Biot-Savart Law and Stokes' theorem

Eric Su
eric.su.mobile@gmail.com
https://sites.google.com/view/physics-news/home

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Biot-Savart law describes magnetic field due to the electric current in a conductive wire. For a long straight wire, the magnetic field is proportional to (I/r). The curl of magnetic field is proportional to (dI/dr). For a constant current, the curl of magnetic field is zero. Consequently, the surface integral of the curl of magnetic field is zero but the line integral of the magnetic field is not. Stokes' theorem can not be applied to the magnetic field vector generated by a constant electric current because the magnetic field is not a differentiable vector.

I. INTRODUCTION

In 1820, Hans Christian Oersted accidentally observed that a current flowing through a wire would move a compass needle placed beside it.[1] This showed that an electric current produced a magnetic field.

Francois Arago brought the news of Oersteds discovery to Paris on September 4, 1820. Andr-Marie Ampre and the team of Jean-Baptiste Biot and Flix Savart quickly set to work to establish a quantitative expression for this effect. However, their approaches were quite different.

The relationship characterizing the magnetic field generated by a electric current was first described by Jean-Baptiste Biot and Flix Savart. Their original experiment involved passing current through a very long vertical wire which moved magnetic some needle distance away from the wire. They built this experiment together in 1820.[2]

Andre Marie Ampere also started to work on a electrodynamic theory in 1820. He showed that two parallel wires, carrying current, attracted each other if the currents flowed in the same direction and opposed each other if the currents flowed in opposite directions. In 1825, Ampere formulated a mathematical expression for magnetic field.[1]

Both mathematical expressions are intended for the magnetic field outside the wire. However, both expressions explicitly also describe the magnetic field inside the wire. A singularity becomes embedded in both expressions unintentionally.

II. PROOF

A. Biot-Savart Law

The magnetic field at a distance of r from the electric current I can be represented as

$$B = \frac{\mu_0 I}{2\pi r} \tag{1}$$

Equation (1) can be derived from Biot-Savart law[2] under the conditions that the electric current is constant throughout a straight conductive wire of infinite length.

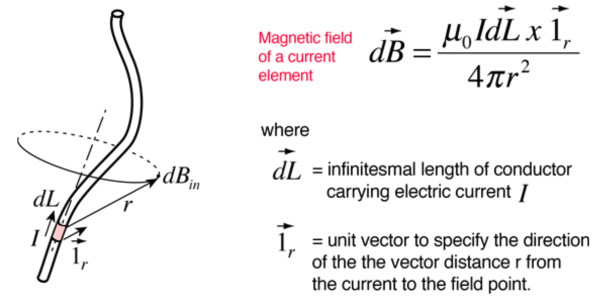

FIG. 1. Biot-Savart Law

Let the wire points in the z direction. The magnetic field vector in the cylindrical coordinates is represented as

$$\vec{B} = \int \frac{\mu_0 I}{4\pi r^3} d\vec{L} \times \vec{r} = \frac{\mu_0 I}{2\pi r}(0,1,0) \tag{2}$$

The curl of a vector $\vec{A}$ in cylindrical coordinates is

$$\vec{\nabla} \times \vec{A} = \frac{1}{r}\begin{vmatrix} \vec{a}_r & r\vec{a}_\phi & \vec{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix} \tag{3}$$

The divergence of a vector $\vec{A}$ in cylindrical coordinates is

$$\vec{\nabla} \bullet \vec{A} = \frac{1}{r}\left(\frac{\partial (rA_r)}{\partial r} + \frac{\partial A_\phi}{\partial \phi} + r\frac{\partial A_z}{\partial z}\right) \tag{4}$$

From equations (2,3),

$$\vec{\nabla} \times \vec{B} = \frac{\mu_0}{2\pi r}\frac{\partial I}{\partial r}(0,0,1) \tag{5}$$

The current is zero outside the wire. Therefore, the curl of magnetic field is also zero outside the wire. If the current I remains constant inside the wire,

$$\vec{\nabla} \times \vec{B} = \frac{\mu_0}{2\pi r}(0,0,0) = (0,0,0) \tag{6}$$

From equations (2,4),

$$\vec{\nabla} \bullet \vec{B} = \frac{1}{r}\frac{\partial B_\phi}{\partial \phi} = \frac{1}{r}(0,0,0) = (0,0,0) \tag{7}$$

At $r = 0$,

$$\vec{\nabla} \times \vec{B} \mid_{r=0} = \frac{\mu_0}{2\pi r}(0,0,0) \mid_{r=0} = (0,0,0) \mid_{r=0} \tag{8}$$

$$\vec{\nabla} \bullet \vec{B} \mid_{r=0} = \frac{1}{r}(0,0,0) \mid_{r=0} = (0,0,0) \mid_{r=0} \tag{9}$$

From equations (8,9),

$$\vec{\nabla} \times \vec{B} = 0 = \vec{\nabla} \bullet \vec{B} \tag{10}$$

B. Stokes' theorem

Stokes' theorem[3] states that a differentiable vector $\vec{A}$ satisfies

$$\iint \vec{\nabla} \times \vec{A} \bullet d\vec{S} = \int \vec{A} \bullet d\vec{l} \tag{11}$$

From equation (10), the curl of magnetic field from a constant current is

$$\vec{\nabla} \times \vec{B} = (0,0,0) \tag{12}$$

The surface integral of the curl is zero.

$$\iint \vec{\nabla} \times \vec{B} \bullet d\vec{S} = 0 \tag{13}$$

The magnetic field exists outside the wire. From equation (2),

$$\vec{B} = \frac{\mu_0 I}{2\pi r}(0,1,0) \tag{14}$$

The line integral is not zero at $r > 0$.

$$\int \vec{B} \bullet d\vec{l} = \mu_0 I \tag{15}$$

From equations (13,15),

$$\iint \vec{\nabla} \times \vec{B} \bullet d\vec{S} \neq \int \vec{B} \bullet d\vec{l} \tag{16}$$

Stokes' theorem can not be applied to the magnetic field vector generated by the constant electric current in a straight conductive wire of infinite length.

Stokes' theorem requires a differentiable vector. The magnetic field vector from Biot-Savart law is not a differentiable vector. Therefore, the surface integral is different from the line integral.

For another example, let $\vec{F}$ be a differentiable vector.

$$\vec{F} = (1,0,0)(\frac{1}{r} - \frac{1}{r}) \tag{17}$$

$\vec{F}$ is zero at r=0 while $\frac{1}{r}$ diverges.

III. CONCLUSION

The cross section of a conductive wire is of finite size. It is not the original intention of Biot-Savart law to describe a magnetic field inside the wire. However, a factor of (1/r) appears in the mathematical expression of the magnetic field. This factor generates singularity in the magnetic field as r becomes zero.

Stokes' theorem can not be applied to the magnetic field vector because it is not a differentiable vector.

It is fabrication of physics to identify the curl of magnetic field with a delta function. An absolutely wrong approach that will result in erroneous theory far from reality.

The correct approach is to establish an accurate expression for the magnetic field. The first step is to clarify the scope of Biot-Savart law that the law is valid only for the magnetic field outside the wire. Stokes' theorem will remain incompatible with Biot-Savart law because the magnetic field vector is not defined within the cross section of the wire.

[1] Andre Koch Torres Assis, J. P. M. C. Chaib. 2015. "Ampere's Electrodynamics". https://www.ifi.unicamp.br/ assis/Amperes-Electrodynamics.pdf

[2] A joint Biot-Savart paper "Note sur le magntisme de la pile de Volta" was published in the Annales de chemie et de physique in 1820

[3] Katz, Victor J. (1979-01-01). "The History of Stokes' Theorem". Mathematics Magazine. 52 (3): 146156. doi:10.2307/2690275. JSTOR 2690275

[4] Maxwell, James Clerk (1865). "A dynamical theory of the electromagnetic field". Philosophical Transactions of the Royal Society of London. 155: 459512. doi:10.1098/rstl.1865.0008.

[5] Eric Su: List of Publications, http://vixra.org/author/eric_su