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On Some New Notions in Nano Ideal Topological Spaces

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Abstract. The purpose of this paper is to introduce the notion of nano ideal topological spaces and investigate the relation between nano topological space and nano ideal topological space. Moreover, we offer some new open and closed sets in the context of nano ideal topological spaces and present some of their basic properties and characterizations.
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1. Introduction

An ideal [6] I on a nonempty set X is a nonempty collection of subsets of X which satisfies (i) $A \in I$ and $B \subset A$ implies $B \in I$ and (ii) $A \in I$ and $B \in I$ implies $A \cup B \in I$. Given a topological space (X, τ) with an ideal I on X and if $P(X)$ is the set of all subsets of X, a set operator $(.)^* : P(X) \to P(X)$, called a local function [5] of A with respect to τ and I is defined as follows: for $A \subset X$, $A^*(I, \tau) = \{x \in X : U \cap A \notin I \text{ for every } U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$. A Kuratowski closure operator $cl^*(.)$ for a topology $\tau^*(I, \tau)$, called the $*$-topology, finer than τ is defined by $cl^*(A) = A \cup A^*(I, \tau)$ [11]. When there is no chance for confusion, we will simply write A^* for $A^*(I, \tau)$ and τ^* for $\tau^*(I, \tau)$. If I is an ideal on X, then the space (X, τ, I) is called an ideal space. A subset A of an ideal space is said to be $*$-dense in itself [3] (resp. τ^*-closed [5]) if $A \subset A^*$ (resp. $A^* \subset A$). By a space (X, τ), we always mean a topological space (X, τ) with no separation properties assumed. If $A \subset X$, then $cl(A)$ and $int(A)$, denote the closure and interior of A in (X, τ), respectively. The interior of A in (X, τ^*) is denoted by $int^*(A)$. The notion of I-open sets was introduced by Jankovic et al. [5], and further it was investigated by T.R. Hamlett et al [2] and Abd El-Monsef et al.[1].
The notion of a nano ideal topological space was introduced by Parimala et al. [10]. They studied its properties and characterizations. In this paper, we introduce some new notions in the context of nano ideal topological spaces and investigate their basic properties.

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2. Preliminaries

The theory of nano topology is established by Thivagar [7] as an extension of theory of sets in order to study the intelligent systems which are characterized by insufficient and incomplete information. Indeed, nano topology has several applications in the real life problems (see [9]). Here, we recall the following definitions which are useful in the sequel.

Definition 2.1. [7] Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as indiscernibility relation. Then U is divided into disjoint equivalence classes. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space.
Let $X \subseteq U$. Then,

(i) The lower approximation of X with respect to R is the set of all objects which can be for certain classified as X with respect to R and is denoted by $L_R(X)$. That is, $L_R(X) = \cup\{R(X) : R(X) \subseteq X, x \in U\}$, where $R(X)$ denotes the equivalence class determined by $x \in U$.

(ii) The upper approximation of X with respect to R is the set of all objects which can be possibly classified as X with respect to R and is denoted by $U_R(X)$. That is, $U_R(X) = \cup\{R(X) : R(X) \cap X \neq \phi, x \in U\}$.

(iii) The boundary region of X with respect to R is the set of all objects which can be classified neither as X nor as not -X with respect to R and is denoted by $B_R(X)$. That is, $B_R(X) = U_R(X) - L_R(X)$.

Property 2.2. [8] If (U, R) is an approximation space and $X, Y \subseteq U$, then

(i) $L_R(X) \subseteq X \subseteq U_R(X)$.

(ii) $L_R(\phi) = U_R(\phi) = \phi$.

(iii) $L_R(U) = U_R(U) = U$.

(iv) $U_R(\mathrm{X}\cup\mathrm{Y}) = U_R(X) \cup U_R(Y)$.

(v) $U_R(\mathrm{X}\cap \mathrm{Y})\subseteq \mathrm{U}_R(\mathrm{X})\cap\mathrm{U}_R(\mathrm{Y})$.

(vi) $L_R(\mathrm{X}\cup\mathrm{Y})\supseteq\mathrm{L}_R(\mathrm{X})\cup \mathrm{L}_R(\mathrm{Y})$.

(vii) $L_R(\mathrm{X}\cap\mathrm{Y}) = L_R(\mathrm{X})\cap \mathrm{L}_R(\mathrm{Y})$.

(viii) $L_R(\mathrm{X})\subseteq \mathrm{L}_R(\mathrm{Y})$ and $\mathrm{U}_R(\mathrm{X})\subseteq\mathrm{U}_R(\mathrm{Y})$ whenever $\mathrm{X}\subseteq\mathrm{Y}$.

(ix) $U_R(X^c) = [L_R(X)]^c$ and $\mathrm{L}_R(X^c) = [U_R(X)]^c$.

(x) $U_R[U_R(X)] = L_R[U_R(X)] = U_R(X)$.

(xi) $L_R[L_R(X)] = U_R[L_R(X)] = L_R(X)$.

Definition 2.3. [7] Let U be the universe, R be an equivalence relation on U and $\tau_R(X)$= $\{U, \phi, L_R(X), U_R(X), B_R(X)\}$, where $X \subseteq U$. Then by property 2.2, $\tau_R(X)$ satisfies the following axioms:

(i) U and $\phi \in \tau_R(X)$.

(ii) The union of the elements of any sub-collection of $\tau_R(X)$ is in $\tau_R(X)$.

(iii) The intersection of the elements of any finite subcollection of $\tau_R(X)$ is in $\tau_R(X)$.

This means that $\tau_R(X)$ is a topology on U called the nano topology on U with respect to X and $(U, \tau_R(X))$ as a nano topological space. The elements of $\tau_R(X)$ are called nano-open sets (briefly n-open sets).

In the rest of the paper, we denote a nano topological space by $(U,\mathcal{N})$, where $\mathcal{N} = \tau_R(X)$. The nano-interior and nano-closure of a subset A of U are denoted by $n\text{-}int(A)$ and $n\text{-}cl(A)$, respectively.

Definition 2.4. [8] Let $(U,\mathcal{N})$ be a nano topological space and $A \subseteq U$. Then A is said to be

(i) nano semi-open if $A \subseteq n\text{-}cl(n\text{-}int(A))$

(ii) nano pre-open if $A \subseteq n\text{-}int(n\text{-}cl(A))$

(iii) nano α-open if $A \subseteq n\text{-}int(n\text{-}cl(n\text{-}int(A)))$

The complement of a nano semi-open (respectively, nano pre-open,nano α-open) set is called a nano semi-closed (respectively, nano pre-closed, nano α-closed) set. The set of all nano open (respectively, nano semi-open, nano pre-open, nano α-open) sets is said to be NO (respectively, NSO,NPO, $N\alpha O$).

A nano topological space $(U,\mathcal{N})$ with an ideal I on U is called [10] a nano ideal topological space and is denoted by $(U,\mathcal{N}$,I). By $G_n(x) = \{G_n \mid x \in G_n, G_n \in \mathcal{N}\}$, we denote [10] the family of nano open sets containing x.

Definition 2.5. [10] Let $(U,\mathcal{N}, I)$ be a nano ideal topological space with $\tau_R(X) = \mathcal{N}$ and an ideal I on U and $(.)^*_n$ be a set operator from $P(U)$ to $P(U)$ ($P(U)$ is the set of all subsets of U). For a subset $A \subset U$, $A^*_n(I,\mathcal{N}) = \{x \in U : G_n \cap A \notin I, \text{for every } G_n \in G_n(x)\}$ is called the nano local function (briefly, n-local function) of A with respect to I and $\mathcal{N}$. We will simply write A^*_n for $A^*_n(I,\mathcal{N})$.

Theorem 2.6. [10] Let $(U,\mathcal{N})$ be a nano topological space with ideals I, I' on U and A, B be subsets of U. Then

(i) $A \subseteq B \Rightarrow A^*_n \subseteq B^*_n$,

(ii) $I \subseteq I' \Rightarrow A^*_n(I') \subseteq A^*_n(I)$,

(iii) $A^*_n = n\text{-}cl(A^*_n) \subseteq n\text{-}cl(A)$ (A^*_n is a nano closed subset of $n\text{-}cl(A)$),

(iv) $(A^*_n)^*_n \subseteq A^*_n$,

(v) $A^*_n \cup B^*_n = (A \cup B)^*_n$,

(vi) $A^*_n - B^*_n = (A - B)^*_n - B^*_n \subseteq (A - B)^*_n$,

(vii) $V \in \mathcal{N} \Rightarrow V \cap A^*_n = V \cap (V \cap A)^*_n \subseteq (V \cap A)^*_n$,

(viii) $J \in I \Rightarrow (A \cup J)^*_n = A^*_n = (A - J)^*_n$.

Theorem 2.7. [10] If $(U, \mathcal{N}, I)$ is a nano topological space with an ideal I and $A \subseteq A_n^*$, then $A_n^* = n\text{-}cl(A_n^*) = n\text{-}cl(A)$.

Definition 2.8. [10] Let $(U, \mathcal{N})$ be a nano topological space with an ideal I on U. The set operator $n\text{-}cl^*$ is called a nano$*$-closure and is defined as $n\text{-}cl^*(A) = A \cup A_n^*$ for $A \subseteq X$.

Theorem 2.9. [10] The set operator $n\text{-}cl^*$ satisfies the following conditions:

(i) $A \subseteq n\text{-}cl^*(A)$,

(ii) $n\text{-}cl^*(\phi) = \phi$ and $n\text{-}cl^*(U) = U$,

(iii) If $A \subset B$, then $n\text{-}cl^*(A) \subseteq n\text{-}cl^*(B)$,

(iv) $n\text{-}cl^*(A) \cup n\text{-}cl^*(B) = n\text{-}cl^*(A \cup B)$.

(v) $n\text{-}cl^*(n\text{-}cl^*(A)) = n\text{-}cl^*(A)$.

3. On nano-I-open and nano-I-closed sets

Definition 3.1. A subset A of a nano ideal topological space $(U, \mathcal{N}, I)$ is said to be nano-I-open (briefly, nI-open) if $A \subseteq n\text{-}int(A_n^*)$.

We denote $NIO(U, \mathcal{N}) = \{A \subseteq U : A \subseteq n\text{-}int(A_n^*)\}$ or simply we write NIO for $NIO(U, \mathcal{N})$ when there is no chance for confusion.

Remark 3.2. It is clear that nI-open and nano-open are independent concepts as it is shown by the following example.

Example 3.3. Let $U = \{a, b, c, d\}$ be the universe, $X = \{a, b\} \subset U$, $U/R = \{\{a\}, \{c\}, \{b, d\}\}$ and $\mathcal{N} = \{\phi, U, \{a\}, \{b, d\}, \{a, b, d\}\}$ and the ideal $I = \phi, \{a\}$.
(i) For $A = \{a, b, d\}$, we have $A_n^* = \{b, c, d\}$ and $n\text{-}int(A_n^*) = \{b, d\} \Rightarrow A \nsubseteq n\text{-}int(A_n^*)$. This shows that $A \in \mathcal{N}$ but $A \notin NIO$.
(ii) For $A = \{b\}$, we have $A_n^* = \{b, c, d\}$ and $n\text{-}int(A_n^*) = \{b, d\} \Rightarrow A \subseteq n\text{-}int(A_n^*)$. This shows that $A \in NIO$ but $A \notin \mathcal{N}$.

Theorem 3.4. Every nI-open set is a nano pre-open set. Also, nI-openness and nano semi-openness are independent concepts.

Proof. Straightforward and it follows from the Example 3.5.

The converse of Theorem 3.4 need not be true in general as shown by the following example.

Example 3.5. Let $U = \{a, b, c, d\}$ be the universe, $X = \{a, b\} \subset U$, $U/R = \{\{a\}, \{c\}, \{b, d\}\}$ and $\mathcal{N} = \{\phi, U, \{a\}, \{b, d\}, \{a, b, d\}\}$ and the ideal $I = \phi, \{a\}$.
(i) For $A = \{a, b, c\}$, we have $A_n^* = \{b, c, d\}$ and $n\text{-}int(A_n^*) = \{b, d\} \nsupseteq A$. But $n\text{-}cl(A) = U \Rightarrow n\text{-}int(n\text{-}cl(A)) = U \supseteq A$. This shows that $A \in NPO$ but $A \notin NIO$.
(ii) For $A = \{b\}$, we have $A_n^* = \{b, c, d\}$ and $n\text{-}int(A_n^*) = \{b, d\} \supseteq A$. But $n\text{-}cl(n\text{-}int(A)) = \phi \nsupseteq A$. This shows that $A \in NIO$ but $A \notin NSO$.
(iii) For $A = \{a, c\}$, we have $A_n^* = \{c\}$ and $n\text{-}int(A_n^*) = \phi \nsupseteq A$. But $n\text{-}cl(n\text{-}int(A)) = \{a, c\} \supseteq A$.

This shows that $A \notin NIO$ but $A \in NSO$.

Theorem 3.6. Arbitrary union of nI-open sets is also nI-open.

Proof. Let $(U, \mathcal{N}, I)$ be any nano ideal topological space and $W_i \in NIO(U, \mathcal{N})$ for $i \in \nabla$. This means that for each $i \in \nabla$, $W_i \in n\text{-}int((W_i)^*_n)$ and so $\cup_i W_{i \in \nabla} n\text{-}int((W_i)^*_n) \subseteq n\text{-}int(\cup_i W_i)^*_n$. Hence $\cup_i W_i \in NIO(U, \mathcal{N})$.

Remark 3.7. The intersection of two nI-open sets need not be nI-open as is illustrated by the following example.

Example 3.8. Let $U = \{a, b, c, d\}$ be the universe, $X = \{a, b, d\} \subset U$, $U/R = \{\{a\}, \{b, d\}, \{c\}\}$, $\mathcal{N} = \{\phi, U, \{a, c\}, \{b, d\}\}$ and the ideal $I = \phi, \{b\}, \{c\}, \{b, c\}$. Then $\{a, c, d\}, \{b, c, d\} \in NIO$, but $\{a, c, d\} \cap \{b, c, d\} = \{c, d\} \notin NIO$.

Theorem 3.9. Let $(U, \mathcal{N}, I)$ be a nano ideal topological space and $A, B \in U$. Then:

(i) If $A \in nIO(U, \mathcal{N})$ and $B \in \mathcal{N}$, then $A \cap B \in NIO(A)$.

(ii) If $A \in nIO(U, \mathcal{N})$ and $B \in NSO(U, \mathcal{N})$, then $A \cap B \in NSO(A)$.

(iii) If $A \in NIO(U, \mathcal{N})$ and $B \in \mathcal{N}$, then $A \cap B \in n\text{-}int(B \cap (B \cap A)^*_n)$.

Proof. (i) $A \cap B \subseteq n\text{-}int(A^*_n) \cap B = n\text{-}int(A^*_n \cap B)$ by Theorem 2.6. (vii), we have $A \cap B \subseteq n\text{-}int(A \cap B)^*_n$.
(ii) It follows from Theorem 2.6 (iii).
(iii) It follows directly from Theorem 2.6 (vii).

Theorem 3.10. For a nano ideal topological space $(U, \mathcal{N}, I)$ and $A \subseteq U$, we have:

(i) If $I = \phi$, then $A^*_n = n\text{-}cl(A)$, and hence the notions of nI-open and nano pre-open sets coincide.

(ii) If $I = P(U)$, then $A^*_n = \phi$ and hence A is nI-open iff $A = \phi$.

Theorem 3.11. For any nI-open set A of a nano topological space $(U, \mathcal{N}, I)$, we have $A^*_n = (n\text{-}int(A^*_n))^*_n$.

Definition 3.12. A subset A of a nano ideal topological space $(U, \mathcal{N}, I)$ is said to be nI-closed (briefly, nI-closed) if its complement is nI-open.

Theorem 3.13. For $A \subseteq (U, \mathcal{N}, I)$, we have $((int(A))^*_n)^c \neq (int(A^c))^*_n$ in general (see Example 3.14.), where A^c denotes the complement of A.

Example 3.14. Let $U = \{a, b, c, d\}$ be the universe, $X = \{a, c\} \subset U$, $U/R = \{\{a\}, \{b, c\}, \{d\}\}$ and $\mathcal{N} = \{U, \phi, \{a\}, \{a, b, c\}, \{b, c\}\}$ and the ideal $I = \phi, \{c\}, \{d\}, \{c, d\}$. Then it is clear that if $A = \{a, b, c\}$, then $((int(A))^*_n)^c = \{c\}$, but $(int(A^c))^*_n = \phi$.

Theorem 3.15. If $A \subseteq (U, \mathcal{N}, I)$ is nI-closed, then $A \supset (int(A))^*_n$.

Proof. It follows from the definition of nI-closed sets and Theorem 2.6 (iii).

Corollary 3.16. (i) The union of nI-closed set and n-closed set is nI-closed.
(ii) The unoin of nI-closed set and an $n\alpha$-closed set is nano pre-closed.

Theorem 3.17. If $A \subseteq (U, \mathcal{N}, I)$ is nI-open and nano semi-closed, then $A = int(A^*_n)$.

Proof. It follows from Theorem 2.6 (iii).

The nano semi-closure of a set A is defined as the intersection of all nano semi-closed sets containing A and is denoted by $n\text{-}scl(A)$ which is the smallest nano semi-closed set containing A.

Theorem 3.18. Let $(U, \mathcal{N}, I)$ be a nono ideal topological space and $A \in U$. Then the following are equivalent.

(i) A is nI-open.

(ii) $A \subset A^*_n$ and $n\text{-}scl(A) = n\text{-}int(n\text{-}cl(A))$.

(iii) $A \subset A^*_n$ and A is nano pre-open.

Proof. $A \in NIO(U)$ if and only if $A \subset A^*_n$ and $A \subset n\text{-}int(A^*_n)$ if and only if $A \subset A^*_n$ and $A \subset n\text{-}int(n\text{-}cl(A))$. Besides $n\text{-}cl(A) = A$ if and only if $A \subset A^*_n$ and $A \cup n\text{-}int(n\text{-}cl(A)) = n\text{-}int(n\text{-}cl(A))$ if and only if $A \subset A^*_n$ and $n\text{-}scl(A) = n\text{-}int(n\text{-}cl(A))$. Therefore, (i) and (ii) are equivalent. It is clear that (i) and (iii) are equivalent.

Theorem 3.19. For a subset $A \subseteq (U, \mathcal{N}, I)$, we have:

(i) If A is $n*$-closed and $A \in NIO(U)$, then $n\text{-}int(A) = n\text{-}int(A^*_n)$.

(ii) A is $n*$-closed iff A is n-open and nI-closed.

(iii) If A is $n*$-perfect, then $A = n\text{-}int(A^*_n)$, for every $A \in NIO(U, \mathcal{N})$.

Proof. Obvious.

4. Quasi-nI-open sets

Definition 4.1. A subset A of a nano ideal topological space $(U, \mathcal{N}, I)$ is quasi-nI-open (briefly, q-nI-open) if $A \subseteq n\text{-}cl(n\text{-}int(A^*_n))$.

Theorem 4.2. Every nI-open set is q-nI-open. Also, q-nI-openness and n-semiopenness (resp., preopenness) are independent concepts (see Examples 4.3.).

The family of all q-nI-open sets is denoted by $QNIO(U, \mathcal{N})$.

Example 4.3. Let $U = \{a, b, c, d\}$ be the universe, $X = \{a, d\} \subset U$, $U/R = \{\{a, c\}, \{b\}, \{d\}\}$ and $\mathcal{N} = \{U, \phi, \{d\}, \{a, c, d\}, \{a, c\}\}$ and the ideal $I = \phi, \{d\}$, then we can show the followings:
(i) $A = \{a, b\}$ is q-nI-open but is not nI- open, nano pre-open and nano semi-open.
(ii) $B = \{a, b, d\}$ is nano pre-open but not q-nI-open.
(iii) $C = \{d\}$ is nano semi-open but not q-nI-open.

Theorem 4.4. Arbitrary union of quasi-nI-open sets is also quasi-nI-open.

Proof. Let $(U, \mathcal{N}, I)$ be any space and $W_i \in QNIO(U, \mathcal{N})$ for $i \in \nabla$. This means that for each $i \in \nabla$, $W_i \in n\text{-}cl(n\text{-}int((W_i)^*_n))$ and so $\cup_{i\in\nabla} n\text{-}cl(n\text{-}int((W_i)^*_n)) \subseteq n\text{-}cl(n\text{-}int(\cup_i W_i)^*_n)$. Hence $\cup_i W_i \in QNIO(U, \mathcal{N})$.

Remark 4.5. A finite intersection of quasi-nI-open sets need not in general be quasi-nI-open, as Example 4.6 shows.

Example 4.6. In Example 4.3., we deduce that the two sets $W_1 = \{a, b\}$ and $W_2 = \{b, c\}$ are quasi-nI-open while their intersection are not.

Remark 4.5 suggests the following result.

Theorem 4.7. In a nano ideal topological space $(U, \mathcal{N}, I)$, if $G \in \mathcal{N}$ and $W \in QNIO(U, \mathcal{N})$, then $G \cap W$ is q-nI-open.

Proof. By hypothesis of the theorem and the fact that $G \cap n\text{-}cl(W) \subseteq n\text{-}cl(G \cap W)$, we have, $G \cap W \subseteq G \cap (n\text{-}cl(n\text{-}int(W)^*_n)) \subseteq n\text{-}cl(G \cap n\text{-}int(W)^*_n)$. Then by Theorem 2.6, we obtain $G \cap W \subseteq n\text{-}cl(n\text{-}int(G \cap W)^*_n)$. Hence the result.

Recall that a subset A of a nano ideal topological space $(U, \mathcal{N}, I)$ is $n*$-dense in itself [10] (resp. $n*$-perfect [10], $*$-closed [10]) if $A \subseteq A^*_n$ (resp. $A = A^*_n$, $A^*_n \subseteq A$).

Proposition 4.8. The following statements hold:

(i) For $(U, \mathcal{N}, P(U))$ then $QNIO(U, \mathcal{N}) = NIO(U, \mathcal{N})$.

(ii) For any nano ideal topological space $(U, \mathcal{N}, I)$, every q-nI-open set which is $n*$-closed, it is nano semi-open.

(iii) For any nano ideal topological space $(U, \mathcal{N}, I)$, every n-semi open which is $n*$-dense-in-itself, it is q-nI-open.

Since the combination of $n*$-dense-in-itself and $n*$-closeness of any $W \in U$ in a nano ideal topological space $(U, \mathcal{N}, I)$ is equivalent with the $n*$-perfect property of W in the same space, then it follows that the two classes $QNIO(U, \mathcal{N})$ and $NSO(U, \mathcal{N})$ coincides with each other.

5. nI-continuous functions and quasi nI-continuous functions

Definition 5.1. A function $f : (U, \mathcal{N}, I) \to (V, \mathcal{N}')$ is called nano-I-continuous (briefly, nI-continuous) function if for every $V \in \mathcal{N}'$, $f^{-1}(V) \in NIO(U, \mathcal{N})$.

Definition 5.2. A function $f : (U, \mathcal{N}, I) \to (V, \mathcal{N}')$ is called quasi nI-continuous (briefly, qnI-continuous) if for every $V \in \mathcal{N}'$, $f^{-1}(V) \in QNIO(U, \mathcal{N})$.

Theorem 5.3. For a function $f : (U, \mathcal{N}, I) \to (Y, \mathcal{N}')$, the following are equivalent:

(i) f is qnI-continuous.

(ii) For each $x \in U$ and each $V \in \mathcal{N}'$ containing $f(x)$, there exists $W \in QnIO(U)$ containing x such that $f(W) \subset V$.

(iii) For each $x \in U$ and each $V \in \mathcal{N}'$ containing $f(x)$, $(f^{-1}(V))^*$ is a neighbourhood of x.

Proof. $(i) \Rightarrow (ii)$ Let $x \in X$ and V be a qnI-open set of Y containing $f(x)$. Since f is nI-continuous, $f^{-1}(V)$ is a qI-open set. Putting $W = f^{-1}(V)$, we have $f(W) \subset V$.
$(ii) \Rightarrow (i)$ Let A be a nI-open set in Y. If $f^{-1}(A) = \phi$, then $f^{-1}(A)$ is clearly a qnI-open set. Assume that $f^{-1}(A) \neq \phi$. Let $x \in f^{-1}(A)$. Then $f(x) \in A$, which implies that there exists a qnI-open W containing x such that $f(W) \subset A$. Thus $W \subset f^{-1}(A)$. Since W is a qnI-open, $x \in W \subset n\text{-}int(W_n^*) \subset n\text{-}int((f^{-1}(A)_n^*))$ and so $f^{-1}(A) \subset n\text{-}int(f^{-1}(A)_n^*)$. Hence $f^{-1}(A)$ is a qnI-open set and so f is qnI-continuous.
$(ii) \Rightarrow (iii)$ Let $x \in X$ and V be a nI-open set of Y containing $f(x)$. Then there exist a qnI-open set W containing x such that $f(W) \subset V$. It follows that $W \subset f^{-1}(f(W)) \subset f^{-1}(V)$. Since W is a qnI-open set, $x \in W \subset n\text{-}int(W_n^*) \subset n\text{-}int(f^{-1}(V)_n^*) \subset f^{-1}(V)_n^*$. Hence $f^{-1}(V)_n^*$ is a qnI-neighborhood of x.
$(iii) \Rightarrow (i)$ Obvious.

In the following example, it is shown that a qnI-continuous function is not nI-continuous.

Example 5.4. Let $U = \{a, b, c, d\}$ be the universe, $X = \{a, d\} \subset U$, $U/R = \{\{a, c\}, \{b\}, \{d\}\}$ and $\mathcal{N} = \{U, \phi, \{d\}, \{a, c, d\}, \{a, c\}\}$ and the ideal $I = \phi, \{d\}$ and $V = \{x, y, z, w\}$ be the universe, $Y = \{x, w\} \subset V$, $V/R' = \{\{x, z\}, \{y\}, \{w\}\}$ and $\mathcal{N}' = \{V, \phi, \{w\}, \{x, z, w\}, \{x, z\}\}$ and the function $f : U \to V$is defined as $f(a) = x, f(b) = y, f(c) = z, f(d) = w$. Thus $A = \{a, b\}$ is q-nI-continuous but not nI-continuous.

In the following two examples we show that n-continuous and nI-continuous are independent notions. Recall that a function $f : (U, \mathcal{N}) \to (V, \mathcal{N}')$ is called nano-continuous function [7] if for every $V \in \mathcal{N}'$, $f^{-1}(V) \in NO(U, \mathcal{N})$.

Example 5.5. Let Let $U = \{a, b, c, d\}$ be the universe, $X = \{a, c\} \subset U$, $U/R = \{\{a\}, \{b, c\}, \{d\}\}$ and $\mathcal{N} = \{U, \phi, \{a\}, \{a, b, c\}, \{b, c\}\}$ and the ideal $I = \phi, \{c\}, \{d\}, \{c, d\}$. and Let $V = \{x, y, z, w\}$ be the universe, $X = \{x, y\} \subset V$, $V/R' = \{\{x\}, \{z\}, \{y, w\}\}$ and $\mathcal{N}' = \{\phi, V, \{x\}, \{y, w\}, \{x, y, w\}\}$ and the function $f : U \to V$ is defined as $f(a) = y, f(b) = w, f(c) = x, f(d) = z$. It is obvious that $A = \{a, b\}$ is nI-continuous but not n-continuous.

Example 5.6. Let $U = \{a, b, c, d\}$ be the universe, $X = \{a, b\} \subset U$, $U/R = \{\{a\}, \{c\}, \{b, d\}\}$ and $\mathcal{N} = \{\phi, U, \{a\}, \{b, d\}, \{a, b, d\}\}$ and the ideal $I = \phi, \{a\}$. Let $V = \{x, y, z, w\}$ be the universe, $X = \{x, z\} \subset V$, $V/R' = \{\{x\}, \{w\}, \{y, z\}\}$ and $\mathcal{N}' = \{\phi, V, \{x\}, \{y, z\}, \{x, y, z\}\}$ and the function $f : U \to V$is defined as $f(a) = x, f(b) = y, f(c) = w, f(d) = z$. One can observe that $A = \{a, b, d\}$ is n-continuous but not nI-continuous.

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