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SOME REMARKS ON LOW SEPARATION AXIOMS VIA ID-SETS

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Abstract. The purpose of this paper is to introduce some new classes of ideal topological spaces by utilizing I-open sets and study some of their fundamental properties.

1. Introduction and Preliminaries

The subject of ideals in topological spaces has been studied by Kuratowski [12] and Vaidyanathasamy [15]. Since then, many mathematicians contributed to this field of research such as M. E. Abd El-Monsef, A. Al-Omari, F. G. Arenas, M. Caldas, J. Dontchev, M. Ganster, D. N. Georgiou, T. R. Hamlett, E. Hatir, S. D. Iliadis, S. Jafari, D. Jankovic, E. F. Lashien, M. Maheswari, , H. Maki, A. C. Megaritis, A. A. Nasef, T. Noiri, B. K. Papadopoulos, M. Parimala, G. A. Prinos, M. L. Puertas, M. Rajamani, N. Rajesh, D. Rose, A. Selvakumar, Jun-Iti Umehara and many others (see [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [14], [13]). An ideal I on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (i) $A \in I$ and $B \subset A$ implies $B \in I$ and (ii) $A \in I$ and $B \in I$ implies $A \cup B \in I$. Given a topological space (X, τ) with an ideal I on X and if $P(X)$ is the set of all subsets of X, a set operator $(.)^*$: $P(X) \rightarrow P(X)$, called the local function [15] of A with respect to τ and I, is defined as follows: for $A \subset X$, $A^*(I, \tau) = \{x \in X | U \cap A \notin I\}$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau | x \in U\}$. A Kuratowski closure operator $\mathrm{Cl}^*(.)$ for a topology $\tau^*(I, \tau)$ called the $*$-topology, finer thatn τ is defined by $\mathrm{Cl}^*(A) = A \cup A^*(I, \tau)$. Where there is no chance of confusion, $A^*(I)$ is denoted by A^*. If I is an ideal on X, then (X, I, τ) is called an ideal space. By a space, we always mean a topological space (X, τ) with no separation properties assumed. If $A \subset X$, $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ will denote the closure and interior of A in (X, τ), respectively. A subset S of an ideal space (X, τ, I) is

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said to be I-open [11] if $S \subset \text{Int}(S^*)$. The family of all I-open sets of (X, τ, I) is denoted by $IO(X)$.

2. ID-sets and associated separation axioms

Definition 2.1. *A subset A of an ideal space (X, τ, I) is called an ID-set if there exist $U, V \in IO(X)$ such that $U \neq X$ and $A = U - V$.*

Observe that every I-open set U different from X is an ID-set with $A = U$ and $V = \varnothing$.

Definition 2.2. *An ideal space (X, τ, I) is called I-D_0 (resp. I-T_0) if for any distinct pair of points x and y of X, there exists an ID-set of (X, τ, I) containing x but not y or an ID-set (resp. I-open set) of (X, τ, I) containing y but not x.*

Definition 2.3. *An ideal space (X, τ, I) is called I-D_1 (resp. I-T_1) if for any distinct pair of points x and y of X, there exists an ID-set (resp. I-open set) of X containing x but not y and an ID-set (resp. I-open set) of X containing y but not x.*

Definition 2.4. *An ideal space (X, τ, I) is called I-D_2 (resp. I-T_2) if for any distinct pair of points x and y of X, there exists disjoint ID-sets (resp. I-open set) of (X, τ, I) containing x and y, respectively.*

Remark 2.5. *(i) If (X, τ, I) is I-T_i, then it is I-D_i, i=0,1,2.*
(ii) If (X, τ, I) is I-D_i, then it is I-D_{i-1}, i=1,2.

Example 2.6. *Let $X = \{a, b, c\}$, $\tau = \{\varnothing, \{a, c\}, X\}$ and $I = \{\varnothing, \{a\}\}$. Then the ideal space (X, τ, I) is both I-D_2 and I-D_1 but none of I-T_2 and I-T_1.*

Problem 2.7. *Find an I-D_0 space which is not I-T_0.*

Problem 2.8. *Find an ideal space I-D_{i-1} which is not I-D_i, where i = 1, 2.*

Theorem 2.9. *For an ideal space (X, τ, I), the following statements are true:*

1. *(X, τ, I) is I-D_0 if and only if it is I-T_0.*
2. *(X, τ, I) is I-D_1 if and only if it is I-D_2.*

Proof. We prove only the necessary condition since the sufficiency is stated in Remark 2.5 (i).
Necessity. Let (X, τ, I) be I-D_0. Then for each distinct pair $x, y \in X$, at least one of x, y say x, belongs to an ID-set G where $y \notin G$. Let $G = U_1 - U_2$ such that $U_1 \neq X$ and U_1, $U_2 \in IO(X)$. Then $x \in U_1$, and for $y \notin G$, we have two cases: (a) $y \notin U_1$; (b) $y \in U_1$ and $y \in U_2$. In case (a), $x \in U_1$ but $y \notin U_1$; In case (b), $y \in U_2$ but $x \notin U_2$. Hence X is I-T_0.

(2) **Sufficiency.** Remark 2.5 (ii).

Necessity. Suppose (X, τ, I) is I-D_1 space. Then for each distinct pair $x, y \in X$, we have ID-sets G_1, G_2 such that $x \in G_1$, $y \notin G_1$; $y \in G_2$, $x \notin G_2$. Let $G_1 = U_1 - U_2$, $G_2 = U_3 - U_4$. From $x \notin G_2$, we have either $x \notin U_3$ or $x \in U_3$ and $x \in U_4$. Now we consider two cases.

(1) $x \notin U_3$. By $y \notin G_1$ we have two subcases:

(a) $y \notin U_1$. By $x \in U_1 - U_2$, it follows that $x \in U_1 - (U_2 \cup U_3)$ and by $y \in U_3 - U_4$ we have $y \in U_3 - (U_2 \cup U_4)$. Hence

$$(U_1 - (U_2 \cup U_3)) \cap (U_3 - (U_1 \cup U_4)) = \varnothing.$$

(b) $y \in U_1$ and $y \in U_2$. We have $x \in U_1 - U_2$, $y \in U_2$ such that $(U_1 - U_2) \cap U_2 = \varnothing$.

(2) $x \in U_3$ and $x \in U_4$. We have $y \in U_3 - U_4$, $x \in U_4$ such that $(U_3 - U_4) \cap U_4 = \varnothing$. Therefore, X is I-D_2.

□

Definition 2.10. *A point $x \in X$ which has only X as the I-neighbourhood is called an I-neat point.*

Theorem 2.11. *For an I-T_0 ideal space (X, τ, I) the following are equivalent:*

(1) *(X, τ, I) is I-D_1;*

(2) *(X, τ, I) has no I-neat point.*

Proof. (1)→(2): Since (X, τ, I) is I-D_1, then each point x of X is contained in a ID-set $O = U - V$ and thus in U. By definition $U \neq X$. This implies that x is not an I-neat point.

(2)→(1): If X is I-T_0, then for each distinct pair of points $x, y \in X$, at least one of them, x (say) has an I-neighbourhood U containing x and not y. Thus U which is different from X is an ID-set. If X has no I-neat point then y is not an I-neat point. This means that there exists an I-neighbourhood V of y such that $V \neq X$. Thus $y \in (V - U)$ but not x and $V - U$ is an ID-set. Hence (X, τ, I) is I-D_1. □

A function $f : (X, \tau, I) \to (Y, \sigma, J)$ is said to be I-irresolute if $f^{-1}(V) \in IO(X)$ for every $V \in IO(Y)$.

Theorem 2.12. *If $f : (X, \tau, I) \to (Y, \sigma, J)$ is an I-irresolute surjective function and E is an ID-set in (Y, σ, J), then the inverse image of E is an ID-set in (X, τ, I).*

Proof. Let E be an ID-set in (Y, σ, J). Then, there are I-open sets U_1 and U_2 in (Y, σ, J) such that $S = U_1 - U_2$ and $U_1 \neq Y$. By the I-irresoluteness of f, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are I-open in (X, τ, I). Since $U_1 \neq Y$, we have $f^{-1}(U_1) \neq X$. Hence $f^{-1}(E) = f^{-1}(U_1) - f^{-1}(U_2)$ is an ID-set. □

Theorem 2.13. *If (Y, σ, J) is I-D_1 and $f : (X, \tau, I) \to (Y, \sigma, J)$ is I-irresolute and bijective, then (X, τ, I) is I-D_1.*

Proof. Suppose that Y is an I-D_1 space. Let x and y be any pair of distinct points in X. Since f is injective and Y is I-D_1, there exist ID-sets G_x and G_y of Y containing $f(x)$ and $f(y)$, respectively, such that $f(y) \notin G_x$ and $f(x) \notin G_y$. By Theorem 2.12, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are ID-sets in (X, τ, I) containing x and y, respectively. This implies that (X, τ, I) is an I-D_1 space. □

Theorem 2.14. *An ideal space (X, τ, I) is I-D_1 if and only if for each pair of distinct points $x, y \in X$, there exists an I-irresolute surjective function $f : (X, \tau, I) \to (Y, \sigma, J)$, where (Y, σ, J) is an I-D_1 space such that $f(x)$ and $f(y)$ are distinct.*

Proof. **Necessity.** For every pair of distinct points of X, it suffices to take the identity function on X.
Sufficiency. Let x and y be any pair of distinct points in X. By hypothesis, there exists an I-irresolute, surjective function f from an ideal space (X, τ, I) onto an I-D_1 space (Y, σ, J) such that $f(x) \neq f(y)$. Therefore, there exist disjoint ID-sets G_x and G_y in Y such that $f(x) \in G_x$ and $f(y) \in G_y$. Since f is I-irresolute and surjective, by Theorem 2.12, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are disjoint ID-sets in X containing x and y, respectively. Hence the space X is an I-D_1 space. □

3. Conclusion

In this paper, we used the notions of I-open sets and ID-set to define some new low sepapartion axioms and presented some of their basic properties. We posed some problems which open up for more research in this direction.

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