PROPERTIES OF β-OPEN SETS IN IDEAL MINIMAL SPACES

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ABSTRACT. In this paper, we introduce and study the class of β-open sets and other related classes of notions in ideal minimal spaces.

1. INTRODUCTION

The notion of minimal structures which is a generalization of a topology on a given nonempty set was introduced by Popa and Noiri [17]. They also introduced the notion of m-continuous function as a function defined between a minimal structure and a topological space. They showed that the m-continuous functions have properties similar to those of continuous functions between topological spaces. However, continuities between two minimal structures have already been in literature (see [8], [9], [10] and [14]). Let X be a topological space and $A \subset X$. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subfamily m of the power set $P(X)$ of a nonempty set X is called a minimal structure [17] on X if $\emptyset$ and X belong to m. By (X, m), we denote a nonempty set X with a minimal structure m on X. Let X be a set with $\mid X \mid \leq 2$, where $\mid X \mid$ denotes the cardinality of X. Then any minimal structure on X is always a topology on X. Thus the study of minimal structure will be interesting when $\mid X \mid > 2$. The members of the minimal structure m are called m-open sets [17], and the pair (X, m) is called an m-space. The complement of an m-open set is said to be m-closed [17]. The concept of ideals in topological spaces has been introduced and studied by Kuratowski [7] and Vaidyanathasamy [19]. An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Given a minimal space (X, m) with an ideal $\mathcal{I}$ on X and if $\mathcal{P}(X)$ is the set of all subsets of X, a set operator $(.)^*_m$: $\mathcal{P}(X) \to \mathcal{P}(X)$, called the local minimal function [18] of A with respect to m and $\mathcal{I}$, is defined as follows: for $A \subset X$, $A^*_m(\mathcal{I}, m) = \{x \in X | U \cap A \notin \mathcal{I}$ for every $U \in m(x)\}$, where $m(x) = \{U \in m | x \in U\}$. The set operator $m\,\mathrm{Cl}^*(.)$ defined as $m\,\mathrm{Cl}^*(A) = A \cup A^*_m$ for $A \subset X$ is called a minimal $*$-closure. The minimal structure $m^*(\mathcal{I}, m)$ generated by

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$m^*(\mathcal{I}, m) = \{U \subset X : mCl^*(X \setminus U) = X \setminus U\}$ is called the $*$-minimal structure which is finer than m. And $m\,\mathrm{Int}^*(A)$ denotes the interior of A in $m^*(\mathcal{I}, m)$. Moreover, if we consider grill [2] on the minimal space and change the set operator by $A^*_m(\mathcal{G}, m) = \{x \in X \mid U \cap A \in \mathcal{G}\}$, where $\mathcal{G}$ is a grill, then we also have the similar study (see [16], [15]). We observe from $(.)^*_m$, the value of A^*_m. In this context, for an ideal topological space $(X, \tau, \mathcal{I})$, the collection $\{A \subseteq X \mid Int(Cl(A)) = \emptyset\}$ forms an ideal and it is called the ideal of nowhere dense sets. Many results in the ideal topological space have been proved through this ideal. But in the field of minimal structure m on a set X, the collection $\{A \subseteq X \mid mInt(mCl(A)) = \emptyset\}$ does not form an ideal on X (see [11]).

2. Preiliminaries

Definition 2.1. [17] *Given $A \subset X$, the m-interior of A and the m-closure of A are defined by $m\,\mathrm{Int}(A) = \cup\{W \mid W \in m, W \subseteq A\}$ and $m\,\mathrm{Cl}(A) = \cap\{F \mid A \subseteq F, X \setminus F \in m\}$, respectively.*

Theorem 2.2. [17] *Let (X, m) be an m-space, and A, B subsets of X. Then $x \in m\,\mathrm{Cl}(A)$ if and only if $U \cap A \neq \varnothing$ for every $U \in m$ containing x. The following properties are satisfied:*

(i) $m\,\mathrm{Cl}(m\,\mathrm{Cl}(A)) = m\,\mathrm{Cl}(A)$.
(ii) $m\,\mathrm{Int}(m\,\mathrm{Int}(A)) = m\,\mathrm{Int}(A)$.
(iii) $m\,\mathrm{Int}(X \setminus A) = X \setminus m\,\mathrm{Cl}(A)$.
(iv) $m\,\mathrm{Cl}(X \setminus A) = X \setminus m\,\mathrm{Int}(A)$.
(v) *If* $A \subset B$, *then* $m\,\mathrm{Cl}(A) \subset m\,\mathrm{Cl}(B)$.
(vi) $m\,\mathrm{Cl}(A \cup B) \supset m\,\mathrm{Cl}(A) \cup m\,\mathrm{Cl}(B)$.
(vii) $A \subset m\,\mathrm{Cl}(A)$ *and* $m\,\mathrm{Int}(A) \subset A$.

A minimal structure m is said to have property $\mathcal{B}$ [17] if m is closed under arbitrary union. Note that $mCl(A \cup B) = mCl(A) \cup mCl(B)$, if $\mid X \mid \leq 2$.

Theorem 2.3. [17] *Let (X, m) be an m-space and m satisfy property $\mathcal{B}$. For a subset A of X, the following properties hold:*

(i) $A \in m$ *if and only if* $m\,\mathrm{Int}(A) = A$.
(ii) A *is m-closed if and only if* $m\,\mathrm{Cl}(A) = A$.
(iii) $m\,\mathrm{Int}(A) \in m$ *and* $m\,\mathrm{Cl}(A)$ *is m-closed.*

Definition 2.4. *A subset A of a minimal space (X, m) is said to be:*

(i) *m-semiopen* [8] *if* $A \subset m\,\mathrm{Cl}(m\,\mathrm{Int}(A))$.
(ii) *m-preopen* [9] *if* $A \subset m\,\mathrm{Int}(m\,\mathrm{Cl}(A))$.

Definition 2.5. *A function $f : (X, m) \to (Y, \tau)$ is said to be:*

(i) *m-semicontinuous* [8] *if the inverse image of every open set of Y is m-semiopen in (X, m).*

(ii) *m-precontinuous* [9] *if the inverse image of every open set of* Y *is m-preopen in* (X, m).

Definition 2.6. *A subset* A *of an ideal minimal space* $(X, m, \mathcal{I})$ *is said to be*

(i) *m-semi-$\mathcal{I}$-open* [4] *if* $A \subset m\,\mathrm{Cl}^*(m\,\mathrm{Int}(A))$.
(ii) *m-pre-$\mathcal{I}$-open* [3] *if* $A \subset m\,\mathrm{Int}(m\,\mathrm{Cl}^*(A))$.
(iii) *m-α-$\mathcal{I}$-open* [5] *if* $A \subset m\,\mathrm{Int}(m\,\mathrm{Cl}^*(m\,\mathrm{Int}(A)))$.
(iv) *m-$\mathcal{I}$-open* [12], [13] *if* $A \subseteq m\,\mathrm{Int}(A^*_m)$.
(v) *M-$\mathcal{I}$-open* [12], [13] *if* $A \subseteq (m\,\mathrm{Int}(A))^*_m$.

Definition 2.7. *A function* $f : (X, m, \mathcal{I}) \to (Y, \tau)$ *is said to be*

(i) *m-pre-$\mathcal{I}$-continuous* [3] *if the inverse image of every open set of* Y *is m-pre-$\mathcal{I}$-open in* X.
(ii) *m-semi-$\mathcal{I}$-continuous* [4] *if the inverse image of every open set of* Y *is m-semi-$\mathcal{I}$-open in* X.
(iii) *m-α-$\mathcal{I}$-continuous* [5] *if the inverse image of every open set of* Y *is m-α-$\mathcal{I}$-open in* X.
(iv) *m-$\mathcal{I}$-continuous* [12], [13] *if the inverse image of every open set of* Y *is m-$\mathcal{I}$-open in* X.
(v) *M-$\mathcal{I}$-continuous* [12], [13] *if the inverse image of every open set of* Y *is M-$\mathcal{I}$-open in* X.

3. m-β-$\mathcal{I}$-open sets

Definition 3.1. *A subset* A *of an ideal minimal space* $(X, m, \mathcal{I})$ *is said to be m-β-$\mathcal{I}$-open if* $A \subset m\,\mathrm{Cl}(m\,\mathrm{Int}(m\,\mathrm{Cl}^*(A)))$. *The complement of m-$\beta$-$\mathcal{I}$-open is called m-$\beta$-$\mathcal{I}$-closed. The family of all m-β-$\mathcal{I}$-open (resp. m-β-$\mathcal{I}$-closed) sets of* $(X, m, \mathcal{I})$ *is denoted by* $\beta\mathcal{I}O(X, m)$ *(resp.* $\beta\mathcal{I}C(X, m)$*). Also, the family of all m-β-$\mathcal{I}$-open sets of* $(X, m, \mathcal{I})$ *containing* x *is denoted by* $m\beta\mathcal{I}O(X, x)$.

Proposition 3.2. (i) *Every m-semi-$\mathcal{I}$-open set is m-β-$\mathcal{I}$-open.*
(ii) *Every m-pre-$\mathcal{I}$-open set is m-β-$\mathcal{I}$-open.*
(iii) *Every m-$\mathcal{I}$-open set is m-β-$\mathcal{I}$-open.*
(iv) *Every M-$\mathcal{I}$-open set is m-β-$\mathcal{I}$-open.*

Proof. This is an immediate consequence of definitions. □

The following examples show that the converses of Proposition 3.2 are not true in general.

Example 3.3. *Let* $X = \{a, b, c\}$, $m = \{\emptyset, \{a\}, \{b\}, X\}$ *and* $\mathcal{I} = \{\emptyset\}$. *Then the set* $\{b, c\}$ *is neither m-pre-$\mathcal{I}$-open nor m-$\mathcal{I}$-open.*

Example 3.4. *Let* $X = \{a, b, c\}$, $m = \{\emptyset, \{a\}, \{a, b\}, \{b, c\}, X\}$ *and* $\mathcal{I} = \{\emptyset\}$. *Then the set* $\{b\}$ *is neither m-semi-$\mathcal{I}$-open nor M-$\mathcal{I}$-open.*

Corollary 3.5. *Every m-α-$\mathcal{I}$-open set is m-β-$\mathcal{I}$-open.*

Proposition 3.6. *Let $(X, m, \mathcal{I})$ be an ideal minimal space, $A \subset X$ and $\mathcal{I} = \mathcal{P}(X)$. Then A is m-β-$\mathcal{I}$-open if and only if A is m-semiopen.*

Proof. The proof follows from the fact that, if $\mathcal{I} = \mathcal{P}(X)$, then $A^*_m = \emptyset$ for every subset A of X. □

Theorem 3.7. *Let $(X, m, \mathcal{I})$ be an ideal minimal space. If $A \subset B \subset m\,\mathrm{Cl}^*(A)$ and B is m-β-$\mathcal{I}$-open, then A is m-β-$\mathcal{I}$-open.*

Proof. Let $A \subset B \subset m\,\mathrm{Cl}^*(A)$ and B be an m-β-$\mathcal{I}$-open set. We have $m\,\mathrm{Cl}^*(A) = m\,\mathrm{Cl}^*(B)$. Thus, $A \subset B \subset m\,\mathrm{Cl}(m\,\mathrm{Int}(m\,\mathrm{Cl}^*(B))) = m\,\mathrm{Cl}(m\,\mathrm{Int}(m\,\mathrm{Cl}^*(A)))$; hence A is m-β-$\mathcal{I}$-open. □

Theorem 3.8. *Let $(X, m, \mathcal{I})$ be an ideal minimal space. If $A \subset B \subset m\,\mathrm{Cl}(A)$ and A is m-β-$\mathcal{I}$-open, then B is m-β-$\mathcal{I}$-open.*

Proof. Let $A \subset B \subset m\,\mathrm{Cl}(A)$ and A be m-β-$\mathcal{I}$-open. Then, $A \subset m\,\mathrm{Cl}(m\,\mathrm{Int}(m\,\mathrm{Cl}^*(A)))$. Since $B \subset m\,\mathrm{Cl}(A)$, then

$$B \subset m\,\mathrm{Cl}(m\,\mathrm{Cl}(\mathrm{Int}(m\,\mathrm{Cl}^*(A)))) = m\,\mathrm{Cl}(m\,\mathrm{Int}(m\,\mathrm{Cl}^*(A))) \subset m\,\mathrm{Cl}(m\,\mathrm{Int}(m\,\mathrm{Cl}^*(B))).$$

Thus, B is m-β-$\mathcal{I}$-open. □

Corollary 3.9. *Let $(X, m, \mathcal{I})$ be an ideal minimal space. If A is m-β-$\mathcal{I}$-open, then $mCl^*(A)$ and $mCl(A)$ are m-β-$\mathcal{I}$-open.*

Theorem 3.10. *If $\{A_\alpha\}_{\alpha\in\Omega}$ is a family of m-β-$\mathcal{I}$-open sets in $(X, m, \mathcal{I})$, then $\bigcup_{\alpha\in\Omega} A_\alpha$ is m-β-$\mathcal{I}$-open in $(X, m, \mathcal{I})$.*

Proof. Since $\{A_\alpha : \alpha \in \Omega\} \subset \beta\mathcal{I}O(X, m)$, then $A_\alpha \subset m\,\mathrm{Cl}(m\,\mathrm{Int}(m\,\mathrm{Cl}^*(A_\alpha)))$ for every $\alpha \in \Omega$. Thus,

$$\cup_{\alpha\in\Omega} A_\alpha \subset \cup_{\alpha\in\Omega} m\,\mathrm{Cl}(m\,\mathrm{Int}(m\,\mathrm{Cl}^*(A_\alpha))) \subset m\,\mathrm{Cl}(m\,\mathrm{Int}(m\,\mathrm{Cl}^*(\cup_{\alpha\in\Omega} A_\alpha))).$$

Therefore, we obtain $\cup_{\alpha\in\Omega} A_\alpha \subset m\,\mathrm{Cl}(m\,\mathrm{Int}(m\,\mathrm{Cl}^*(\cup_{\alpha\in\Omega} A_\alpha)))$. Hence any union of m-β-$\mathcal{I}$-open sets is m-β-$\mathcal{I}$-open. □

Note that for an ideal minimal space $(X, m, \mathcal{I})$, $\emptyset$, $X \in \beta\mathcal{I}O(X, m)$. Thus $\beta\mathcal{I}O(X, m)$ forms a minimal structure on X having property $\mathcal{B}$.

Theorem 3.11. *A subset A of an ideal minimal space $(X, m, \mathcal{I})$ is m-β-$\mathcal{I}$-open if and only if $m\,\mathrm{Cl}(A) = m\,\mathrm{Cl}(m\,\mathrm{Int}(m\,\mathrm{Cl}^*(A)))$.*

Proof. Let A be an m-β-$\mathcal{I}$-open set of X. Then, we have $A \subset m\,\mathrm{Cl}(m\,\mathrm{Int}(m\,\mathrm{Cl}^*(A)))$ and hence

$$m\,\mathrm{Cl}(A) \subset m\,\mathrm{Cl}(m\,\mathrm{Int}(m\,\mathrm{Cl}^*(A))) \subset m\,\mathrm{Cl}(m\,\mathrm{Int}(m\,\mathrm{Cl}(A))) \subset m\,\mathrm{Cl}(A).$$ Therefore, $m\,\mathrm{Cl}(A) = m\,\mathrm{Cl}(m\,\mathrm{Int}(m\,\mathrm{Cl}^*(A)))$.

The converse is obvious. □

Definition 3.12. *An ideal minimal space $(X, m, \mathcal{I})$ is said to be minimal extremally disconnected if $m\,\mathrm{Cl}(m\,\mathrm{Int}(m\,\mathrm{Cl}^*(A))) \in m$ for every $A \in m$.*

Example 3.13. [1] *Consider the minimal space (X, m) where $X = \{a, b, c, d\}$, $m = \{\emptyset, \{a\}, \{a, c\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. Clearly, (X, m) is a minimal extremally disconnected space.*

Proposition 3.14. *Let $(X, m, \mathcal{I})$ be an minimal extremally disconnected space and m have property $\mathcal{B}$. Then if A is m-β-$\mathcal{I}$-open, then A is m-preopen in X.*

Proof. Let A be an m-β-$\mathcal{I}$-open set of X, $A \subset m\,\mathrm{Cl}(m\,\mathrm{Int}(m\,\mathrm{Cl}^*(A)))$. Since X is minimal extremally disconnected, for $m\,\mathrm{Int}(m\,\mathrm{Cl}^*(A)) \in m$, $m\,\mathrm{Cl}(m\,\mathrm{Int}(m\,\mathrm{Cl}^*(A))) \in m$. Hence, $A \subset m\,\mathrm{Cl}(m\,\mathrm{Int}(m\,\mathrm{Cl}^*(A))) \subset m\,\mathrm{Int}(m\,\mathrm{Cl}(m\,\mathrm{Int}(m\,\mathrm{Cl}^*(A)))) \subset m\,\mathrm{Int}(m\,\mathrm{Cl}(A))$; hence A is m-preopen in X. □

Definition 3.15. *In an ideal minimal space $(X, m, \mathcal{I})$, $A \subset X$ is said to be m-β-$\mathcal{I}$-closed if $X \backslash A$ is m-β-$\mathcal{I}$-open in X.*

Theorem 3.16. *Let $(X, m, \mathcal{I})$ be an ideal minimal space. $A \subset X$ is an m-β-$\mathcal{I}$-closed set if and only if $m\,\mathrm{Int}(m\,\mathrm{Cl}(m\,\mathrm{Int}^*(A))) \subset A$.*

Proof. The proof follows from the definitions. □

Theorem 3.17. *If a subset A of an ideal minimal space $(X, m, \mathcal{I})$ is m-β-$\mathcal{I}$-closed, then $m\,\mathrm{Int}(m\,\mathrm{Cl}^*(m\,\mathrm{Int}(A))) \subset A$*

Proof. The proof follows from the fact that $m\,\mathrm{Cl}^*(A) \subset m\,\mathrm{Cl}(A)$ and $mInt(A) \subset mInt^*(A)$ for every subset A of X. □

Theorem 3.18. *Arbitrary intersection of m-β-$\mathcal{I}$-closed sets is always m-β-$\mathcal{I}$-closed.*

Proof. This follows from Theorem 3.10. □

It is obvious that $\emptyset$ and X are m-β-$\mathcal{I}$-closed. Thus the collection of all m-β-$\mathcal{I}$-closed sets in an ideal minimal space forms a minimal structure again.

Definition 3.19. *Let $(X, m, \mathcal{I})$ be an ideal minimal space, S a subset of X and x be a point of X. Then*

(i) *x is called an m-β-$\mathcal{I}$-interior point of S if there exists $V \in \beta\mathcal{I}O(X, m)$ such that $x \in V \subset S$.*

ii) *the set of all m-β-$\mathcal{I}$-interior points of S is called the m-β-$\mathcal{I}$-interior of S and is denoted by $m\beta\mathcal{I}\,\mathrm{Int}(S)$.*

Theorem 3.20. *Let A and B be subsets of $(X, m, \mathcal{I})$. The following properties hold:*

(i) *$m\beta\mathcal{I}\,\mathrm{Int}(A) = \cup\{T : T \subset A$ and $T \in \beta\mathcal{I}O(X, m)\}$.*

(ii) $m\beta\mathcal{I}\,\mathrm{Int}(A)$ *is the largest* m-β-$\mathcal{I}$-*open subset of* X *contained in* A.
(iii) A *is* m-β-$\mathcal{I}$-*open if and only if* $A = m\beta\mathcal{I}\,\mathrm{Int}(A)$.
(iv) $m\beta\mathcal{I}\,\mathrm{Int}(m\beta\mathcal{I}\,\mathrm{Int}(A)) = m\beta\mathcal{I}\,\mathrm{Int}(A)$.
(v) *If* $A \subset B$, *then* $m\beta\mathcal{I}\,\mathrm{Int}(A) \subset m\beta\mathcal{I}\,\mathrm{Int}(B)$.
(vi) $m\beta\mathcal{I}\,\mathrm{Int}(A) \cup m\beta\mathcal{I}\,\mathrm{Int}(B) \subset m\beta\mathcal{I}\,\mathrm{Int}(A \cup B)$.
(vii) $m\beta\mathcal{I}\,\mathrm{Int}(A \cap B) \subset m\beta\mathcal{I}\,\mathrm{Int}(A) \cap m\beta\mathcal{I}\,\mathrm{Int}(B)$.

Proof. Obvious from the fact that the collection $\beta\mathcal{I}O(X,m)\}$ is a minimal structure having property $\mathcal{B}$. □

Definition 3.21. *Let* $(X, m, \mathcal{I})$ *be an ideal minimal space,* S *a subset of* X *and* x *be a point of* X. *Then*

(i) x *is called an* m-β-$\mathcal{I}$-*cluster point of* S *if* $V \cap S \neq \emptyset$ *for every* $V \in m\beta\mathcal{I}O(X, x)$.
(ii) *the set of all* m-β-$\mathcal{I}$-*cluster points of* S *is called the* m-β-$\mathcal{I}$-*closure of* S *and is denoted by* $m\beta\mathcal{I}\,\mathrm{Cl}(S)$.

Theorem 3.22. *Let* A *and* B *be subsets of* $(X, m, \mathcal{I})$. *Then the following properties hold:*

(i) $m\beta\mathcal{I}\,\mathrm{Cl}(A) = \cap\{F : A \subset F$ *and* $F \in \beta\mathcal{I}C(X,m)\}$.
(ii) $m\beta\mathcal{I}\,\mathrm{Cl}(A)$ *is the smallest* m-β-$\mathcal{I}$-*closed subset of* X *containing* A.
(iii) A *is* m-β-$\mathcal{I}$-*closed if and only if* $A = m\beta\mathcal{I}\,\mathrm{Cl}(A)$.
(iv) $m\beta\mathcal{I}\,\mathrm{Cl}(m\beta\mathcal{I}\,\mathrm{Cl}(A)) = m\beta\mathcal{I}\,\mathrm{Cl}(A)$.
(v) *If* $A \subset B$, *then* $m\beta\mathcal{I}\,\mathrm{Cl}(A) \subset m\beta\mathcal{I}\,\mathrm{Cl}(B)$.
(vi) $m\beta\mathcal{I}\,\mathrm{Cl}(A) \cup m\beta\mathcal{I}\,\mathrm{Cl}(B) \subset m\beta\mathcal{I}\,\mathrm{Cl}(A \cup B)$.
(vii) $m\beta\mathcal{I}\,\mathrm{Cl}(A \cap B) \subset m\beta\mathcal{I}\,\mathrm{Cl}(A) \cap m\beta\mathcal{I}\,\mathrm{Cl}(B)$.

Proof. The proof follows from the definitions. □

Theorem 3.23. *Let* $(X, m, \mathcal{I})$ *be an ideal minimal space and* $A \subset X$. *A point* $x \in m\beta\mathcal{I}\,\mathrm{Cl}(A)$ *if and only if* $U \cap A \neq \emptyset$ *for every* $U \in m\beta\mathcal{I}O(X, x)$.

Proof. Suppose that $x \in m\beta\mathcal{I}\,\mathrm{Cl}(A)$. We shall show that $U \cap A \neq \emptyset$ for every $U \in m\beta\mathcal{I}O(X, x)$. Suppose that there exists $U \in m\beta\mathcal{I}O(X, x)$ such that $U \cap A = \emptyset$. Then $A \subset X\backslash U$ and $X\backslash U$ is m-β-$\mathcal{I}$-closed. Since $A \subset X\backslash U$, $m\beta\mathcal{I}\,\mathrm{Cl}(A) \subset m\beta\mathcal{I}\,\mathrm{Cl}(X\backslash U)$. Since $x \in m\beta\mathcal{I}\,\mathrm{Cl}(A)$, we have $x \in m\beta\mathcal{I}\,\mathrm{Cl}(X\backslash U)$. Since $X\backslash U$ is m-β-$\mathcal{I}$-closed, we have $x \in X\backslash U$; hence $x \notin U$, which is contrary to the fact that $x \in U$. Therefore, $U \cap A \neq \emptyset$. Conversely, suppose that $U \cap A \neq \emptyset$ for every $U \in m\beta\mathcal{I}O(X, x)$. We shall show that $x \in m\beta\mathcal{I}\,\mathrm{Cl}(A)$. Suppose that $x \notin m\beta\mathcal{I}\,\mathrm{Cl}(A)$. Then there exists $U \in m\beta\mathcal{I}O(X, x)$ such that $U \cap A = \emptyset$. This is contrary to $U \cap A \neq \emptyset$; hence $x \in m\beta\mathcal{I}\,\mathrm{Cl}(A)$. □

Theorem 3.24. *Let* $(X, m, \mathcal{I})$ *be an ideal minimal space and* $A \subset X$. *Then the following propeties hold:*

(i) $m\beta\mathcal{I}\,\mathrm{Int}(X\backslash A) = X\backslash m\beta\mathcal{I}\,\mathrm{Cl}(A)$;
(ii) $m\beta\mathcal{I}\,\mathrm{Cl}(X\backslash A) = X\backslash m\beta\mathcal{I}\,\mathrm{Int}(A)$.

Proof. (i). Let $x \notin m\beta\mathcal{I}\,\mathrm{Cl}(A)$. There exists $V \in m\beta\mathcal{I}O(X, x)$ such that $V \cap A = \emptyset$; hence we obtain $x \in m\beta\mathcal{I}\,\mathrm{Int}(X\backslash A)$. This shows that $X\backslash m\beta\mathcal{I}\,\mathrm{Cl}(A) \subset m\beta\mathcal{I}\,\mathrm{Int}(X\backslash A)$. Let $x \in m\beta\mathcal{I}\,\mathrm{Int}(X\backslash A)$. Since $m\beta\mathcal{I}\,\mathrm{Int}(X\backslash A)\cap A = \emptyset$, we obtain $x \notin m\beta\mathcal{I}\,\mathrm{Cl}(A)$; hence $x \in X\backslash m\beta\mathcal{I}\,\mathrm{Cl}(A)$. Therefore, we obtain $m\beta\mathcal{I}\,\mathrm{Int}(X\backslash A) = X\backslash m\beta\mathcal{I}\,\mathrm{Cl}(A)$.
(ii). This follows from (i). □

Definition 3.25. *Let $(X, m, \mathcal{I})$ be an ideal minimal space and $A \subset X$.*

(i) *The set defined by $\{x :$ for every m-β-$\mathcal{I}$-open set U containing x, $U \cap (A\backslash\{x\}) \neq \emptyset\}$ is called the m-β-$\mathcal{I}$-derived set of A and is denoted by $m\beta\mathcal{I}D(A)$."*
(ii) *The m-β-$\mathcal{I}$-frontier of A, denoted by $m\beta\mathcal{I}Fr(A)$, is defined as $m\beta\mathcal{I}\,\mathrm{Cl}(A) \cap m\beta\mathcal{I}\,\mathrm{Cl}(X\backslash A)$.*
(iii) *The m-frontier of A, denoted by $mFr(A)$, is defined as $m\,\mathrm{Cl}(A)\cap m\,\mathrm{Cl}(X\backslash A)$.*

Proposition 3.26. *Let $(X, m, \mathcal{I})$ be an ideal minimal space and $A \subset X$. Then the following propeties hold:*

(i) $m\beta\mathcal{I}\,\mathrm{Int}(A) = A\backslash m\beta\mathcal{I}D(X\backslash A)$.
(ii) $m\beta\mathcal{I}\,\mathrm{Cl}(A) = A \cup m\beta\mathcal{I}D(A)$.

Proof. Obvious. □

4. m-β-$\mathcal{I}$-continuous functions

Definition 4.1. *A function $f : (X, m, \mathcal{I}) \to (Y, \tau)$ is said to be m-β-$\mathcal{I}$-continuous if the inverse image of every open set of Y is m-β-$\mathcal{I}$-open in X .*

Proposition 4.2. (i) *Every m-semi-$\mathcal{I}$-continuous function is m-β-$\mathcal{I}$-continuous but not conversely.*
(ii) *Every m-pre-$\mathcal{I}$-continuous function is m-β-$\mathcal{I}$-continuous but not conversely.*
(iii) *Every m-$\mathcal{I}$-continuous function is m-β-$\mathcal{I}$-continuous but not conversely.*
(iv) *Every M-$\mathcal{I}$-continuous function is m-β-$\mathcal{I}$-continuous but not conversely.*

Proof. The proof follows from Proposition 3.2 and Examples 3.3, 3.4. □

Corollary 4.3. *Every m-α-$\mathcal{I}$-continuous function is m-β-$\mathcal{I}$-continuous but not conversely.*

Theorem 4.4. *For a function $f : (X, m, \mathcal{I}) \to (Y, \tau)$, the following statements are equivalent:*

(i) *f is m-β-$\mathcal{I}$-continuous;*

(ii) *For each point x in X and each open set F in Y such that $f(x) \in F$, there is an m-β-$\mathcal{I}$-open set A in X such that $x \in A$, $f(A) \subset F$;*
(iii) *The inverse image of each closed set in Y is m-β-$\mathcal{I}$-closed in X;*
(iv) *For each subset B of Y, $f^{-1}(Int(B)) \subset m\beta\mathcal{I}Int(f^{-1}(B))$;*
(v) *For each subset A of X, $f(m\beta\mathcal{I}\,\mathrm{Cl}(A)) \subset \mathrm{Cl}(f(A))$;*
(vi) *For each subset A of X, $f(m\beta\mathcal{I}D(A)) \subset \mathrm{Cl}(f(A))$;*
(vii) *For any subset B of Y, $f^{-1}(Int(B)) \subset m\beta\mathcal{I}Int(f^{-1}(B))$;*
(viii) *For each subset C of Y, $m\beta\mathcal{I}Fr(f^{-1}(C)) \subset f^{-1}(Fr(C))$.*

Proof. The proof follows from the definitions. □

Theorem 4.5. *Let $f : (X, m, \mathcal{I}) \to (Y, \tau)$ be a function. Then $X \backslash m\beta\mathcal{I}C(f) = \cup\{m\beta\mathcal{I}Fr(f^{-1}(V)) : V \in \tau, f(x) \in V, x \in X\}$, where $m\beta\mathcal{I}C(f)$ denotes the set of points at which f is m-β-$\mathcal{I}$-continuous.*

Proof. Let $x \in X\backslash m\beta\mathcal{I}C(f)$. Then there exists $V \in \tau$ containing $f(x)$ such that $f(U)$ is not a subset of V for every m-β-$\mathcal{I}$-open set U containing x. Hence $U \cap (X\backslash f^{-1}(V)) \neq \emptyset$ for every m-β-$\mathcal{I}$-open set U containing x. Therefore, by Theorem 3.23, $x \in m\beta\mathcal{I}\,\mathrm{Cl}(X\backslash f^{-1}(V))$. Then $x \in f^{-1}(V) \cap m\beta\mathcal{I}\,\mathrm{Cl}(X\backslash f^{-1}(V)) \subset m\beta\mathcal{I}Fr(f^{-1}(V))$. So,

$$X\backslash m\beta\mathcal{I}C(f) \subset \cup\{m\beta\mathcal{I}Fr(f^{-1}(V)) : V \in \tau, f(x) \in V, x \in X\}.$$

Conversely, let $x \notin X\backslash m\beta\mathcal{I}C(f)$. Then for each $V \in \tau$ containing $f(x)$, $f^{-1}(V)$ contains an m-β-$\mathcal{I}$-open set U containing x. Thus, $x \in m\beta\mathcal{I}\,\mathrm{Int}(f^{-1}(V))$ and hence $x \notin m\beta\mathcal{I}Fr(f^{-1}(V))$ for every $V \in \tau$ containing $f(x)$. Therefore,

$$X\backslash m\beta\mathcal{I}C(f) \supset \cup\{m\beta\mathcal{I}Fr(f^{-1}(V)) : V \in \tau, f(x) \in V, x \in X\}.$$

□

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