Point-free topological monoids and Hopf algebras on locales and frames

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Abstract

In this note, we are intended to offer some theoretical consideration concerning the introduction of point-free topological monoids on the locales and frames. Moreover, we define a quantum group on locales by utilizing the Drinfeld-Jimbo group.

1 Introduction

Topological groups(or monoids) are very important mathematical objects with not only applications in mathematics, for example in Lie group theory, but also in physics. Topological groups are defined on topological spaces that involve points as fundamental objects of such topological spaces. In the past decades, a new form of topology which does not use the notion of points was developed. This approach is called point-free topology. Instead of a collection of sets, frames and locales are the basic objects used in this theory. In topology, continuous groups(or monoids) defined on topological spaces are also well-known. An example of such groups are Lie groups. The open question is whether topological groups(or monoids) can be defined in point-free topology. Also most properties of such groups are unknown. It is known that localic groups can be defined on locales in [1].

2 Theoretical Considerations

A frame is a lattice that satisfies the following distributive law for N lattices L_k with $k \in \{1, \dots, N\}$ and another lattice M:

$$M \vee \wedge_{k=1}^{N} L_k = \wedge_{k=1}^{N} (M \vee L_k). \tag{1}$$

A locale is a frame where map $c : P \to Q$ between locales P and Q induces a map $d : R \to S$ between the frames R and S, i.e. the category of locales is the opposite category of the category of frames. Let A be a locale. Then this locale can be represented as a meet of n distinct sublocales $A_i \subset A$ with $i \in \{1, \dots, n\}$ since locales must be closed under finite meet operations (analogous to a topological space that must be closed under finite intersections). In set-theoretic topology, intersections and unions must be preserved if a continuous map $f^{-1} : X \to Y$ between topological spaces X and Y is applied. On the other hand, if the inverse of f^{-1} is continuous, also unions and intersections must be preserved. Analogously, if there is a map $g : A \to B$ between the locales A and B, meet and join operations must be preserved. Let Ω be a functor that maps from the category of topological spaces to locales. Then the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\Omega}\downarrow & & {\scriptstyle\Omega}\downarrow \\ B & \xrightarrow{f^*} & A \end{array} \tag{2}$$

We note that the commutative diagram (2) can also be formulated in terms of its inverse maps. Therefore, the map g lies in a group G if also the identity element e lies in this group. Hence, maps between locales that represent continuous maps between topological space in its corresponding spectrum are topological groups. This is true since the corresponding spectrum of these topological groups are of continuous maps and topological groups. Obviously it must be continuous groups.

3 Quantum groups on locales

In this section, we will define a quantum group on locales using the Drinfeld-Jimbo group (Se [2] and [3]).

Let $A = [a_{ij}]$ be the Cartan matrix of the Kac-Moody algebra, and let $q \neq 1$ be a non-zero complex number. Then $U_q(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra whose Cartan matrix is A, is defined by the following generators k_λ, where λ is an element of the weight lattice, i.e. $2(\lambda, a_i)/(a_i, a_i)$ is an integer for all i, and e_i and f_i for associated simple roots a_i, and relations

$$k_0 = 1, \quad k_\lambda k_\mu = k_{\lambda+\mu},$$

$$k_\lambda e_i k_\lambda^{-1} = q^{(\lambda, \alpha_i)} e_i, \quad k_\lambda f_i k_\lambda^{-1} = q^{-(\lambda, \alpha_i)} f_i,$$

$$[e_i, f_j] = \delta_{ij} \frac{k_i - {k_i}^{-1}}{q_i - {q_i}^{-1}}.$$

For $i \neq j$, we have

$$\sum_{n=0}^{1-a_{ij}} (-1)^n \frac{[1-a_{ij}]_{q_i}!}{[1-a_{ij}-n]_{q_i}![n]_{q_i}!} {e_i}^n e_j {e_i}^{1-a_{ij}-n} = 0,$$

$$\sum_{n=0}^{1-a_{ij}} (-1)^n \frac{[1-a_{ij}]_{q_i}!}{[1-a_{ij}-n]_{q_i}![n]_{q_i}!} {f_i}^n f_j {f_i}^{1-a_{ij}-n} = 0,$$

where $k_i = k_{\alpha_i}, q_i = q^{\frac{1}{2}(\alpha_i,\alpha_i)}, [0]_{q_i}! = 1, [n]_{q_i}! = \prod_{m=1}^{n}[m]_{q_i}$ for all positive integers n, and $[m]_{q_i} = \frac{q_i^m - q_i^{-m}}{q_i - q_i^{-1}}$. These are the q-factorial and q-number, respectively, the q-analogs of the ordinary factorial. The last two relations above are the q-Serre relations, the deformations of the Serre relations.

In the limit as $q \to 1$, these relations approach the relations for the universal enveloping algebra $U(G)$, where $k \to 1$ and $\frac{k_\lambda - k_{-\lambda}}{q - q^{-1}} \to t_\lambda$ as $q \to 1$, where the element, t_λ, of the Cartan subalgebra satisfies $(t_\lambda, h) = \lambda(h)$ for all h in the Cartan subalgebra.

There are various coassociative coproducts under which these algebras are Hopf algebras, for example,

$$\Delta_1(k_\lambda) = k_\lambda \otimes k_\lambda, \quad \Delta_1(e_i) = 1 \otimes e_i + e_i \otimes k_i, \quad \Delta_1(f_i) = k_i^{-1} \otimes f_i + f_i \otimes 1,$$

$$\Delta_2(k_\lambda) = k_\lambda \otimes k_\lambda, \quad \Delta_2(e_i) = k_i^{-1} \otimes e_i + e_i \otimes 1, \quad \Delta_2(f_i) = 1 \otimes f_i + f_i \otimes k_i,$$

$$\Delta_3(k_\lambda) = k_\lambda \otimes k_\lambda, \quad \Delta_3(e_i) = k_i^{-\frac{1}{2}} \otimes e_i + e_i \otimes k_i^{1/2}, \quad \Delta_3(f_i) = k_i^{-\frac{1}{2}} \otimes f_i + f_i \otimes k_i^{1/2},$$

where the set of generators has been extended, if required, to include k_λ for λ which is expressible as the sum of an element of the weight lattice and half an element of the root lattice.

In addition, any Hopf algebra leads to another with reversed coproduct $T \circ \Delta$, where T is given by $T(x \otimes y) = y \otimes x$, giving three more possible versions.

The counit on $U_q(A)$ is the same for all these coproducts: $\varepsilon(k_\lambda) = 1, \varepsilon(e_i) = \varepsilon(f_i) = 0$, and the respective antipodes for the above coproducts are given by

$$S_1(k_\lambda) = k_{-\lambda},\ S_1(e_i) = -e_i k_i^{-1},\ S_1(f_i) = -k_i f_i,$$
$$S_2(k_\lambda) = k_{-\lambda},\ S_2(e_i) = -k_i e_i,\ S_2(f_i) = -f_i k_i^{-1},$$
$$S_3(k_\lambda) = k_{-\lambda},\ S_3(e_i) = -q_i e_i,\ S_3(f_i) = -q_i^{-1} f_i.$$

Alternatively, the quantum group $U_q(G)$ can be regarded as an algebra over the field $\mathbf{C}(q)$, the field of all rational functions of an indeterminate q over $\mathbf{C}$.

Similarly, the quantum group $U_q(G)$ can be regarded as an algebra over the field $\mathbf{Q}(q)$, the field of all rational functions of an indeterminate q over $\mathbf{Q}$.

We will regard all elements (e.g. the generators) of this Drinfeld-Jimbo algebra as elements of a locale. A Locale element satisfying above relations is a locale, where a product between locales exists.

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