A) \subset \beta Cl_\theta(A)$ and $\beta Cl(A) = \beta Cl_\theta(A)$ if $A \in \beta O(X)$.

A function $f : X \to Y$ is said to be:
(i) $\beta\theta$-continuous [10] If $f^{-1}(V)$ is $\beta\theta$-closed for every closed set V in Y, equivalently if the inverse image of every open set V in Y is $\beta\theta$-open in X.
(ii) weakly β-irresolute ([10], Theorem 4.5) if $f^{-1}(V)$ is $\beta\theta$-open in X for every $\beta\theta$-open set V in Y.

2. Maps and $\beta\theta$-regular spaces

Definition 2.1. A topological space is said to be $\beta\theta$-regular if for each closed set F of X and each point $x \in X\backslash F$, there exist disjoint $\beta\theta$-open sets U and V such that $F \subset U$ and $x \in V$.

Theorem 2.2. For a topological space X, the following statements are equivalent:
(1) X is $\beta\theta$-regular.
(2) For each $x \in X$ and each open set U of X containing x, there exists $V \in \beta\theta O(X)$ such that $x \in V \subset \beta Cl_\theta(V) \subset U$.
(3) For each closed set F of X, $\cap\{\beta Cl_\theta(V) \mid F \subset V \in \beta\theta O(X)\} = F$.
(4) For each subset A of X and each open set U of X such that $A \cap U \neq \emptyset$, there exists $V \in \beta\theta O(X)$ such that $A \cap V \neq \emptyset$ and $\beta Cl_\theta(V) \subset U$.
(5) For each nonempty subset A of X and each closed set F of X such that $A \cap F = \emptyset$, there exist $V, W \in \beta\theta O(X)$ such that $A \cap V \neq \emptyset$, $F \subset W$ and $V \cap W = \emptyset$.

Proof. (1)$\Rightarrow$(2): Let U be an open set containing x, then $X\backslash U$ is closed in X and $x \notin X\backslash U$. By (1), there exist $V, W \in \beta\theta O(X)$ such that $x \in V$, $X\backslash U \subset W$ and $V \cap W = \emptyset$. Hence, we have $\beta Cl_\theta(V) \cap W = \emptyset$ and therefore $x \in V \subset \beta Cl_\theta(V) \subset U$.

(2)$\Rightarrow$(3): Let F be a closed set of X. It is obvious that

$$\cap\{\beta Cl_\theta(V) \mid F \subset V \in \beta\theta O(X)\} \supset F.$$

Conversely, let $x \notin F$. Then $X\backslash F$ is an open set containing x. By (2), there exists $U \in \beta\theta O(X)$ such that $x \in U \subset \beta Cl_\theta(U) \subset X\backslash F$. Put $V = X\backslash\beta Cl_\theta(U)$. It follows from Theorem 1.2 that $F \subset V \in \beta\theta O(X)$ and $x \notin \beta Cl_\theta(V)$. This implies that $\cap\{\beta Cl_\theta(V) \mid F \subset V \in \beta\theta O(X)\} \subset F$.

(3)$\Rightarrow$(4): Let $A\cap U \neq \emptyset$ and U be an open set in X. Let $x \in A\cap U$, then $X\backslash U$ is a closed set not containing x. By (3), there exists $W \in \beta\theta O(X)$ such that $X\backslash U \subset W$ and $x \notin \beta Cl_\theta(W)$. Put $V = X\backslash\beta Cl_\theta(W)$. By using Theorem 1.2, we obtain $V \in \beta\theta O(X)$, $x \in V\cap A$ and $\beta Cl_\theta(V) \subset \beta Cl_\theta(X\backslash W) = X\backslash W \subset U$.

(4)$\Rightarrow$(5): Let $A\cap F = \emptyset$ and F be closed in X, where $A \neq \emptyset$. Since $X\backslash F$ is open in X and $A \neq \emptyset$, by (4) there exists $V \in \beta\theta O(X)$ such that $A \cap V \neq \emptyset$ and $\beta Cl_\theta(V) \subset X\backslash F$. Put $W = X\backslash\beta Cl_\theta(V)$. Then, we have $F \subset W \in \beta\theta O(X)$ and $V \cap W = \emptyset$.

(5)$\Rightarrow$(1): The proof is obvious. ■

Definition 2.3. A topological space is said to be:
(1) (β, θ)-regular if for each $\beta\theta$-clopen set F of X and each point $x \in X\backslash F$,

there exist disjoint open sets U and V such that $F \subset U$ and $x \in V$.
(2) Extremally $\beta\theta$-disconnected (briefly, ED^*) if $\beta Cl_\theta(U)$ is $\beta\theta$-open in X for every $\beta\theta$-open set U of X.

Theorem 2.4. For a topological space X, the following statements are equivalent:
(1) X is (β, θ)-regular.
(2) For each $x \in X$ and any $\beta\theta$-clopen set U of X containing x, there exists a $V \in O(X)$ such that $x \in V \subset Cl(V) \subset U$.

Proof. (1)⇒(2): Let U be a $\beta\theta$-clopen set containing x, then $X \backslash U$ is a $\beta\theta$-clopen in X and $x \notin X \backslash U$. By (1), there exist $V, W \in O(X)$ such that $x \in V$, $X \backslash U \subset W$ and $V \cap W = \emptyset$. Hence, we have $Cl(V) \cap W = \emptyset$ and therefore $x \in V \subset Cl(V) \subset U$.

(2)⇒(1): Let F be a $\beta\theta$-clopen set of X and $x \in X \backslash F = U$ (say). Hence $x \in U$ and U is $\beta\theta$-clopen, by (2) there exists a $V \in O(X)$ such that $x \in V \subset Cl(V) \subset U$. Therefore $X \backslash U \subset X \backslash Cl(V)$. Hence $V \cap X \backslash Cl(V) = \emptyset$. Thus shows that X is (β, θ)-regular. ■

Example 2.5. Let (X, τ) be a topological space such that, $X = \{a, b, c\}$, $\tau = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$. Clearly $\beta\theta O(X, \tau) = \{\emptyset, \{b\}, \{c\}, \{a, c\}, \{b, c\}, X\}$. Then (X, τ) is $\beta\theta$-regular, and it is not (β, θ)-regular and not regular.

Theorem 2.6. If a space X is $\beta\theta$-regular, ED^* and (β, θ)-regular, then it is regular.

Proof. Let U be any open subset X and $x \in U$. Since X is $\beta\theta$-regular, there exists $V \in \beta\theta O(X)$ such that $x \in V \subset \beta Cl_\theta(V) \subset U$ (Theorem 2.2). Since X is ED^*, $\beta Cl_\theta(V)$ is $\beta\theta$-clopen and since X is (β, θ)-regular, there exists an open subset O of X such that $x \in O \subset Cl(O) \subset \beta Cl_\theta(V)$. Hence $x \in O \subset Cl(O) \subset U$. Thus shows that X is regular. ■

Definition 2.7. [4,7] A function $f : X \to Y$ is said to be:
(1) $\beta\theta$-closed (resp. pre-$\beta\theta$-closed), if the image of each closed (resp. $\beta\theta$-closed) set F in X is $\beta\theta$-closed in Y.
(2) $\beta\theta$-open (resp. pre-$\beta\theta$-open), if the image of each open (resp. $\beta\theta$-open) set U in X is $\beta\theta$-open in Y.

Pre-$\beta\theta$-open functions are independent of $\beta\theta$-open functions as it can be seen from the following example.

Example 2.8. (i) Let (X, τ) be a topological space such that $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, X\}$. Let $f : (X, \tau) \to (X, \tau)$ be the identity function. Then f is pre-$\beta\theta$-open but it is not $\beta\theta$-open.

(ii) Let (X, τ) be a topological space such that $X = \{a, b, c\}$ and

$$\tau = \{\emptyset, \{c\}, \{a, b\}, X\}.$$

Let $f : (X, \tau) \to (X, \tau)$ be defined by $f(a) = c$, $f(c) = a$ and $f(b) = b$. Then f is $\beta\theta$-open but it is not pre-$\beta\theta$-open.

Lemma 2.9. A function $f : X \to Y$ is $\beta\theta$-closed (resp. pre-$\beta\theta$-closed) if and only if for each subset B of Y and each open (resp. $\beta\theta$-open) set U containing $f^{-1}(B)$ there exists a $\beta\theta$-open set V of Y containing B such that $f^{-1}(V) \subset U$.

Proof. Necessity: Suppose that f is $\beta\theta$-closed. Let $B \subset Y$ and $U \in O(X)$ containing $f^{-1}(B)$. Put $V = Y \backslash f(X \backslash U)$. Then we obtain a $\beta\theta$-open set V of Y containing B such that $f^{-1}(V) \subset U$.
Sufficiency: Let F be any closed set of X. Set $f(F) = B$, then $F \subset f^{-1}(B)$ and $f^{-1}(Y \backslash B) \subset X \backslash F \in O(X)$. By hypothesis, there exists $V \in \beta\theta O(Y)$ such that $Y \backslash B \subset V$ and $f^{-1}(V) \subset X \backslash F$. Therefore we obtain $Y \backslash V \subset f(F) \subset Y \backslash V$. Hence $f(F) = Y \backslash V$ and $f(F)$ is $\beta\theta$-closed in Y. Therefore f is $\beta\theta$-closed.
The other case is analogous. ■

Theorem 2.10. Let $f : X \to Y$ be a continuous $\beta\theta$-closed surjection with compact point inverses. If X is regular, then Y is $\beta\theta$-regular.

Proof. Let F be a closed set of Y and $y \notin F$. Then $f^{-1}(F)$ is closed in X and $f^{-1}(y)$ is a compact set. Moreover $f^{-1}(F)$ and $f^{-1}(y)$ are disjoint in the regular space X. Hence, there exist disjoint open sets U_y and U_F such that $f^{-1}(y) \subset U_y$ and $f^{-1}(F) \subset U_F$. Since f is $\beta\theta$-closed by Lemma 2.9, there exist $V_y, V_F \in \beta\theta O(X)$ such that $y \in V_y$ and $F \subset V_F$, $f^{-1}(V_y) \subset U_y$ and $f^{-1}(V_F) \subset U_F$. Since $U_y \cap U_F = \emptyset$ and f is surjective, we obtain $V_y \cap V_F = \emptyset$. This show that Y is $\beta\theta$-regular. ■

A subset A of X is said to be g$\beta\theta$-closed if $\beta Cl_\theta(A) \subset U$ whenever $A \subset U$ and U is open in X. A subset of a space (X, τ) is said to be g$\beta\theta$-open if $X \backslash A$ is g$\beta\theta$-closed.

Lemma 2.11. A subset A of a space (X, τ) is g$\beta\theta$-open if and only if $F \subset \beta Int_\theta(A)$ whenever $F \subset A$ and F is closed.

Proof. Necessity. Suppose that A is g$\beta\theta$-open. Let $F \subset A$ and F be closed in (X, τ). Then $X \backslash A \subset X \backslash F$ and $X \backslash F$ is open. Therefore, $\beta Cl_\theta(X \backslash A) \subset X \backslash F$ and hence $F \subset \beta Int_\theta(A)$.
Sufficiency. If F is a closed set with $F \subset \beta Int_\theta(A)$ whenever $F \subset A$. Then it follows that $X \backslash A \subset X \backslash F$ and $X \backslash \beta Int_\theta(A) \subset X \backslash F$, i.e. $\beta Cl_\theta(X \backslash A) \subset X \backslash F$. Therefore $X \backslash A$ is g$\beta\theta$-closed. Thus, A is g$\beta\theta$-open. ■

A function $f : X \to Y$ is said to be:
(1) generalized $\beta\theta$-closed (briefly g$\beta\theta$-closed) if for each closed set F of X, $f(F)$ is g$\beta\theta$-closed.

(2) pre generalized $\beta\theta$-closed (briefly pre g$\beta\theta$-closed) if for each g$\beta\theta$-closed set F of X, $f(F)$ is g$\beta\theta$-closed.
(3) pre generalized $\beta\theta$-open (briefly pre g$\beta\theta$-open) if for each g$\beta\theta$-open set F of X, $f(F)$ is g$\beta\theta$-open.

Lemma 2.12. A function $f : X \to Y$ is g$\beta\theta$-closed (resp. pre g$\beta\theta$-closed) if and only if for each subset B of Y and each open (resp. g$\beta\theta$-open) set U containing $f^{-1}(B)$, there exists a g$\beta\theta$-open set V of Y containing B such that $f^{-1}(V) \subset U$.

Proof. The proof is similar to that of Lemma 2.9. ■

Theorem 2.13. If $f : X \to Y$ is a continuous $\beta\theta$-open and g$\beta\theta$-closed surjection from a regular space X onto a space Y, then Y is $\beta\theta$-regular.

Proof. Let U be a open set containing a point y in Y. Let x be a point of X such that $y = f(x)$. It follows from assumptions that there is an open set V such that $x \in V \subset Cl(V)) \subset f^{-1}(U)$. Then since $y \in f(V) \subset f(Cl(V) \subset U$ and $f(Cl(V))$ is g$\beta\theta$-closed, we have $\beta Cl_\theta(f(Cl(V)) \subset U$. Therefore $y \in f(V) \subset \beta Cl_\theta(f(V) \subset U$ and $f(V)$ is $\beta\theta$-open Y. Hence by Theorem 2.2, Y is $\beta\theta$-regular. ■

3. Maps and $\beta\theta$-normal spaces

Definition 3.1. A topological space is said to be $\beta\theta$-normal if for any pair of disjoint closed sets F_1 and F_2 of X, there exist disjoint $\beta\theta$-open sets U_1 and U_2 such that $F_1 \subset U_1$ and $F_2 \subset U_2$.

We have the following characterizations of $\beta\theta$-normality.

Theorem 3.2. For a topological space X, the following statements are equivalent:
(1) X is $\beta\theta$-normal.
(2)For every pair of open sets U and V whose union is X, there exist $\beta\theta$-closed sets A and B such that $A \subset U$, $B \subset V$ and $A \cup B = X$.
(3) For each closed set F and every open set G containing F, there exists a $\beta\theta$-open set U such that $F \subset U \subset \beta Cl_\theta(U) \subset G$.

Proof. (1)⇒(2): Let U and V be a pair of open sets in a $\beta\theta$-normal space X such that $X = U \cup V$. Then $X\backslash U$, $X\backslash V$ are disjoint closed sets. Since X is $\beta\theta$-normal, there exist disjoint $\beta\theta$-open sets U_1 and V_1 such that $X\backslash U \subset U_1$ and $X\backslash V \subset V_1$. Let $A = X\backslash U_1$, $B = X\backslash V_1$. Then A and B are $\beta\theta$-closed sets such that $A \subset U$, $B \subset V$ and $A \cup B = X$.

(2)⇒(3): Let F be a closed set and G be an open set containing F. Then $X\backslash F$

and G are open sets whose union is X. Then by (2), there exist $\beta\theta$-closed sets W_1 and W_2 such that $W_1 \subset X\backslash F$ and $W_2 \subset G$ and $W_1 \cup W_2 = X$. Then $F \subset X\backslash W_1$, $X\backslash G \subset X\backslash W_2$ and $(X\backslash W_1) \cap (X\backslash W_2) = \emptyset$. Let $U = X\backslash W_1$ and $V = X\backslash W_2$. Then U and V are disjoint $\beta\theta$-open sets such that $F \subset U \subset X\backslash V \subset G$. As $X\backslash V$ is a $\beta\theta$-closed set, we have $\beta Cl_\theta(U) \subset X\backslash V$ and $F \subset U \subset \beta Cl_\theta(U) \subset G$.

(3)$\Rightarrow$(1): Let F_1 and F_2 be any two disjoint closed sets of X. Put $G = X\backslash F_2$, then $F_1 \subset G$ where G is an open set. Thus by (3) there exists a $\beta\theta$-open set U of X such that $F_1 \subset U \subset \beta Cl_\theta(U) \subset G$. It follows that $F_2 \subset X\backslash \beta Cl_\theta(U) = V$ (say), then V is $\beta\theta$-open and $U \cap V = \emptyset$. Hence F_1 and F_2 are separated by $\beta\theta$-open sets U and V. Therefore X is $\beta\theta$-normal. ■

Theorem 3.3. For a topological space (X, τ), the following properties are equivalent:
(1) X is $\beta\theta$-normal;
(2) For each pair of open sets U and V whose union is X, there exist g$\beta\theta$-closed sets A and B such that $A \subset U$, $B \subset V$ and $A \cup B = X$;
(3) For each closed set F and every open set G containing F, there exists a g$\beta\theta$-open set U such that $F \subset U \subset \beta Cl_\theta(U) \subset G$;
(4) For each pair of disjoint closed sets F_1 and F_2 of X, there exist disjoint g$\beta\theta$-open sets U_1 and U_2 such that $F_1 \subset U_1$ and $F_2 \subset U_2$.

Proof. (1) $\Rightarrow$ (2): The proof is obvious by Theorem 3.2 since every $\beta\theta$-closed set is g$\beta\theta$-closed.

(2) $\Rightarrow$ (3): Let F be a closed set and G an open set containing F. Then $X\backslash F$ is open and $(X\backslash F) \cup G = X$. By (2), there exist g$\beta\theta$-closed sets W_1 and W_2 such that $W_1 \subset X\backslash F$, $W_2 \subset G$ and $W_1 \cup W_2 = X$. Then $F \subset X\backslash W_1$, $X\backslash G \subset X\backslash W_2$ and $(X\backslash W_1) \cap (X\backslash W_2) = \emptyset$. Since $X\backslash W_1$ and $X\backslash W_2$ are g$\beta\theta$-open, put $U = \beta Int_\theta(X\backslash W_1)$ and $V = \beta Int_\theta(X\backslash W_2)$. Then U and V are $\beta\theta$-open sets such that $F \subset U$, $X\backslash G \subset V$ and $U \cap V = \emptyset$. Since $\beta Cl_\theta(U) \cap V = \emptyset$, we obtain $F \subset U \subset \beta Cl_\theta(U) \subset X\backslash V \subset G$.

(3) $\Rightarrow$ (4): Let F_1 and F_2 be disjoint closed sets. Then $X\backslash F_2$ is an open set containing F_1. By (3), there exists a g$\beta\theta$-open set U_1 such that $F \subset U_1 \subset \beta Cl_\theta(U_1) \subset X\backslash F_2$. Now, put $U_2 = X\backslash \beta Cl_\theta(U_1)$. Then U_2 is $\beta\theta$-open and hence g$\beta\theta$-open. Moreover, we have $F_2 \subset U_2$ and $U_1 \cap U_2 = \emptyset$.

(4) $\Rightarrow$ (1): Let F_1 and F_2 be disjoint closed sets. By (4), there exist disjoint g$\beta\theta$-open sets U_1 and U_2 such that $F_1 \subset U_1$ and $F_2 \subset U_2$. By Lemma 2.11, we have $F_1 \subset \beta Int_\theta(U_1)$ and $F_2 \subset \beta Int_\theta(U_2)$. Put $V_1 = \beta Int_\theta(U_1)$ and $V_2 = \beta Int_\theta(U_2)$. Then V_1, V_2 are $\beta\theta$-open, $F_1 \subset V_1$, $F_2 \subset V_2$ and $V_1 \cap V_2 = \emptyset$. This shows that (X, τ) is $\beta\theta$-normal. ■

Recall that a topological space (X, τ) is said to be $\beta\theta$-R_0 [4] if every open set of the space contains the $\beta\theta$-closure of each of its singletons.

Theorem 3.4. If X is $\beta\theta$-normal and $\beta\theta$-R_0, then X is $\beta\theta$-regular.

Proof. Let F be closed and $x \notin F$. Then $x \in X \backslash F \in O(X)$ which implies $\beta Cl_\theta(\{x\}) \subset X \backslash F$ and there exist disjoint $\beta\theta$-open sets U and V such that $x \in \beta Cl_\theta(\{x\}) \subset U$ and $F \subset V$. ■

Definition 3.5. A function $f : X \to Y$ is said to be almost-$\beta\theta$-irresolute if $f(\beta Cl_\theta(U)) = \beta Cl_\theta(f(U))$ for every $U \in \beta\theta O(X)$.

Theorem 3.6. If $f : X \to Y$ is a pre-$\beta\theta$-open continuous almost-$\beta\theta$-irresolute function from a $\beta\theta$-normal space X onto a space Y, then Y is $\beta\theta$-normal.

Proof. Let A be a closed subset of Y and B be an open set containing A. Then by continuity of f, $f^{-1}(A)$ is closed and $f^{-1}(B)$ is an open set of X such that $f^{-1}(A) \subset f^{-1}(B)$. Since X is $\beta\theta$-normal, there exists a $\beta\theta$-open U in X such that $f^{-1}(A) \subset U \subset \beta Cl_\theta(U)) \subset f^{-1}(B)$ by Theorem 3.2. Then $f(f^{-1}(A)) \subset f(U) \subset f(\beta Cl_\theta(U)) \subset f(f^{-1}(B))$. Since f is pre-$\beta\theta$-open and almost-$\beta\theta$-irresolute surjection, we obtain $A \subset f(U) \subset \beta Cl_\theta(f(U)) \subset B$. Then again by Theorem 3.2 the space Y is $\beta\theta$-normal. ■

Theorem 3.7. If $f : X \to Y$ is a pre-g$\beta\theta$-closed continuous function from a $\beta\theta$-normal space X onto a space Y, then Y is $\beta\theta$-normal.

Proof. Let A and B be disjoint closed sets in Y. Since X is $\beta\theta$-normal and f continuous, there exist disjoint $\beta\theta$-open sets U and V of X such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. By Lemma 2.12, there exist g$\beta\theta$-open sets G and H of Y such that $A \subset G$, $B \subset H$, $f^{-1}(G) \subset U$ and $f^{-1}(H) \subset V$. Since U and V are disjoint, so are G and H. It follows from Theorem 3.3 that Y is $\beta\theta$-normal. ■

Theorem 3.8. If $f : X \to Y$ is a g$\beta\theta$-closed continuous function from a normal space X onto a space Y, then Y is $\beta\theta$-normal.

Proof. The proof is almost analogous to Theorem 3.7. ■

Definition 3.9. [5,8] A topological space (X, τ) is said to be:
1) $\beta\theta$-T_0 (resp. $\beta\theta$-T_1) if for any distinct pair of points x and y in X, there is a $\beta\theta$-open U in X containing x but not y or (resp. and) a $\beta\theta$-open set V in X containing y but not x.
2) $\beta\theta$-T_2 (resp. β-T_2 [9]) if for every pair of distinct points x and y, there exist two $\beta\theta$-open (resp. β-open) sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Theorem 3.10. [5,7,8]. For a topological space (X, τ), the following properties are equivalent:
1) (X, τ) is $\beta\theta$-T_0;
2) (X, τ) is $\beta\theta$-T_1;

3) (X, τ) is $\beta\theta$-T_2;
4) (X, τ) is β-T_2;
5) For every pair of distinct points $x, y \in X$, there exist $U, V \in \beta O(X)$ such that $x \in U$, $y \in V$ and $\beta Cl(U) \cap \beta Cl(V) = \emptyset$;
6) For every pair of distinct points $x, y \in X$, there exist $U, V \in \beta R(X)$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.
7) For every pair of distinct points $x, y \in X$, there exist $U \in \beta\theta O(X, x)$ and $V \in \beta\theta O(X, y)$ such that $\beta Cl_\theta(U) \cap \beta Cl_\theta(V) = \emptyset$.

Definition 3.11. (i) A topological space (X, τ) is said to be weakly Hausdorff [13] (briefly weakly-T_2) if every point of X is an intersection of regular closed sets of X.
(ii) A subset A of a space X is said to be S-closed relative to X [11] if for every cover $\{V_\alpha \mid \alpha \in \nabla\}$ of A by semi-open sets of X, there exists a finite subset ∇_0 of ∇ such that $A \subset \bigcup\{Cl(V_\alpha) \mid \alpha \in \nabla_0\}$. A space X is said to be S-closed if X is S-closed relative to X.

Now in view of ([12], Lemma 2.2) and Lemma 2.9, we prove the following result.

Theorem 3.12. If $f : X \to Y$ is a pre-$\beta\theta$-closed function from a weakly Hausdorff $\beta\theta$-normal space X onto a space Y and $f^{-1}(y)$ is S-closed relative to X for each $y \in Y$, then Y is $\beta\theta$-T_2.

Proof. Let y_1 and y_2 be any two distinct points of Y. Since X is weakly Hausdorff and $f^{-1}(y_i)$ is S-closed relative to X for $i = 1, 2$, by Lemma 2.2 of [12] $f^{-1}(y_1)$ and $f^{-1}(y_2)$ are disjoint closed subsets of X. As X is $\beta\theta$-normal, there exist disjoint $\beta\theta$-open sets V_1 and V_2 of X such that $f^{-1}(y_i) \subset V_i$ for $i = 1, 2$. Since f is pre-$\beta\theta$-closed, by Lemma 2.9 there exist $\beta\theta$-open sets U_1 and U_2 of Y containing y_1 and y_2 respectively such that $f^{-1}(U_i) \subset V_i$ for $i = 1, 2$. Then it follows that $U_1 \cap U_2 = \emptyset$. Hence Y is $\beta\theta$-T_2. ■

Theorem 3.13. If $f : X \to Y$ is a weakly β-irresolute closed injection from a space X to a $\beta\theta$-normal space Y, then X is $\beta\theta$-normal.

Proof. Let Y be $\beta\theta$-normal. Let A and B be two closed subsets of X. Since f is closed and injective, $f(A)$ and $f(B)$ are disjoint closed subsets of Y. Therefore, there exist disjoint $\beta\theta$-open subsets U and V of Y such that $f(A) \subset U$ and $f(B) \subset V$. Now $f^{-1}(U)$ and $f^{-1}(V)$ are $\beta\theta$-open subsets of X such that $A \subset f^{-1}(U)$, $B \subset f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Thus X is $\beta\theta$-normal. ■

Theorem 3.14. If $f : X \to Y$ is a $\beta\theta$-continuous closed injection from a space X to a normal space Y, then X is $\beta\theta$-normal.

Proof. The proof is analogous to Theorem 3.13. ■

4. $(\beta, \theta)^*$-normal spaces

Definition 4.1. A topological space is said to be $(\beta, \theta)^*$-normal if for any pair of disjoint $\beta\theta$-closed sets F_1 and F_2 of X, there exist disjoint $\beta\theta$-open sets U_1 and U_2 such that $F_1 \subset U_1$ and $F_2 \subset U_2$.

Definition 4.2. A subset A of a topological space (X, τ) is said to be (β, θ)-closed if $\beta Cl_\theta(A) \subset U$ whenever $A \subset U$ and U is $\beta\theta$-open. The complement of a (β, θ)-closed set is said to be (β, θ)-open.

Lemma 4.3. A subset A of a space (X, τ) is (β, θ)-open if and only if $F \subset \beta Int_\theta(A)$ whenever $F \subset A$ and F is $\beta\theta$-closed.

Proof. The proof is similar to that of Lemma 2.9. ■

Theorem 4.4. For a topological space X, the following statements are equivalent:
(1) X is $(\beta, \theta)^*$-normal;
(2) For every pair of $\beta\theta$-open sets U and V whose union is X, there exist $\beta\theta$-closed sets A and B such that $A \subset U$, $B \subset V$ and $A \cup B = X$;
(3) For each $\beta\theta$-closed set F and every $\beta\theta$-open set G containing F, there exists a $\beta\theta$-open set U such that $F \subset U \subset \beta Cl_\theta(U) \subset G$;
(4) For each pair of $\beta\theta$-open sets U and V whose union is X, there exist (β, θ)-closed sets A and B such that $A \subset U$, $B \subset V$ and $A \cup B = X$;
(5) For each $\beta\theta$-closed set F and every $\beta\theta$-open set G containing F, there exists a (β, θ)-open set U such that $F \subset U \subset \beta Cl_\theta(U) \subset G$;
(6) For each pair of disjoint $\beta\theta$-closed sets F_1 and F_2 of X, there exist disjoint (β, θ)-open sets U_1 and U_2 such that $F_1 \subset U_1$ and $F_2 \subset U_2$.

Proof. The proof is similar to proofs of Theorem 3.2 and Theorem 3.3. ■

Definition 4.5. A function $f : X \to Y$ is said to be $(\beta, \theta)^*$-closed if for any $\beta\theta$-closed set F of X, $f(F)$ is (β, θ)-closed in Y.

Lemma 4.6. A function $f : X \to Y$ is $(\theta, \beta)^*$-closed if and only if for each subset B of Y and each $\beta\theta$-open set U containing $f^{-1}(B)$, there exists a (β, θ)-open set V of Y containing B such that $f^{-1}(V) \subset U$.

Proof. It is similar to the proof of Lemma 2.9. ■

Theorem 4.7. If a function $f : X \to Y$ is $(\theta, \beta)^*$-closed and weakly β-irresolute surjection and X is $(\beta, \theta)^*$-normal, then Y is $(\beta, \theta)^*$-normal.

Proof. Let K and L be disjoint $\beta\theta$-closed sets of Y. Since f is weakly β-irresolute, $f^{-1}(K)$ and $f^{-1}(L)$ are disjoint $\beta\theta$-closed sets. Since X is $(\beta, \theta)^*$-normal, there exist disjoint $\beta\theta$-open sets U and V such that $f^{-1}(K) \subset U$ and $f^{-1}(L) \subset V$. By Lemma 4.6, there exist (β, θ)-open sets G and H of Y such that $K \subset G$, $L \subset H$, $f^{-1}(G) \subset U$ and $f^{-1}(H) \subset V$. Since U and V are disjoint, G and H are disjoint. By Theorem 4.4, Y is $(\beta,\theta)^*$-normal. ■

Theorem 4.8. If a function $f : X \to Y$ is pre $\beta\theta$-closed, weakly β-irresolute injection and Y is $(\beta,\theta)^*$-normal, then X is $(\theta,\beta)^*$-normal.

Proof. Let A and B be disjoint $\beta\theta$-closed sets of X. Since f is pre $\beta\theta$-closed and injective, $f(A)$ and $f(B)$ are disjoint $\beta\theta$-closed sets of Y. Since Y is $(\beta,\theta)^*$-normal, there exist disjoint $\beta\theta$-open sets U and V of Y such that $f(A) \subset U$ and $f(B) \subset V$. Therefore, we have $A \subset f^{-1}(U)$ and $B \subset f^{-1}(V)$. Since f is weakly β-irresolute, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint $\beta\theta$-open sets of X. Therefore, X is $(\beta,\theta)^*$-normal. ■

A well known characterization of a normal space is that a space is normal if and only if every point-finite open cover is shrinkable. We prove a parallel characterization for $(\beta,\theta)^*$-normal spaces.

Definition 4.9. A cover $\mho = \{U_i : i \in I\}$ of a space X is said to be (β,θ)-shrinkable if there exists a $\beta\theta$-open cover $\surd = \{V_i : i \in I\}$ of X such that $\beta Cl_\theta(V_i) \subset U_i$ for each $i \in I$. Then the cover $\surd$ is called a (β,θ)-shrinking of $\mho$.

A covering $\mho$ is point-finite provided each $x \in X$ belongs to only finitely many elements of $\mho$.

Theorem 4.10. A space X is $(\beta,\theta)^*$-normal if and only if every point-finite $\beta\theta$-open cover is (β,θ)-shrinkable.

Proof. Suppose that X is $(\beta,\theta)^*$-normal. Let $\mho = \{U_i : i \in I\}$ be a point-finite $\beta\theta$-open cover of X. Well order the set $I = \{i_1, i_2, ..., i, ...\}$. Now construct $\{V_i : i \in I\}$ by transfinite induction as follows: Let $F_{i_1} = X \backslash (\bigcup_{i>i_1} U_i)$. Each U_i being $\beta\theta$-open, $\bigcup_{i>i_1} U_i$ is $\beta\theta$-open and hence F_{i_1} is $\beta\theta$-closed. Also $F_{i_1} \subset U_{i_1}$. Therefore in view of Theorem 4.2, there exists a $\beta\theta$-open set V_{i_1} such that $F_{i_1} \subset V_{i_1} \subset \beta Cl_\theta(V_{i_1}) \subset U_{i_1}$. Let $F_{i_2} = X \backslash \{V_{i_1} \cup (\bigcup_{i>i_2} U_i)\}$. Then F_{i_2} is a $\beta\theta$-closed set such that $F_{i_2} \subset U_{i_2}$. Therefore there exists a $\beta\theta$-open set V_{i_2} such that $F_{i_2} \subset V_{i_2} \subset \beta Cl_\theta(V_{i_2}) \subset U_{i_2}$. Let us suppose that similarly V_j has been defined for each $j < i$. Let $F_i = X \backslash \{(\bigcup_{j<i} V_j) \cup (\bigcup_{k>i} U_k)\}$. Then F_i is a $\beta\theta$-closed set such that $F_i \subset U_i$. Hence we have a $\beta\theta$-open set V_i such that $F_i \subset V_i \subset \beta Cl_\theta(V_i) \subset U_i$. Now consider the family $\surd = \{V_i : i \in I\}$. Let $\surd$ be a cover of X. Let $x \in X$. Then, since $\mho$ is a point-finite cover, $x \in U_i$ for finitely many i's , say $i_1, i_2, ..., i_n$. Let $k = max\{i_1, i_2, ..., i_n\}$. Then $x \notin U_j$ for $j > k$. So if $x \notin V_j$ for $j < k$, then $x \in F_k = X \backslash \{(\bigcup_{j<k} V_j) \cup (\bigcup_{i>k} U_i)\} \subset V_k$. Hence in any case $x \in V_i$ for $i \leq k$. Also each V_i is $\beta\theta$-open and $\beta Cl_\theta(V_i) \subset U_i$ for each $i \in I$. Thus $\surd$ is a (β,θ)-shrinking of $\mho$.
For the converse, let F_1 and F_2 be disjoint $\beta\theta$-closed subsets of X. Then

$\{X\backslash F_1, X\backslash F_2\}$ is a point-finite cover of X. But any (β, θ)-shrinking $\{V_1, V_2\}$ of $\{X\backslash F_1, X\backslash F_2\}$ induces a separation $X\backslash \beta Cl_\theta(V_1)$, $X\backslash \beta Cl_\theta(V_2)$ of F_1 and F_2. ■

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