Three circle chains arising from three lines

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Abstract. We generalize a problem in Wasan geometry involving a triangle and its incircle and get simple relationships between the three chains arising from three lines.

Keywords. circle chain, incircle of a triangle.

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1. Introduction

In this paper we generalize the following problem in Wasan geometry (Japanese geometry) [5] (see Figure 1).

Problem 1. For the incircle δ of a triangle PQR, let α, β and γ be the incircles of the curvilinear triangles made by δ and two sides of PQR. If a, b, c, d are the radii of the circles α, β, γ, δ, respectively, show that the following relation holds.

$$d = \sqrt{ab} + \sqrt{bc} + \sqrt{ca}.$$

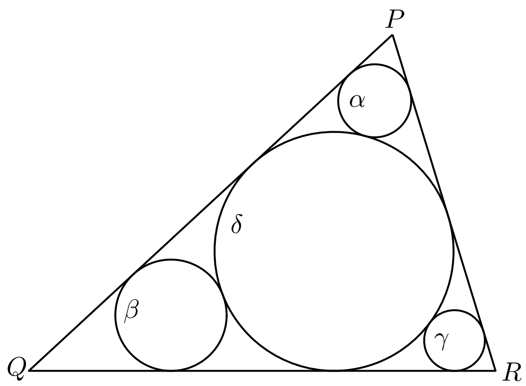


Figure 1.

The problem is involving three lines which make a triangle with incircle δ. In this paper we show that the same relation is also true if the three lines make a triangle with excircle δ or two of the three lines are parallel, and generalize the result to three infinite chains of circles.

2. Generalization

We generalize the problem. Let p, q and r be three tangents of a circle δ of radius d. We assume that α (resp. β, γ) is the circle of radius a (resp, b, c) touching δ externally and the lines q and r (resp. r and p, p and q) from the same sides as δ. In this case we say that α touches δ in the same sense if α and the line p have no points in common (see Figures 2, 3 and 4), otherwise in the opposite sense (see Figures 5, 6 and 7). The notions of touching in the same sense and touching in the opposite sense also apply to the circles β and γ. Let $P = q \cap r$, $Q = r \cap p$ $R = p \cap q$.

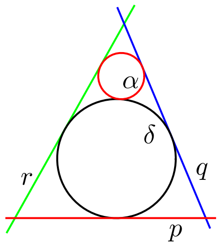
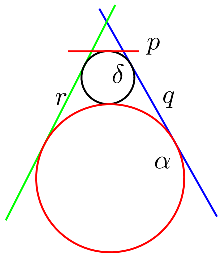


Figure 2.



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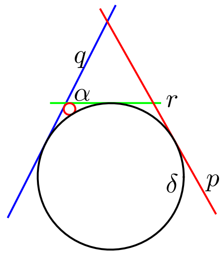

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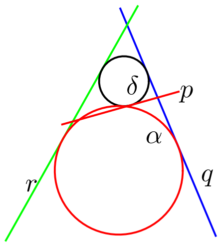

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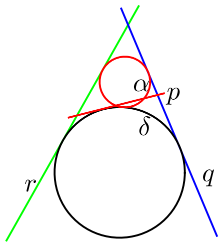


Figure 6.

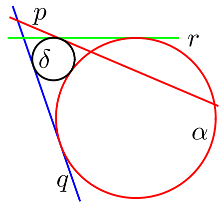


Figure 7.

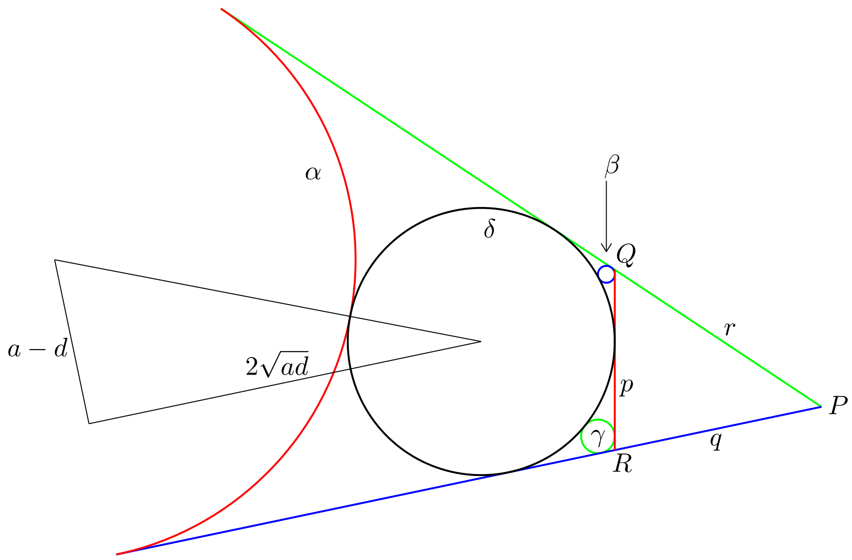


Figure 8: δ is the excircle of PQR touching p from the side opposite to P, where the circles α, β and γ touch δ in the same sense.

Theorem 1. *The following statements hold.*
(i) *If the circles α, β and γ touch the circle δ in the same sense, then the following relation holds.*

$$d = \sqrt{ab} + \sqrt{bc} + \sqrt{ca}. \tag{1}$$

(ii) *If the circles α, β and γ touch the circle δ in the opposite sense, then the following relation holds.*

$$\frac{1}{d} = \frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}}. \tag{2}$$

Proof. We assume that δ is the excircle of the triangle PQR touching p from the side opposite to P (see Figure 8). We assume that the circles α, β and γ touch δ in the same sense. Then we have

$$\tan\frac{P}{2}=\frac{a-d}{2\sqrt{ad}},\quad \tan\frac{Q}{2}=\frac{2\sqrt{bd}}{d-b},\quad \tan\frac{R}{2}=\frac{2\sqrt{cd}}{d-c}. \tag{3}$$

While from $\angle P+\angle Q+\angle R=\pi$, we get

$$\tan\frac{P}{2}\tan\frac{Q}{2}+\tan\frac{Q}{2}\tan\frac{R}{2}+\tan\frac{R}{2}\tan\frac{P}{2}=1. \tag{4}$$

Substituting (3) in (4) and rearranging, we have

$$\frac{(d-(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}))(d(\sqrt{a}+\sqrt{b}+\sqrt{c})-\sqrt{abc})}{\sqrt{a}(d-b)(d-c)}=0.$$

Therefore we get (1) or (2). Meanwhile we get $2d>b+c$ from $d>b$ and $d>c$. Hence (2) implies

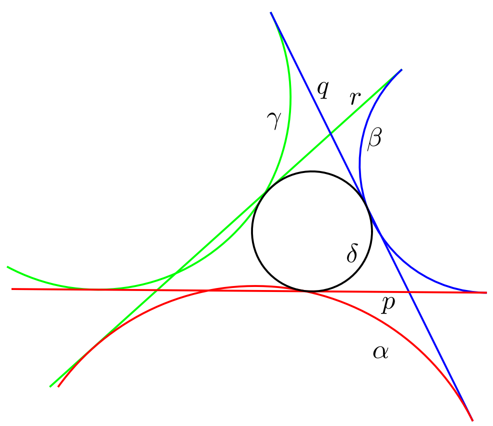
$$2d-b-c=-\frac{\sqrt{a}(\sqrt{b}-\sqrt{c})^2+(\sqrt{b}+\sqrt{c})(b+c)}{\sqrt{a}+\sqrt{b}+\sqrt{c}}<0,$$

a contradiction. Therefore we get (1). The same proof is also valid in the case the lines q and r being parallel if we consider $\tan(P/2)=0$. The case where δ is the incircle of the triangle PQR can be proved similarly. The proof of (i) is now complete.

We assume again that δ is the excircle of the triangle PQR touching p from the side opposite to P. We now assume that the circles α, β and γ touches δ in the opposite sense. Then we have

$$\tan\frac{P}{2}=-\frac{a-d}{2\sqrt{ad}},\quad \tan\frac{Q}{2}=-\frac{2\sqrt{bd}}{d-b},\quad \tan\frac{R}{2}=-\frac{2\sqrt{cd}}{d-c}.$$

Therefore we also get (1) or (2). Meanwhile we get $bc>d^2$ from $b>d$ and $c>d$. Hence (1) implies $bc-d^2<0$, a contradiction. Therefore we get (2). The same proof is also valid if the lines q and r are parallel. The case in which δ is the incircle of the triangle PQR can be proved in a similar way. □



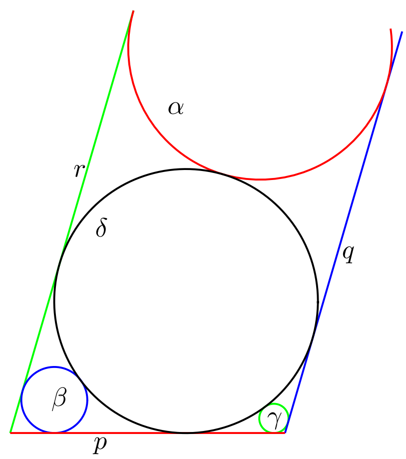


Figure 9: Touching in the opposite sense. Figure 10: Touching in the same sense.

A sangaku problem considering the part (ii) of the theorem in the case δ being the incircle of PQR can be found in [7] (see Figure 9). We consider the special case in which the lines q and r are parallel (see Figure 10).

Corollary 1. *If the lines q and r are parallel, then the following statements hold.*
(i) *If the circles β and γ touch the circle δ in the same sense, then the following relation holds.*

$$\left(\sqrt{\frac{d}{b}}-1\right)\left(\sqrt{\frac{d}{c}}-1\right)=2. \tag{5}$$

(ii) *If the circles β and γ touch the circle δ in the opposite sense, then the following relation holds.*

$$\left(\sqrt{\frac{b}{d}}-1\right)\left(\sqrt{\frac{c}{d}}-1\right)=2.$$

Proof. If the circle α also touches δ in the same sense, then we get $a = d$. Therefore we get $d = \sqrt{db}+\sqrt{bc}+\sqrt{cd} = \sqrt{d}(\sqrt{b}+\sqrt{c})+\sqrt{bc}$ by Theorem 1. This implies $\sqrt{d/b}\sqrt{d/c}-\sqrt{d/b}-\sqrt{d/c}+1 = 2$. Hence we get (5). The part (ii) follows from the part (i). □

3. Three infinite chains of circles

Theorem 1 and Corollary 1 are obtained if $n = 1$ in the next theorem (see Figure 11).

Theorem 2. *Let p, q, r, α, β, γ and δ be as in the previous section. Let $\{\cdots,\alpha_{-2},\alpha_{-1},\alpha_0=\delta,\alpha_1=\alpha,\alpha_2,\cdots\}$ be the chains of circles touching the lines q and r. Similarly the chains $\{\cdots,\beta_{-2},\beta_{-1},\beta_0=\delta,\beta_1=\beta,\beta_2,\cdots\}$ and $\{\cdots,\gamma_{-2},\gamma_{-1},\gamma_0=\delta,\gamma_1=\gamma,\gamma_2,\cdots\}$ are defined. Let a_n, b_n, c_n be the radii of the circles α_n, β_n, γ_n, respectively for an integer n, and let $r_a = a_1/d$, $r_b = b_1/d$, $r_c = c_1/d$.*
(i) *If the circles α, β and γ touch the circle δ in the same sense, then we have*

$$\sqrt{r_ar_b}+\sqrt{r_br_c}+\sqrt{r_cr_a}=1, \tag{6}$$

$$d^{\frac{1}{n}}=(a_nb_n)^{\frac{1}{2n}}+(b_nc_n)^{\frac{1}{2n}}+(c_na_n)^{\frac{1}{2n}}. \tag{7}$$

(ii) *If the lines q and r are parallel and the circles β and γ touch the circle δ in the same sense, then we have*

$$(\sqrt{r_b}+1)(\sqrt{r_c}+1)=2, \tag{8}$$

$$\left(\left(\frac{d}{b_n}\right)^{\frac{1}{2n}}-1\right)\left(\left(\frac{d}{c_n}\right)^{\frac{1}{2n}}-1\right)=2.$$

(iii) *If the circles α, β and γ touch the circle δ in the opposite sense, then we have*

$$\frac{1}{\sqrt{r_ar_b}}+\frac{1}{\sqrt{r_br_c}}+\frac{1}{\sqrt{r_cr_a}}=1,$$

$$\frac{1}{d^{\frac{1}{n}}}=\frac{1}{(a_nb_n)^{\frac{1}{2n}}}+\frac{1}{(b_nc_n)^{\frac{1}{2n}}}+\frac{1}{(c_na_n)^{\frac{1}{2n}}}.$$

(iv) *If the lines q and r are parallel and the circles β and γ touch the circle δ in the opposite sense, then we have*

$$\left(\frac{1}{\sqrt{r_b}}+1\right)\left(\frac{1}{\sqrt{r_c}}+1\right)=2,$$

$$\left(\left(\frac{b_n}{d}\right)^{\frac{1}{2n}}-1\right)\left(\left(\frac{c_n}{d}\right)^{\frac{1}{2n}}-1\right)=2.$$

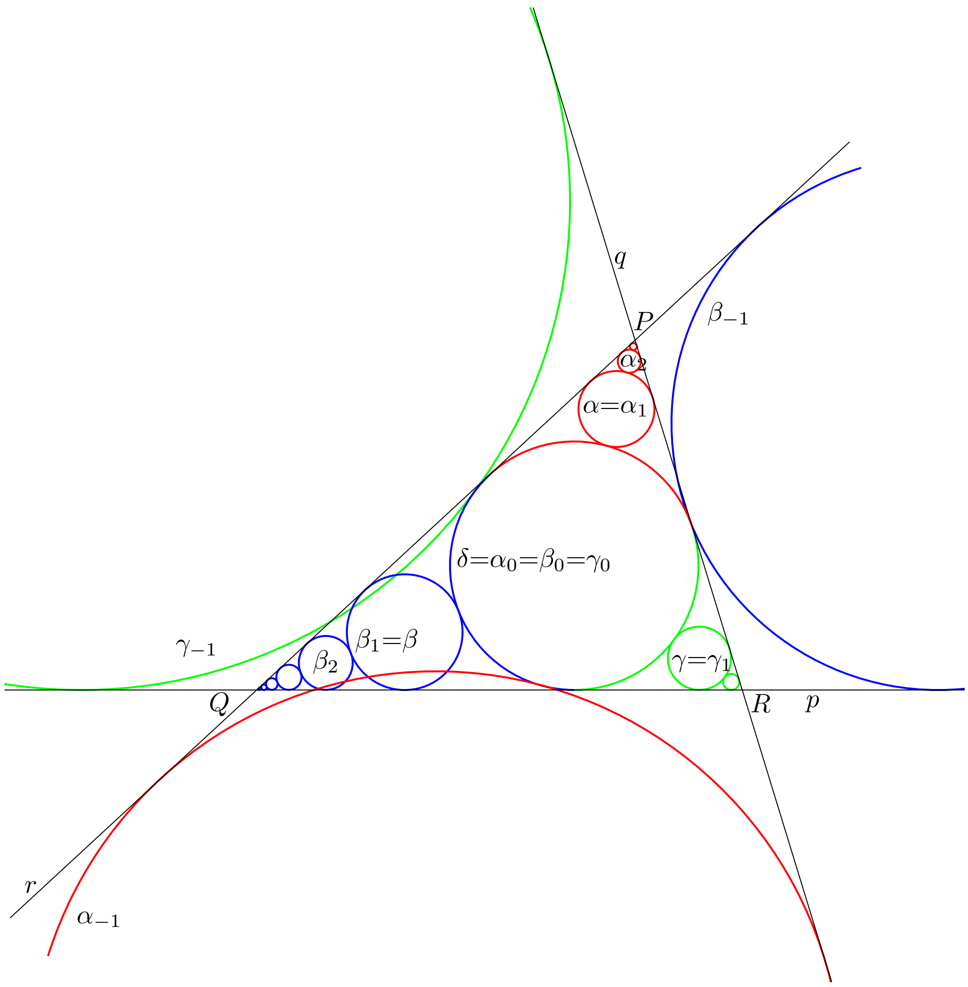


Figure 11: δ is the incircle of PQR and α, β, γ touch δ in the same sense.

Proof. Obviously we have $a_n = r_a^n d$, $b_n = r_b^n d$, $c_n = r_c^n d$. We prove (i). We have $d = \sqrt{a_1 b_1} + \sqrt{b_1 c_1} + \sqrt{c_1 a_1} = d(\sqrt{r_a r_b} + \sqrt{r_b r_c} + \sqrt{r_c r_a})$ by Theorem 1. Hence we get (6). Then the right side of (7) equals

$$(r_a^n d \cdot r_b^n d)^{\frac{1}{2n}} + (r_b^n d \cdot r_c^n d)^{\frac{1}{2n}} + (r_c^n d \cdot r_a^n d)^{\frac{1}{2n}} = d^{\frac{1}{n}}((r_a r_b)^{\frac{1}{2}} + (r_b r_c)^{\frac{1}{2}} + (r_c r_a)^{\frac{1}{2}}) = d^{\frac{1}{n}}.$$

This proves (i). If the lines q and r are parallel and the circles β and γ touch δ in the same sense, we also assume that the circle α touch δ in the same sense. Then we get (6) and (7) with $r_a = 1$ and $d = a$, which imply (ii). The first halves of (iii) and (iv) follow from (6) and (8), respectively. The rest of the theorem is proved similarly. □

4. An open problem involving Malfatti circles

Italian mathematician G. F. Malfatti proposed the following problem in 1803 [3] (see Figure 12):

Problem 2. Draw mutually touching three circles inside of a given triangle such that each of which touches two sides of the triangle.

The three circles are called Malfatti circles. Those circles were popular in Wasan geometry. We can find problems involving Malfatti circles in several Wasan books or manuscripts as in [1], [2], [4], [5, 6], [9], [10], [11] and even in a popular book of formulas [12], where the triangles of the problems in [9] and [11] are not general triangles but right triangles. As far as the author knows at the present time of writing, 1738 book [11] is the earliest one concerning with Malfatti circles in Wasan geometry.

For problems involving Malfatti circles in a general triangle, [8] stated that the manuscript [2] was made during 1771-1773 thereby Malfatti circles were considered in Wasan geometry earlier than Malfatti. While [6] was made earlier than [2]. The author considers that Fujita's work should be noted, since there seems no expository writing referring [6] for Malfatti circles.

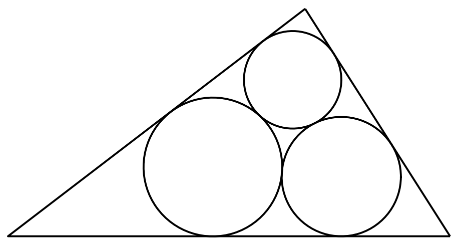


Figure 12.

Most problems involving Malfatti circles in Wasan geometry are concerning with the relationships between the radii of Malfatti circles and the lengths of the sides or the inradius of the triangle, one of which may be described as follows [5]:

$$\frac{1}{d} = \frac{1}{2}\left(\frac{1}{\sqrt{r_1r_2}} + \frac{1}{\sqrt{r_2r_3}} + \frac{1}{\sqrt{r_3r_1}} - \sqrt{\frac{1}{r_1r_2} + \frac{1}{r_2r_3} + \frac{1}{r_3r_1}}\right),$$

where d is the inradius of a given triangle and r_1, r_2 and r_3 are the radii of the the Malfatti circles. Bearing in mind the last equation and comparing Theorem 1 with Theorem 2, we pose the following problem (see Figure 13).

Problem 3. For given three lines, consider three infinite chains of circles touching two lines such that each of the chains contains one of the Malfatti circles of the triangle made by the three lines as in Figure 13. Find some interesting relationships between the radii of the circles of the chains and the inradius or the exradii or the lengths of the three sides of the triangle.

The three chains in Theorem 2 have the circle δ in common, but the chains in Problem 3 have no such circle, which makes the problem difficult.

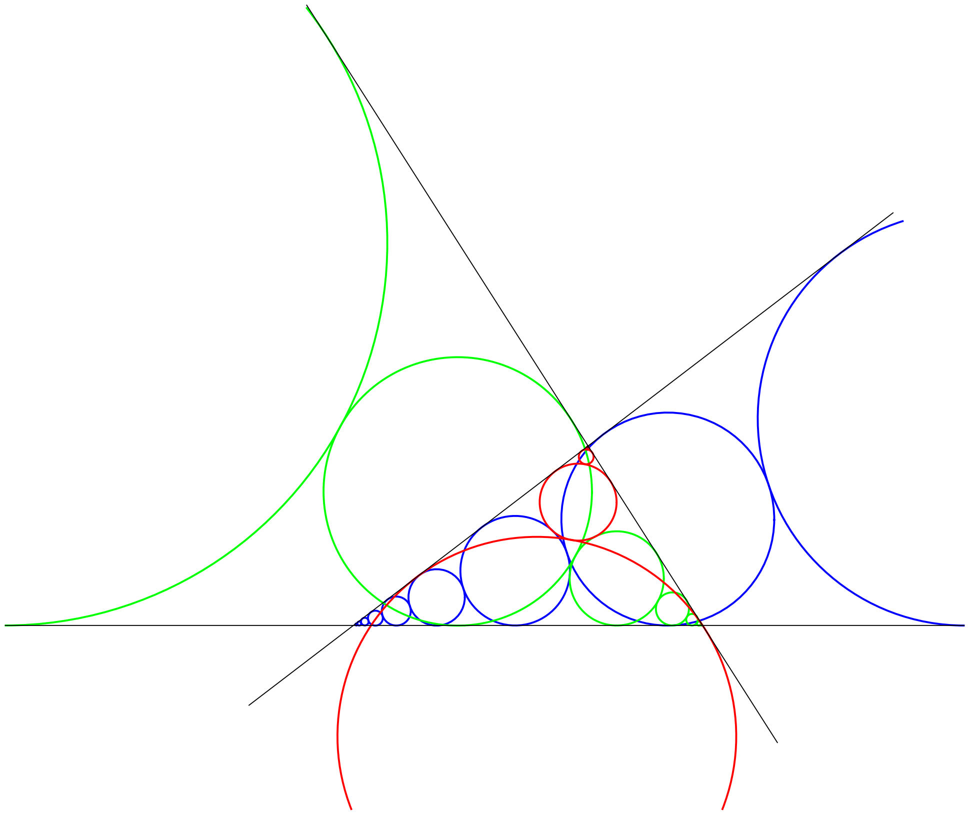


Figure 13.

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