Compact Manifolds Representation, a New Approach

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Abstract

In this paper we discuss a new method for representing Compact Manifolds.

Key Words: compact manifold, topology.

1 Introduction

This is not a formal mathematical paper but rather the outline of a possible method that requires further mathematical research. Some reasonable propositions are provided but, till a formal proof is given and as always happen in math, appearances may be deceiving. However, if those propositions turn out to be true, the method we propose may result very interesting.

In this paper we discuss a new method for representing Compact Manifolds. The methods will be applied to 2-dimensional and 3-dimensional manifolds but it may be applied to higher dimensions manifolds.

One advantage of this method is that it reduced by one the dimension of the problem. For example, 3D closed manifolds are represented by 2D closed surfaces, which are in general not manifolds.

2 Two Dimensional Manifolds

2.1 Definitions

Definition 2.1: Thick line
Let X a closed 1-dimensional Δ-complex.

1. *For each vertex embed X locally in a disk centred in the vertex and with radius smaller of the lengths of the edges so that other vertices are not in it.*

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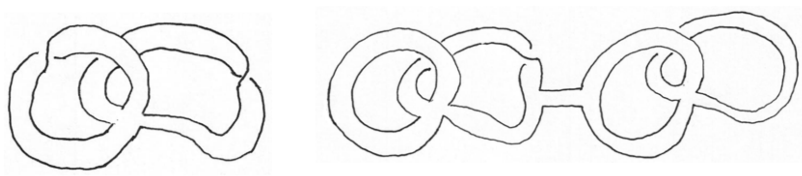
2. *Give an orientation to each disk and choose the order you encounter edges by going around clockwise. Since we locally embed X in each disk, edges cannot cross each other in the disk and the order we give them is part of the data to define the Thick Line.*

3. *Give a second dimension to X by expanding it by a small δL orthogonally to each edges and keeping the final space locally embedded in each disk. This will turn X in a 2D manifold with boundaries (i.e. circles), the edges in strips and the vertices in polygons.*

4. *To each edge (i.e. now a strip) decide which one has to be twisted. Cut midway all the edges (i.e. strip) to be twisted and glue back two sides of the cut by identifying them in the opposite direction.*

The result is what we call a Thick Line $\hat{X}$.

The Fig. 1 shows an examples of Thick Lines:



(a) **(b)**

Figure 1: Thick Lines

Thick Lines are two dimensional manifolds with boundaries. Compare the above definition with the relevant definition given in [1].

Definition 2.2: Associated Manifold
Let $\hat{X}$ be a Thick Line. Let γ_i be the path connected subsets of the boundary of $\hat{X}$. Note that the γ_i are circles. For each γ_i we take a disk D_i and we identify its boundary with γ_i. By doing so we get a compact two dimensional manifold $\Omega(\hat{X})$ which we will call the associated compact manifold to the Thick Line $\hat{X}$.

Definition 2.3: Equivalent Thick Lines
Two thick lines are said to be equivalent if their associated manifolds are homeomorphic.

Note that, given a Thick Line $\hat{X}$, if we can split $\hat{X}$ in two parts A and B such that B can be embedded in $\mathbb{R}^2$, then the thick line we get by removing B from $\hat{X}$ is equivalent to $\hat{X}$. See [1] and compare with next definition.

Definition 2.4: Trivial Intersection and Unsplittable Thick Line
Let $\hat{X}$ be a Thick Line. If we can split $\hat{X}$ in two parts A and B which overlap in a region such that A and B cannot be embedded in $\mathbb{R}^2$ while C can, then $\hat{X}$ can be decomposed in two Thick Lines $\hat{X}_A$ and $\hat{X}_A$ that cross in a

trivial intersection. A Thick Line that cannot be further decomposed is called an Unsplittable Thick Line.

Fig, 2 shows an example of a trivial intersection. Note that, given the two Thick Lines in the definition above we have that $\Omega(\hat{X}_A)\#\Omega(\hat{X}_B)$ (i.e. connected sum) is homeomorphic to $\Omega(\hat{X})$ (See [1]).

In previus paper we have used the word prime rather then unsplittable (See [1] revision v3), however, the word prime is misleading because it suggests that the associated manifold is prime, which is not the case.

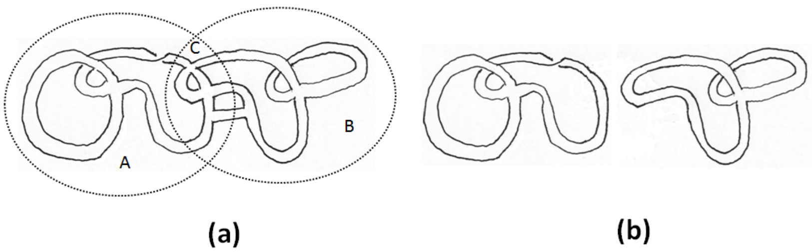


Figure 2: Thick Lines Decomposition

Definition 2.5: Singular Regions.

Let $\hat{X}$ be a Thick Line. In general it is not possible to embed $\hat{X}$ in $\mathbb{R}^2$. However, if we remove a finite number of disks from $\mathbb{R}^2$ we can embed part of $\hat{X}$ letting the non embeddable parts out from $\mathbb{R}^2$ through the boundary of the relevant disks. Let suppose we maximise the number of disks in order to maximise the part of $\hat{X}$ that are embedded in $\mathbb{R}^2$. We will call the regions of $\hat{X}$ that are not embedded in $\mathbb{R}^2$ the Singular Regions of $\hat{X}$. A Singular Region is a 2D manifold having part of its boundary laying on the boundary of the disk.

Fig, 3 shows an example of Singular Regions. Note that in two dimensions there are only two types of Singular Regions: 1- Missed intersection (regions a and b in the figure), 2- Twisted strip (region C in the figure).

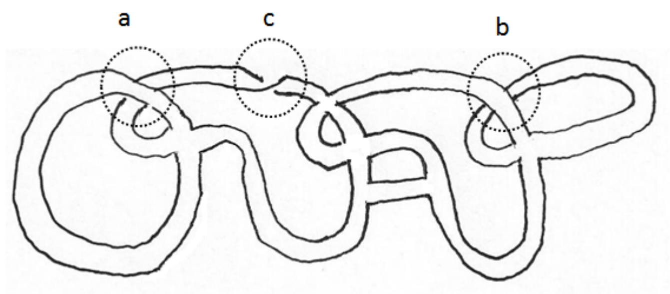


Figure 3: Singular Regions

Definition 2.6: Type of a Singular Region and Completing Thick Line.

We say that two Singular Regions S_1 and S_2 are of the same type is there exist an homeomorphism between the two Singular Region and the restriction of this homeomorphism to the part of the boundary of S_1 and S_2 laying on the boundary of the 2-disck in which they are defined (i.e. their interface with $\mathbb{R}^2$) is an homeomorphism.

We say that a Thick Line complete a Singular Region S if it has only a Singular Region and this region is of type S.

Definition 2.7: Prime Thick Line.

A Prime Thick Line is a Thick Line with only one Singular Region.

Note that the word prime is used in a different way with respect to [1], compare with definition 2.4 above.

2.2 Propositions

Proposition 2.1: *Let $\hat{X}$ be a Unsplittable Thick Line with k different types of Singular Regions each of which present in a number of $n_1, n_2, ...n_k$. Let $\hat{Y}$ be a Thick Line with the same number k of different types of Singular Regions but this time each of which present only once. Then $\Omega(\hat{Y})$ is homeomorphic to $\Omega(\hat{X})$.*

This theorem says that an Unsplittable Tick Line will fit (i.e. will have associated manifold) in the most simple possible manifold. For example, the Thick Line of Fig. 1a and the left Thick Line of Fig. 2b have the same associated manifold although the one of Fig. 1a has an additional "twisted strip" type of Singular Region in it (See [1]). This Manifold is the Klein bottle which can embed Unsplittable Tick Line with any number of "twisted strip" type and "missed Strip Crossing" type Singular Region.

The proof of this proposition is not easy. However several example show that it is likely to be true. Of course, without a formal proof, intuition from a few examples may be deceiving.

Proposition 2.2: *Let $\hat{X}$ and be $\hat{Y}$ be two Prime Thick Lines. If $\Omega(\hat{X})$ is homeomorphic to $\Omega(\hat{Y})$ then the Singular Regions of $\hat{X}$ and $\hat{Y}$ are of the same type.*

Proof of this theorem is trivial since there are only two types of Singular Regions in two dimensions.

Proposition 2.3: *There is a one to one map between Prime Thick Lines and Prime two dimensional closed manifolds.*

Proof of this theorem is trivial since there are only two types of Prime Thick Lines in two dimensions corresponding to the real projective plane and the torus.

3 Three Dimensional Manifolds

3.1 Definitions

Definition 3.1: Thick Surface
Let X a closed 2-dimensional Δ-complex.

1. *For each edge embed X locally in a 3-disk shaped as a cylinder which axis is on the edge and with radius smaller of the lengths of other edges so that other vertices are not in it. The cylinders shall be slightly longer of the edge so that the two vertices at the ends of the edge are in the internal of it.*

2. *Give an orientation to each edge and choose the order you encounter surfaces by going around the edge clockwise. Since we locally embed X in each cylinder, edges cannot cross each other in the 3-disk and the order we give them is part of the data to define the Thick Surface.*

3. *Give a third dimension to X by expanding it by a small δL orthogonally to each surface and keeping the final space locally embedded in each 3-disk. This will turn X in a 3D manifold with boundaries, the faces in tiles with a thickness, the edges in cylinders with polygonal base and the vertices in polyhedra.*

The result is what we call a Thick Surface $\hat{X}$.

Thick Surfaces are three dimensional manifold with boundaries. Note that in the above definition we do not need the last step in which we change some strips in twisted strips. This is because a surface has already a property of orientability, which does not make sense for lines and twisting a strip is equivalent to change the general orientability of the space it is attached if this is an oriented surface. Compare the above definition with the relevant definition given in [1].

Definition 3.2: Associated Manifold
Let $\hat{X}$ be a Thick Surface. Let γ_i be the path connected subsets of the boundary of $\hat{X}$. For each γ_i we define a Δ-complex on it. We attach a 3-simplex on each 2-simplex of the above defined complex. We attach the remaining three faces of each simplex to each other following the same pattern of which the relevant 2-complex on γ_i it is attached to its neighbour simplices. By doing so we get a compact three dimensional manifold $\Omega(\hat{X})$ which we will call the associated compact manifold to the Thick Surface $\hat{X}$.

Definition 3.3: Equivalent Thick Surfaces
Two Thick Surfaces are said to be equivalent if their associated manifolds are homeomorphic.

Note that, given a Thick Surface $\hat{X}$, if we can split $\hat{X}$ in two parts A and B such that B can be embedded in $\mathbb{R}^3$, then the Thick Surface we get by removing B from $\hat{X}$ is equivalent to $\hat{X}$. See [1] and compare with next definition.

Definition 3.4: Trivial Intersection and Unsplittable Thick Line

Let $\hat{X}$ be a Thick Surface. If we can split $\hat{X}$ in two parts A and B which overlap in a region such that A and B cannot be embedded in $\mathbb{R}^3$ while C can, then $\hat{X}$ can be decomposed in two Thick Surfaces $\hat{X}_A$ and $\hat{X}_A$ that cross in a trivial intersection. A Thick Surfaces that cannot be further decomposed is called an Unsplittable Thick Surface.

Note that, given the two Thick Surfaces in the definition above we have that $\Omega(\hat{X}_A)\#\Omega(\hat{X}_B)$ (i.e. connected sum) is homeomorphic to $\Omega(\hat{X})$ See [1]. In previous papers we have used the word prime rather then unsplittable (See [1] revision v3), however, the word prime is misleading because it suggests that the associated manifold is prime, which is not the case.

Definition 3.5: Singular Regions.

Let $\hat{X}$ be a Thick Surface. In general it is not possible to embed $\hat{X}$ in $\mathbb{R}^3$. However, if we remove a finite number of 3-disks from $\mathbb{R}^3$ we can embed part of $\hat{X}$ letting the non embeddable parts out from $\mathbb{R}^3$ through the boundary of the relevant 3-disks. Let suppose we maximise the number of 3-disks in order to maximise the part of $\hat{X}$ that are embedded in $\mathbb{R}^3$. We will call the regions of $\hat{X}$ that are not embedded in $\mathbb{R}^3$ the Singular Regions of $\hat{X}$. A Singular Region is a 3D manifold having part of its boundary laying on the boundary of the 3-disk.

Fig. 4 shows an example of Singular Region and in particular a singular region composed of non path connected manifolds. Note that in three dimensions there is only one type of "manifold like" Singular Regions, which is the missed intersection between a torus and a plane showed in the figure. All other Singular Regions have at least a part of them which is not a manifold and there are an infinite number of them.

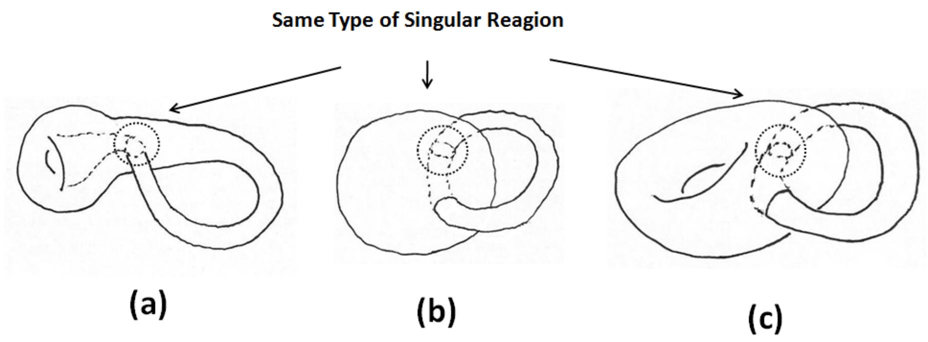


Figure 4: Singular Region for Thick Surfaces

Definition 3.6: Type of a Singular Region and Completing Thick Surface.

We say that two Singular Regions S_1 and S_2 are of the same type is there exist an homeomorphism between the two Singular Region and the restriction of this homeomorphism to the part of the boundary of S_1 and S_2 laying on the boundary of the 3-disck in which they are defined (i.e. their interface with $\mathbb{R}^3$) is an homeomorphism.

We say that a Thick Surface complete a Singular Region S if it has only a Singular Region and this region is of type S.

Note that the same Singular Region can be completed to Thick Surfaces which which are not homoemorphic to each other (See Fig. 4).

Definition 3.7: Prime Thick Surface.
A Prime Thick Surface is a Thick Surface with only one Singular Region.

For example, the three Tick Surfaces in Fig. 4 are all Prime Thick Surfaces. Note that the word prime is used in a different way with respect to [1], compare with definition 3.4 above.

3.2 Propositions

Proposition 3.1: *Let $\hat{X}$ be a Unsplittable Thick Surface with k different types of Singular Regions each of which present in a number of $n_1, n_2, ...n_k$, then there exist a Thick Surface $\hat{Y}$ with the same number k of different types of Singular Regions, this time each of which present only once, so that $\Omega(\hat{Y})$ is homeomorphic to $\Omega(\hat{X})$.*

This theorem is derived by analogy from the two dimensional case where proof was not given. If the proposition can be considered fairly reasonable in two dimensions, in three dimension there are too many things that may go wrong and therefore any final judgement on this proposition is suspended till further analysis.

Proposition 3.2: *Let $\hat{X}$ and be $\hat{Y}$ two Prime Thick Surfaces. If $\Omega(\hat{X})$ is homeomorphic to $\Omega(\hat{Y})$ then the Singular Region of $\hat{X}$ and $\hat{Y}$ are of the same type.*

This theorem is derived by analogy from the two dimensional case where although a proof was not given, it is clear that the proposition is true. However, in three dimension there are too many things that may go wrong and therefore any final judgement on this proposition is suspended till further analysis.

Note that the proposition states that a prime three dimensional compact manifold can be generated only by a Singular Region but a Singular region may generate more than one manifold.

Proposition 3.3: *There is a one to one map between Prime Thick Surfaces and prime three dimensional closed manifolds.*

This theorem is derived by analogy from the two dimensional case where although a proof was not given, it is clear that the proposition is true. However, in three dimension there are too many things that may go wrong and therefore any final judgement on this proposition is suspended till further analysis.

4 Conclusions

If proposition 3.1 is true the method proposed in this paper is interesting. If proposition 3.2 and 3.3 are true then the method proposed in this paper is very interesting. We believe that this method deserves further formal mathematical research.

Appendix

A.1 Introduction

The Poincaré homology sphere and lens spaces are classic example of 3-Manifolds defined in the very early years of topology.

In this paper we want to find and discuss Thick surfaces of both the Poincaré homology sphere and two examples of 3-dimensional lens spaces.

A.2 The Poincaré Sphere

The Poincaré homology sphere, first introduced by Henri Poincaré, is an example of a closed 3-manifold with homology groups homologous to a 3-sphere but which is not homeomorphic to it. As a matter of fact it has a finite fundamental group of order 120 known as the binary icosahedral group. There are many ways to construct the Poincaré homology sphere. Among all, the simplest construction is by identifying opposite faces of a dodecahedron using the minimal clockwise twist to line up the faces.

A.2.1 Cell Complex Definition

We want to find the solid strip configuration of the Poincaré homology sphere. In order to do it we have to define the Poincaré homology sphere Cell complex first.

As mentioned in the introduction a possible Cell complex of the Poincaré homology sphere is composed by a single 3-cell which is a dodecahedron where opposite faces of the dodecahedron are identified using the minimal clockwise twist to line up them. By doing so some edges an vertices of the original dodecahedron get identified themselves. We have worked out identified edges (labelled by letters from A to J) and vertices (labelled by numbers from 1 to 5) and the result is shown in Fig. 1a where an orientation for each edge is also given.
By doing so we get a final Cell complex with:

$$\begin{cases} 5 & Vertices \\ 10 & Edges \\ 6 & Faces \\ 1 & 3-Cell \end{cases} \tag{1}$$

Which gives an Euler characteristics equal to $\chi = 0$, as expected since all the homology groups of the Poincaré sphere are trivial.

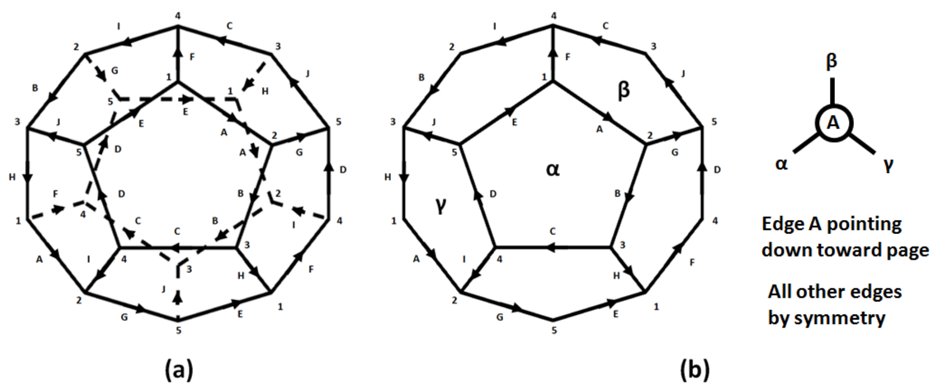


Figure 5: Poincaré Sphere Cell complex and Thick Surface

A.2.2 Thick Surface

We want to find the Thick surface of the Poincaré Sphere. In this case, it is obvious that the relevant surface is the one we get identifying opposite pentagons of the original tetrahedron. This is because to each pentagon there is a pentagonal base pyramid attached to it and all these pyramids make the internal of the manifold under study. However, this data itself it is not enough. We also need to define the order surfaces are met when going around each oriented edge clockwise. This can be done by going from one surface to another from the inside of the original dodecahedron. The result is also shown in Fig. 5. The two pieces of data, the base surface and the order of surfaces are met going around edges, define the thick surface ξ_p (p for Poincaré). This is a prime Thick Surface.

The Poincaré homology sphere is the manifold associated to ξ_p (i.e. $\Omega(\xi_p)$ see [1]) and it is the simplest manifold in which ξ_p can be embedded.

A.3 Lens Spaces

Lens spaces are manifold first introduced by Heinrich Tietze for the 3-dimensional case. Alexander showed that the lens spaces $L(5;1)$ and $L(5;2)$ are not homeomorphic even though they have the same fundamental groups and the same homology.

There are many ways to construct lens spaces $L(p,q)$. Among all, the simplest construction is by using a bi-pyramid with a polygonal base having p edges and identifying faces of the up pyramid with faces of the down pyramid in a specific way.

A.3.1 Cell Complex Definition

We need to construct a Cell complex of lens spaces. A way to do it is to start from a solid bi-pyramid having for base a p-polygon. Let N and S be the two vertices of the bi-pyramid on the two opposite sides of the base. Moreover, let $V_0 ... V_{p-1}$ br the vertices of the bi-pyramid located on the vertices of the polygonal base. If we now identify the faces of the bi-pyramid, which are triangles, by identifying N to S, V_i with V_{i+q} and V_{i+1} with V_{i+q+1}, the resulting space is homeomorphic to the lens space $L(p;q)$.

The Fig. 6 shows the construction for $L(5;1)$ and $L(5;2)$ which are the two lens spaces we are focusing on.

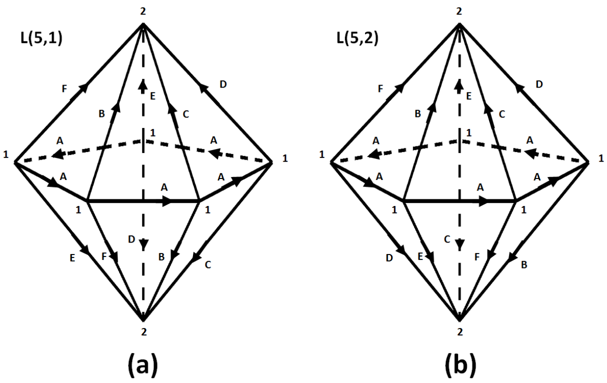


Figure 6: Lens Spaces Cell complexes

A.3.2 Thick surface

We want to find the Thick Surface of the two above defined lens spaces. We can proceed in this case exactly as we did for the Poincaré Sphere since also in this case it is obvious that the relevant surface is the one we get identifying opposite faces of the bi-pyramid. Also in this case, we need to define the order surfaces are met when going around each oriented edge clockwise (this is necessary for one edge only of the proposed complex since other edges connect only two surfaces). The result is shown in the figure below.

Spaces $L(5;1)$ and $L(5;2)$ are the manifolds associated to the above defined Tick Surface and they are the simplest manifolds in which the above Thick Surfaces can be embedded. This are Prime Thick Surfaces.

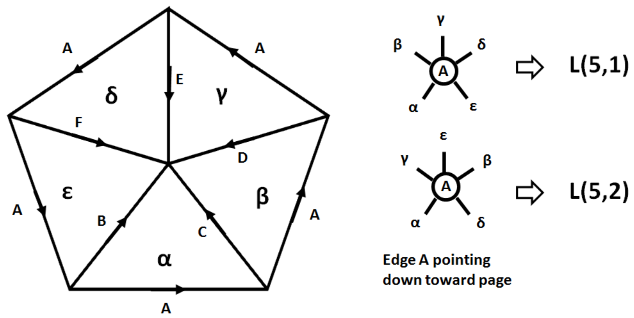


Figure 7: Lens Spaces Thick Surfaces

We note that the two lens spaces are generated by the same basic surface

(Fig. 7a) but they differ by the order faces are encountered when going around the edge A (Fig 6b). This order get frozen when we thicken the surface given as a result two different Thick Surfaces which are not homeomorphic each other.

References

[1] V. Nardozza. *Solid Strips Configurations.* http://vixra.org/abs/1912.0161 (2019)