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Some Fundamental Properties of β-Open Sets in Ideal Bitopological Spaces

M. Caldas[1], S. Jafari[2] and N. Rajesh[3,*]

[1] *Departamento de Matematica Aplicada, Universidade Federal Fluminense, Rua Mario Santos Braga, S/n, 24020-140, Niteroi, RJ Brasil*
[2] *College of Vestsjaelland South, Herrestraede 11, 4200 Slagelse, Denmark*
[3] *Department of Mathematics, Rajah Serfoji Govt. College, Thanjavur-613005, Tamilnadu, India*

Abstract. In this paper we introduce and characterize the concepts of β-open sets and their related notions in ideal bitopological spaces.

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1. Introduction

Kuratowski [7] and Vaidyanathasamy [9] introduced and investigated the concept of ideals in topological spaces. An ideal $\mathscr{I}$ on a topological space (X,τ) is a nonempty collection of subsets of X which satisfies (i) $A \in \mathscr{I}$ and $B \subset A$ implies $B \in \mathscr{I}$ and (ii) $A \in \mathscr{I}$ and $B \in \mathscr{I}$ implies $A \cup B \in \mathscr{I}$. Given a bitopological space (X,τ_1,τ_2) with an ideal $\mathscr{I}$ on X and if $\mathscr{P}(X)$ is the set of all subsets of X, a set operator $(.)_i^*$: $\mathscr{P}(X) \to \mathscr{P}(X)$, called the local function [9] of A with respect to τ_i and $\mathscr{I}$, is defined as follows: for $A \subset X$, $A_i^*(\tau_i,\mathscr{I}) = \{x \in X | U \cap A \notin \mathscr{I}$ for every $U \in \tau_i(x)\}$, where $\tau_i(x) = \{U \in \tau_i | x \in U\}$. For every ideal topological space $(X,\tau,\mathscr{I})$, there exists a topology $\tau^*(\mathscr{I})$, finer than τ, generated by the base $\beta(\mathscr{I},\tau) = \{U \backslash I | U \in \tau$ and $I \in \mathscr{I}\}$, but in general $\beta(\mathscr{I},\tau)$ is not always a topology [4]. Observe additionally that $\tau_i - \mathrm{Cl}^*(A) = A \cup A_i^*(\tau_i,\mathscr{I})$ defines a Kuratowski closure operator for $\tau^*(\mathscr{I})$, when there is no chance of confusion, $A_i^*(\mathscr{I})$ is denoted by A_i^* and $\tau_i - \mathrm{Int}^*(A)$ denotes the interior of A in $\tau_i^*(\mathscr{I})$. In this paper we introduce and characterize the concepts of β-open sets and their related notions in ideal bitopological spaces.

*Corresponding author.

Email addresses: `gmamccs@vm.uff.br` (M. Caldas), `jafari@stofanet.dk` (S. Jafari), `nrajesh_topology@yahoo.co.in` (N. Rajesh)

2. Preiliminaries

For a subset A of a bitopological space (X, τ_1, τ_2), we denote the closure of A and the interior of A with respect to τ_i by $\tau_i - \text{Cl}(A)$ and $\tau_i - \text{Int}(A)$, respectively.

Definition 1. *A subset A of a bitopological space (X, τ_1, τ_2) is said to be (i, j)-semiopen [5] (resp. (i, j)-preopen [5], (i, j)-semi-preopen [6]) if $A \subset \tau_j - \text{Cl}(\tau_i - \text{Int}(A))$ (resp. $A \subset \tau_i - \text{Int}(\tau_j - \text{Cl}(A))$, $A \subset \tau_j - \text{Cl}(\tau_i - \text{Int}(\tau_j - \text{Cl}(A)))$), where $i, j = 1, 2$ and $i \neq j$.*

Definition 2. *A subset A of an ideal bitopological space $(X, \tau_1, \tau_2, \mathscr{I})$ is said to be*

(i) *(i, j)-semi-$\mathscr{I}$-open [3] if $A \subset \tau_j - \text{Cl}^*(\tau_i - \text{Int}(A))$.*

(ii) *(i, j)-pre-$\mathscr{I}$-open [2] if $A \subset \tau_i - \text{Int}(\tau_j - \text{Cl}^*(A))$.*

(iii) *$(i, j) - b - \mathscr{I}$-open [3] if $A \subset \tau_i - \text{Int}(\tau_j - \text{Cl}^*(A)) \cup \tau_j - \text{Cl}^*(\tau_i - \text{Int}(A))$.*

(iv) *$(i, j) - \alpha - \mathscr{I}$-open [3] if $A \subset \tau_i - \text{Int}(\tau_j - \text{Cl}^*(\tau_i - \text{Int}(A)))$.*

Definition 3. *A function $f : (X, \tau_1, \tau_2, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be*

(i) *(i, j)-pre-$\mathscr{I}$-continuous [2] if the inverse image of every σ_i-open set of Y is (i, j)-pre-$\mathscr{I}$-open in X, where $i \neq j$, i, j=1, 2.*

(ii) *(i, j)-semi-$\mathscr{I}$-continuous [3] if the inverse image of every σ_i-open set of Y is (i, j)-semi-$\mathscr{I}$-open in X, where $i \neq j$, i, j=1, 2.*

(iii) *$(i, j) - b - \mathscr{I}$-continuous [3] if the inverse image of every σ_i-open set of Y is $(i, j) - b - \mathscr{I}$-open in X, where $i \neq j$, i, j=1, 2.*

(iv) *$(i, j) - \alpha - \mathscr{I}$-continuous [3] if the inverse image of every σ_i-open set of Y is $(i, j) - \alpha - \mathscr{I}$-open in X, where $i \neq j$, i, j=1, 2.*

(v) *pairwise semi-precontinuous [6] if the inverse image of every σ_i-open set in (Y, σ_1, σ_2) is (i, j)-semi-preopen in (X, τ_1, τ_2), where $i \neq j$, i, j=1, 2.*

3. Properties of $(i, j) - \beta - \mathscr{I}$-open Sets

Definition 4. *A subset A of an ideal bitopological space $(X, \tau_1, \tau_2, \mathscr{I})$ is said to be $(i, j) - \beta - \mathscr{I}$-open if $A \subset \tau_j - \text{Cl}(\tau_i - \text{Int}(\tau_j - \text{Cl}^*(A)))$, where $i, j = 1, 2$ and $i \neq j$.*
The family of all $(i, j) - \beta - \mathscr{I}$-open sets of $(X, \tau_1, \tau_2, \mathscr{I})$ is denoted by $\beta\mathscr{I}O(X, \tau_1, \tau_2)$ or $(i, j) - \beta\mathscr{I}O(X)$. Also, The family of all $(i, j) - \beta - \mathscr{I}$-open sets of $(X, \tau_1, \tau_2, \mathscr{I})$ containing x is denoted by $(i, j) - \beta\mathscr{I}O(X, x)$.

Remark 1. *Let $\mathscr{I}$ and $\mathscr{J}$ be two ideals on (X, τ_1, τ_2). If $\mathscr{I} \subset \mathscr{J}$, then $\beta\mathscr{J}O(X, \tau_1, \tau_2) \subset \beta\mathscr{I}O(X, \tau_1, \tau_2)$.*

Proposition 1. *(i) Every $(i, j) - b - \mathscr{I}$-open set is $(i, j) - \beta - \mathscr{I}$-open.*

(ii) Every $(i,j)-\beta-\mathscr{I}$-open set is (i,j)-semi-preopen.

Proof. The proof follows from the definitions.

The following example shows that the converses of Proposition 1 is not true in general.

Example 1. *Let $X=\{a,b,c\}$, $\tau_1=\{\emptyset,\{a\},X\}$, $\tau_2=\{\emptyset,\{a\},\{a,b\},X\}$ and $\mathscr{I}=\{\emptyset,\{a\}\}$. Then the set $\{a,c\}$ is $(i,j)-\beta-\mathscr{I}$-open but not $(i,j)-b-\mathscr{I}$-open.*

Corollary 1. *(i) Every $(i,j)-\alpha-\mathscr{I}$-open set is $(i,j)-\beta-\mathscr{I}$-open.*

(ii) Every (i,j)-semi-$\mathscr{I}$-open set is $(i,j)-\beta-\mathscr{I}$-open.

(iii) Every (i,j)-pre-$\mathscr{I}$-open set is $(i,j)-\beta-\mathscr{I}$-open.

Proposition 2. *For an ideal bitopological space $(X,\tau_1,\tau_2,\mathscr{I})$ and $A\subset X$, we have:*

(i) If $\mathscr{I}=\{\emptyset\}$, then A is $(i,j)-\beta-\mathscr{I}$-open if and only if A is (i,j)-semi-preopen.

(ii) If $\mathscr{I}=\mathscr{P}(X)$, then A is $(i,j)-\beta-\mathscr{I}$-open if and only if A is (i,j)-semiopen.

Proof. The proof follows from the fact that

(i) If $\mathscr{I}=\{\emptyset\}$, then $A^*=\text{Cl}(A)$.

(ii) If $\mathscr{I}=\mathscr{P}(X)$, then $A^*=\emptyset$ for every subset A of X.

Remark 2. *The intersection of any two $(i,j)-\beta-\mathscr{I}$-open sets is not an $(i,j)-\beta-\mathscr{I}$-open set as it can be seen from the following example.*

Example 2. *Let $X=\{a,b,c,d\}$, $\tau_1=\{\varnothing,\{a\},\{b\},\{a,b\},\{a,b,c\},X\}$, $\tau_2=\{\varnothing,X\}$ and $\mathscr{I}=\{\varnothing,\{c\},\{d\},\{c,d\}\}$. Then the sets $\{a,c\}$ and $\{b,c\}$ are $(1,2)-\beta-\mathscr{I}$-open sets of $(X,\tau_1,\tau_2,\mathscr{I})$ but their intersection $\{c\}$ is not an $(1,2)-\beta-\mathscr{I}$-open set of $(X,\tau_1,\tau_2,\mathscr{I})$.*

Theorem 1. *If $\{A_\alpha\}_{\alpha\in\Omega}$ is a family of $(i,j)-\beta-\mathscr{I}$-open sets in $(X,\tau_1,\tau_2,\mathscr{I})$, then $\bigcup_{\alpha\in\Omega}A_\alpha$ is $(i,j)-\beta-\mathscr{I}$-open in $(X,\tau_1,\tau_2,\mathscr{I})$.*

Proof. Since $\{A_\alpha:\alpha\in\Omega\}\subset(i,j)-\beta\mathscr{I}O(X)$, then $A_\alpha\subset\tau_j-\text{Cl}(\tau_i-\text{Int}(\tau_j-\text{Cl}^*(A_\alpha)))$ for every $\alpha\in\Omega$. Thus,

$$\begin{aligned}\underset{\alpha\in\Omega}{\cup}A_\alpha&\subset\underset{\alpha\in\Omega}{\cup}\tau_j-\text{Cl}(\tau_i-\text{Int}(\tau_j-\text{Cl}^*(A_\alpha)))\subset\tau_j-\text{Cl}(\tau_i-\text{Int}(\underset{\alpha\in\Omega}{\cup}\tau_j-\text{Cl}^*(A_\alpha)))\\&=\tau_j-\text{Cl}(\tau_i-\text{Int}(\tau_j-\text{Cl}^*(\underset{\alpha\in\Omega}{\cup}A_\alpha))).\end{aligned}$$

Therefore, we obtain $\underset{\alpha\in\Omega}{\cup}A_\alpha\subset\tau_j-\text{Cl}(\tau_i-\text{Int}(\tau_j-\text{Cl}^*(\underset{\alpha\in\Omega}{\cup}A_\alpha)))$. Hence any union of $(i,j)-\beta-\mathscr{I}$-open sets is $(i,j)-\beta-\mathscr{I}$-open.

Theorem 2. *A subset A of an ideal bitopological space $(X,\tau_1,\tau_2,\mathscr{I})$ is $(i,j)-\beta-\mathscr{I}$-open if and only if $\tau_j-\text{Cl}(A)=\tau_j-\text{Cl}(\tau_i-\text{Int}(\tau_j-\text{Cl}^*(A)))$.*

Proof. Let A be an $(i,j)-\beta-\mathscr{I}$-open subset of X. Then, we have $A\subset\tau_j-\mathrm{Cl}(\tau_i-\mathrm{Int}(\tau_j-\mathrm{Cl}^*(A)))$ and hence

$$\tau_j-\mathrm{Cl}(A)\subset\tau_j-\mathrm{Cl}(\tau_i-\mathrm{Int}(\tau_j-\mathrm{Cl}^*(A)))\subset\tau_j-\mathrm{Cl}(\tau_i-\mathrm{Int}(\tau_j-\mathrm{Cl}(A)))\subset\tau_j-\mathrm{Cl}(A).$$

Therefore, $\tau_j-\mathrm{Cl}(A)=\tau_j-\mathrm{Cl}(\tau_i-\mathrm{Int}(\tau_j-\mathrm{Cl}^*(A)))$. The converse is obvious.

Definition 5. *A bitopological space (X,τ_1,τ_2) is said to be pairwise extremally disconnected [1] if $\tau_j-\mathrm{Cl}(A)\in\tau_i$ for every $A\in\tau_i$.*

Proposition 3. *Let $(X,\tau_1,\tau_2,\mathscr{I})$ be a pairwise extremally disconnected space. If A is $(i,j)-\beta-\mathscr{I}$-open, then it is (i,j)-preopen in X.*

Proof. Let A be $(i,j)-\beta-\mathscr{I}$-open set of X, we have $A\subset\tau_j-\mathrm{Cl}(\tau_i-\mathrm{Int}(\tau_j-\mathrm{Cl}^*(A)))$. Since X is pairwise extremally disconnected, for $\tau_i-\mathrm{Int}(\tau_j-\mathrm{Cl}^*(A))\in\tau_i$, we have $\tau_j-\mathrm{Cl}(\tau_i-\mathrm{Int}(\tau_j-\mathrm{Cl}^*(A)))\in\tau_i$. So, we have

$$\begin{aligned}A\subset&\tau_j-\mathrm{Cl}(\tau_i-\mathrm{Int}(\tau_j-\mathrm{Cl}^*(A)))\subset\tau_i-\mathrm{Int}(\tau_j-\mathrm{Cl}(\tau_i-\mathrm{Int}(\tau_j-\mathrm{Cl}^*(A))))\\ \subset&\tau_i-\mathrm{Int}(\tau_j-\mathrm{Cl}(\tau_j-\mathrm{Cl}^*(A)))\subset\tau_i-\mathrm{Int}(\tau_j-\mathrm{Cl}(A\cup A^*))\\ =&\tau_i-\mathrm{Int}(\tau_j-\mathrm{Cl}(A)\cup\tau_j-\mathrm{Cl}(A^*))\subset\tau_i-\mathrm{Int}(\tau_j-\mathrm{Cl}(A));\end{aligned}$$

hence A is (i,j)-preopen in X.

An ideal bitopological space is said to satisfy the condition $(\mathscr{A})$ if $U\cap\tau_j-\mathrm{Cl}^*(A)\subset\tau_j-\mathrm{Cl}^*(U\cap A)$ for every $U\in\tau_i$.

Theorem 3. *Let $(X,\tau_1,\tau_2,\mathscr{I})$ be a pairwise extremally disconnected space which satisfies the condition $\mathscr{A}$. If A is (i,j)-semi-$\mathscr{I}$-open and B is (i,j)-pre-$\mathscr{I}$-open, then $A\cap B$ is $(i,j)-\beta-\mathscr{I}$-open.*

Proof. Let A be (i,j)-semi-$\mathscr{I}$-open and B an (i,j)-pre-$\mathscr{I}$-open set of X. Then

$$\begin{aligned}A\cap B\subset&\tau_j-\mathrm{Cl}^*(\tau_i-\mathrm{Int}(A))\cap\tau_i-\mathrm{Int}(\tau_j-\mathrm{Cl}^*(B))\subset\tau_j-\mathrm{Cl}^*(\tau_i-\mathrm{Int}(A)\cap\tau_i-\mathrm{Int}(\tau_j-\mathrm{Cl}^*(B))\\ =&\tau_j-\mathrm{Cl}^*(\tau_i-\mathrm{Int}(\tau_i-\mathrm{Int}(A))\cap\tau_j-\mathrm{Cl}^*(B)))\subset\tau_j-\mathrm{Cl}^*(\tau_i-\mathrm{Int}(\tau_j-\mathrm{Cl}^*(\tau_i-\mathrm{Int}(A)\cap B)))\\ \subset&\tau_j-\mathrm{Cl}^*(\tau_i-\mathrm{Int}(\tau_j-\mathrm{Cl}^*(A\cap B)))\subset\tau_j-\mathrm{Cl}(\tau_i-\mathrm{Int}(\tau_j-\mathrm{Cl}^*(A\cap B))).\end{aligned}$$

Thus, $A\cap B$ is $(i,j)-\beta-\mathscr{I}$-open in X.

Definition 6. *In an ideal bitopological space $(X,\tau_1,\tau_2,\mathscr{I})$, $A\subset X$ is said to be $(i,j)-\beta-\mathscr{I}$-closed if $X\backslash A$ is $(i,j)-\beta-\mathscr{I}$-open in X, $i,j=1,2$ and $i\neq j$.*

Theorem 4. *If A is an $(i,j)-\beta-\mathscr{I}$-closed set in an ideal bitopological space $(X,\tau_1,\tau_2,\mathscr{I})$ if and only if $\tau_j-\mathrm{Int}(\tau_i-\mathrm{Cl}(\tau_j-\mathrm{Int}^*(A)))\subset A$.*

Proof. The proof follows from the definitions.

Theorem 5. *A subset A of an ideal bitopological space $(X, \tau_1, \tau_2, \mathscr{I})$ is $(i, j)-\beta-\mathscr{I}$-closed, then* $\tau_j - \mathrm{Int}(\tau_i - \mathrm{Cl}^*(\tau_j - \mathrm{Int}(A))) \subset A$

Proof. The proof follows from the fact that $\mathrm{Cl}^*(A) \subset \mathrm{Cl}(A)$ for every subset A of X.

Theorem 6. *Arbitrary intersection of $(i, j)-\beta-\mathscr{I}$-closed sets is always $(i, j)-\beta-\mathscr{I}$-closed.*

Proof. Follows from Theorems 1 and 5.

Definition 7. *Let $(X, \tau_1, \tau_2, \mathscr{I})$ be an ideal bitopological space, S a subset of X and x be a point of X. Then*

(i) x is called an $(i, j)-\beta-\mathscr{I}$-interior point of S if there exists $V \in (i, j)-\beta\mathscr{I}O(X, \tau_1, \tau_2)$ such that $x \in V \subset S$.

(ii) the set of all $(i, j)-\beta-\mathscr{I}$-interior points of S is called $(i, j)-\beta-\mathscr{I}$-interior of S and is denoted by $(i, j)-\beta\mathscr{I}\,\mathrm{Int}(S)$.

Theorem 7. *Let A and B be subsets of $(X, \tau_1, \tau_2, \mathscr{I})$. Then the following properties hold:*

(i) $(i, j)-\beta\mathscr{I}\,\mathrm{Int}(A) = \cup\{T : T \subset A \text{ and } T \in (i, j)-\beta\mathscr{I}O(X)\}$.

(ii) $(i, j)-\beta\mathscr{I}\,\mathrm{Int}(A)$ is the largest $(i, j)-\beta-\mathscr{I}$-open subset of X contained in A.

(iii) A is $(i, j)-\beta-\mathscr{I}$-open if and only if $A = (i, j)-\beta\mathscr{I}\,\mathrm{Int}(A)$.

(iv) $(i, j)-\beta\mathscr{I}\,\mathrm{Int}((i, j)-\beta\mathscr{I}\,\mathrm{Int}(A)) = (i, j)-\beta\mathscr{I}\,\mathrm{Int}(A)$.

(v) If $A \subset B$, then $(i, j)-\beta\mathscr{I}\,\mathrm{Int}(A) \subset (i, j)-\beta\mathscr{I}\,\mathrm{Int}(B)$.

(vi) $(i, j)-\beta\mathscr{I}\,\mathrm{Int}(A \cap B) \subset (i, j)-\beta\mathscr{I}\,\mathrm{Int}(A) \cap (i, j)-\beta\mathscr{I}\,\mathrm{Int}(B)$.

(vii) $(i, j)-\beta\mathscr{I}\,\mathrm{Int}(A \cup B) \supset (i, j)-\beta\mathscr{I}\,\mathrm{Int}(A) \cup (i, j)-\beta\mathscr{I}\,\mathrm{Int}(B)$.

Proof. (vi). Since $A \cap B \subset A$ and $A \cap B \subset B$, by (iv), we have $(i, j)-\beta\mathscr{I}\,\mathrm{Int}(A \cap B) \subset (i, j)-\beta\mathscr{I}\,\mathrm{Int}(A)$ and $(i, j)-\beta\mathscr{I}\,\mathrm{Int}(A \cap B) \subset (i, j)-\beta\mathscr{I}\,\mathrm{Int}(B)$. Therefore, $(i, j)-\beta\mathscr{I}\,\mathrm{Int}(A \cap B) \subset (i, j)-\beta\mathscr{I}\,\mathrm{Int}(A) \cap (i, j)-\beta\mathscr{I}\,\mathrm{Int}(B)$.
(vii). We have $(i, j)-\beta\mathscr{I}\,\mathrm{Int}(A) \subset (i, j)-\beta\mathscr{I}\,\mathrm{Int}(A \cup B)$ and $(i, j)-\beta\mathscr{I}\,\mathrm{Int}(B) \subset (i, j)-\beta\mathscr{I}\,\mathrm{Int}(A \cup B)$. Then we obtain $(i, j)-\beta\mathscr{I}\,\mathrm{Int}(A) \cup (i, j)-\beta\mathscr{I}\,\mathrm{Int}(B) \subset (i, j)-\beta\mathscr{I}\,\mathrm{Int}(A \cup B)$.
The other proofs are obvious.

Definition 8. *Let $(X, \tau_1, \tau_2, \mathscr{I})$ be an ideal bitopological space, S a subset of X and x be a point of X. Then*

(i) x is called an $(i, j)-\beta-\mathscr{I}$-cluster point of S if $V \cap S \neq \emptyset$ for every $V \in (i, j)-\beta\mathscr{I}O(X, x)$.

(ii) the set of all $(i, j)-\beta-\mathscr{I}$-cluster points of S is called $(i, j)-\beta-\mathscr{I}$-closure of S and is denoted by $(i, j)-\beta\mathscr{I}\,\mathrm{Cl}(S)$.

Theorem 8. *Let A and B be subsets of $(X, \tau_1, \tau_2, \mathscr{I})$. Then the following properties hold:*

(i) $(i,j)-\beta\mathscr{I}\,\mathrm{Cl}(A) = \cap\{F : A \subset F \text{ and } F \in (i,j)-\beta\mathscr{I}C(X)\}$.

(ii) $(i,j)-\beta\mathscr{I}\,\mathrm{Cl}(A)$ *is the smallest* $(i,j)-\beta-\mathscr{I}$*-closed subset of* X *containing* A.

(iii) A *is* $(i,j)-\beta-\mathscr{I}$*-closed if and only if* $A = (i,j)-\beta\mathscr{I}\,\mathrm{Cl}(A)$.

(iv) $(i,j)-\beta\mathscr{I}\,\mathrm{Cl}((i,j)-\beta\mathscr{I}\,\mathrm{Cl}(A) = (i,j)-\beta\mathscr{I}\,\mathrm{Cl}(A)$.

(v) *If* $A \subset B$*, then* $(i,j)-\beta\mathscr{I}\,\mathrm{Cl}(A) \subset (i,j)-\beta\mathscr{I}\,\mathrm{Cl}(B)$.

(vi) $(i,j)-\beta\mathscr{I}\,\mathrm{Cl}(A \cup B) \supset (i,j)-\beta\mathscr{I}\,\mathrm{Cl}(A) \cup (i,j)-\beta\mathscr{I}\,\mathrm{Cl}(B)$.

(vii) *]* $(i,j)-\beta\mathscr{I}\,\mathrm{Cl}(A \cap B) \subset (i,j)-\beta\mathscr{I}\,\mathrm{Cl}(A) \cap (i,j)-\beta\mathscr{I}\,\mathrm{Cl}(B)$.

Proof. The proofs follows from the definitions.

Theorem 9. *Let $(X, \tau_1, \tau_2, \mathscr{I})$ be an ideal bitopological space and $A \subset X$. A point* $x \in (i,j)-\beta\mathscr{I}\,\mathrm{Cl}(A)$ *if and only if* $U \cap A \neq \emptyset$ *for every* $U \in (i,j)-\beta\mathscr{I}O(X,x)$.

Proof. Suppose that $x \in (i,j)-\beta\mathscr{I}\,\mathrm{Cl}(A)$. We shall show that $U \cap A \neq \emptyset$ for every $U \in (i,j)-\beta\mathscr{I}O(X,x)$. Suppose that there exists $U \in (i,j)-\beta\mathscr{I}O(X,x)$ such that $U \cap A = \emptyset$. Then $A \subset X\backslash U$ and $X\backslash U$ is $(i,j)-\beta-\mathscr{I}$-closed. Since $A \subset X\backslash U$, $(i,j)-\beta\mathscr{I}\,\mathrm{Cl}(A) \subset (i,j)-\beta\mathscr{I}\,\mathrm{Cl}(X\backslash U)$. Since $x \in (i,j)-\beta\mathscr{I}\,\mathrm{Cl}(A)$, we have $x \in (i,j)-\beta\mathscr{I}\,\mathrm{Cl}(X\backslash U)$. Since $X\backslash U$ is $(i,j)-\beta-\mathscr{I}$-closed, we have $x \in X\backslash U$; hence $x \notin U$, which is a contradiction that $x \in U$. Therefore, $U \cap A \neq \emptyset$. Conversely, suppose that $U \cap A \neq \emptyset$ for every $U \in (i,j)-\beta\mathscr{I}O(X,x)$. We shall show that $x \in (i,j)-\beta\mathscr{I}\,\mathrm{Cl}(A)$. Suppose that $x \notin (i,j)-\beta\mathscr{I}\,\mathrm{Cl}(A)$. Then there exists $U \in (i,j)-\beta\mathscr{I}O(X,x)$ such that $U \cap A = emptyset$. This is a contradiction to $U \cap A \neq \emptyset$; hence $x \in (i,j)-\beta\mathscr{I}\,\mathrm{Cl}(A)$.

Theorem 10. *Let $(X, \tau_1, \tau_2, \mathscr{I})$ be an ideal bitopological space and $A \subset X$. Then the following propeties hold:*

(i) $(i,j)-\beta\mathscr{I}\,\mathrm{Int}(X\backslash A) = X\backslash(i,j)-\beta\mathscr{I}\,\mathrm{Cl}(A)$;

(ii) $(i,j)-\beta\mathscr{I}\,\mathrm{Cl}(X\backslash A) = X\backslash(i,j)-\beta\mathscr{I}\,\mathrm{Int}(A)$.

Proof. (i). Let $x \in (i,j)-\beta\mathscr{I}\,\mathrm{Cl}(A)$. There exists $V \in (i,j)-\beta\mathscr{I}O(X,x)$ such that $V \cap A \neq \emptyset$; hence we obtain $x \in (i,j)-\beta\mathscr{I}\,\mathrm{Int}(X\backslash A)$. This shows that $X\backslash(i,j)-\beta\mathscr{I}\,\mathrm{Cl}(A) \subset (i,j)-\beta\mathscr{I}\,\mathrm{Int}(X\backslash A)$. Let $x \in (i,j)-\beta\mathscr{I}\,\mathrm{Int}(X\backslash A)$. Since $(i,j)-\beta\mathscr{I}\,\mathrm{Int}(X\backslash A) \cap A = \emptyset$, we obtain $x \notin (i,j)-\beta\mathscr{I}\,\mathrm{Cl}(A)$; hence $x \in X\backslash(i,j)-\beta\mathscr{I}\,\mathrm{Cl}(A)$. Therefore, we obtain $(i,j)-\beta\mathscr{I}\,\mathrm{Int}(X\backslash A) = X\backslash(i,j)-\beta\mathscr{I}\,\mathrm{Cl}(A)$.
(ii). Follows from (i).

Proposition 4. *The product of two* $(i,j)-\beta-\mathscr{I}$*-open sets is* $(i,j)-\beta-\mathscr{I}$*-open.*

Proof. The proof follows from Lemma 3.3 of [10].

4. $(i,j)-\beta-\mathscr{I}$-continuous Functions

Definition 9. *A function $f:(X,\tau_1,\tau_2,\mathscr{I})\to(Y,\sigma_1,\sigma_2)$ is said to be $(i,j)-\beta-\mathscr{I}$-continuous if the inverse image of every σ_i-open set of Y is $(i,j)-\beta-\mathscr{I}$-open in X, where $i\neq j$, $i,j=1,2$.*

Proposition 5. *Every $(i,j)-b-\mathscr{I}$-continuous function is $(i,j)-\beta-\mathscr{I}$-continuous but not conversely.*

Proof. The proof follows from Proposition 1.

The following example shows that the converse of Proposition 5 is not true, in general.

Example 3. *Let $X=\{a,b,c\}$, $\tau_1=\{\emptyset,\{a\},X\}$, $\tau_2=\{\emptyset,\{a\},\{a,b\},X\}$, $\sigma_1=\{\emptyset,\{a,c\},X\}$, $\sigma_2=\{\emptyset,\{a\},X\}$ and $\mathscr{I}=\{\emptyset,\{a\}\}$. Then the identity function $f:(X,\tau_1,\tau_2,\mathscr{I})\to(Y,\sigma_1,\sigma_2)$ is $(1,2)-\beta-\mathscr{I}$-continuous but not $(1,2)-b-\mathscr{I}$-continuous.*

Corollary 2.

(i) Every $(i,j)-\alpha-\mathscr{I}$-continuous function is $(i,j)-\beta-\mathscr{I}$-continuous but not conversely.

(ii) Every (i,j)-semi-$\mathscr{I}$-continuous function is $(i,j)-\beta$-continuous but not conversely.

(iii) Every (i,j)-pre-$\mathscr{I}$-continuous function is $(i,j)-\beta-\mathscr{I}$-continuous but not conversely.

Theorem 11. *For a function $f:(X,\tau_1,\tau_2,\mathscr{I})\to(Y,\sigma_1,\sigma_2)$, the following statements are equivalent:*

(i) f is $(i,j)-\beta-\mathscr{I}$-continuous.

(ii) For each point x in X and each σ_i-open set F in Y such that $f(x)\in F$, there exists an $(i,j)-\beta-\mathscr{I}$-open set A in X such that $x\in A$, $f(A)\subset F$.

(iii) The inverse image of each σ_i-closed set in Y is $(i,j)-\beta-\mathscr{I}$-closed in X.

(iv) For each subset A of X, $f((i,j)-\beta\mathscr{I}\,\mathrm{Cl}(A))\subset\sigma_i-\mathrm{Cl}(f(A))$.

(v) For each subset B of Y, $(i,j)-\beta\mathscr{I}\,\mathrm{Cl}(f^{-1}(B))\subset f^{-1}(\sigma_i-\mathrm{Cl}(B))$.

(vi) For each subset C of Y, $f^{-1}(\sigma_i-\mathrm{Int}(C))\subset(i,j)-\beta\mathscr{I}\,\mathrm{Int}(f^{-1}(C))$.

Proof. (i)⇒(ii): Let $x\in X$ and F be a σ_i-open set of Y containing $f(x)$. By (i), $f^{-1}(F)$ is $(i,j)-\beta-\mathscr{I}$-open in X. Let $A=f^{-1}(F)$. Then $x\in A$ and $f(A)\subset F$.
(ii)⇒(i): Let F be σ_i-open in Y and let $x\in f^{-1}(F)$. Then $f(x)\in F$. By (ii), there is an $(i,j)-\beta-\mathscr{I}$-open set U_x in X such that $x\in U_x$ and $f(U_x)\subset F$. Then $x\in U_x\subset f^{-1}(F)$. Hence $f^{-1}(F)$ is $(i,j)-\beta-\mathscr{I}$-open in X.
(i)⇔(iii): This follows due to the fact that for any subset B of Y, $f^{-1}(Y\backslash B)=X\backslash f^{-1}(B)$.
(iii)⇒(iv): Let A be a subset of X. Since $A\subset f^{-1}(f(A))$ we have $A\subset f^{-1}(\sigma_i-\mathrm{Cl}(f(A)))$. Now, $\sigma_i-\mathrm{Cl}(f(A))$ is σ_i-closed in Y and hence $(i,j)-\beta\mathscr{I}\,\mathrm{Cl}(A)\subset f^{-1}(\sigma_i-\mathrm{Cl}(f(A)))$ for $(i,j)-\beta\mathscr{I}\,\mathrm{Cl}(A)$ is the smallest $(i,j)-\beta-\mathscr{I}$-closed set containing A. Then
$f((i,j)-\beta\mathscr{I}\,\mathrm{Cl}(A))\subset\sigma_i-\mathrm{Cl}(f(A))$.

(iv)⇒(iii): Let F be any $(i,j)-\beta-\mathscr{I}$-closed subset of Y. Then $f((i,j)-\beta\mathscr{I}\,\mathrm{Cl}(f^{-1}(F)))\subset\sigma_i-\mathrm{Cl}(f(f^{-1}(F)))=\sigma_i-\mathrm{Cl}(F)=F$. Therefore, $(i,j)-\beta\mathscr{I}\,\mathrm{Cl}(f^{-1}(F))\subset f^{-1}(F)$. Consequently, $f^{-1}(F)$ is $(i,j)-\beta-\mathscr{I}$-closed in X.
(iv)⇒(v): Let B be any subset of Y. Now,

$$f((i,j)-\beta\mathscr{I}\,\mathrm{Cl}(f^{-1}(B)))\subset\sigma_i-\mathrm{Cl}(f(f^{-1}(B)))\subset\sigma_i-\mathrm{Cl}(B).$$

Consequently, $(i,j)-\beta\mathscr{I}\,\mathrm{Cl}(f^{-1}(B))\subset f^{-1}(\sigma_i-\mathrm{Cl}(B))$.
(v)⇒(iv): Let $B=f(A)$, where A is a subset of X. Then,

$$(i,j)-\beta\mathscr{I}\,\mathrm{Cl}(A)\subset(i,j)-\beta\mathscr{I}\,\mathrm{Cl}(f^{-1}(B))\subset f^{-1}(\sigma_i-\mathrm{Cl}(B))=f^{-1}(\sigma_i-\mathrm{Cl}(f(A))).$$

This shows that $f((i,j)-\beta\mathscr{I}\,\mathrm{Cl}(A))\subset\sigma_i-\mathrm{Cl}(f(A))$.
(i)⇒(vi): Let B be a σ_i-open set in Y. Clearly, $f^{-1}(\sigma_i-\mathrm{Int}(B))$ is $(i,j)-\beta-\mathscr{I}$-open and we have $f^{-1}(\sigma_i-\mathrm{Int}(B))\subset(i,j)-\beta\mathscr{I}\,\mathrm{Int}(f^{-1}(\sigma_i-\mathrm{Int}(B)))\subset(i,j)-\beta\mathscr{I}\,\mathrm{Int}(f^{-1}(B))$.
(vi)⇒(i): Let B be a σ_i-open set in Y. Then $\sigma_i-\mathrm{Int}(B)=B$ and $f^{-1}(B)\backslash f^{-1}(\sigma_i-\mathrm{Int}(B))\subset(i,j)-\beta\mathscr{I}\,\mathrm{Int}(f^{-1}(B))$. Hence we have $f^{-1}(B)=(i,j)-\beta\mathscr{I}\,\mathrm{Int}(f^{-1}(B))$. This shows that $f^{-1}(B)$ is $(i,j)-\beta-\mathscr{I}$-open in X.

If $\mathscr{I}=\{\emptyset\}$ in Theorem 11, we get the following

Corollary 3 ([6, Theorem 5.1]). *For a function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$, the following statements are equivalent:*

(i) f is pairwise semi-precontinuous;

(ii) For each point x in X and each σ_i-open set F in Y such that $f(x)\in F$, there is an (i,j)-semi-preopen set A in X such that $x\in A$, $f(A)\subset F$;

(iii) The inverse image of each σ_i-closed set in Y is (i,j)-semi-preclosed in X;

(iv) For each subset A of X, $f((i,j)-sp\,\mathrm{Cl}(A))\subset\sigma_i-\mathrm{Cl}(f(A))$;

(v) For each subset B of Y, $(i,j)-sp\,\mathrm{Cl}(f^{-1}(B))\subset f^{-1}(\sigma_i-\mathrm{Cl}(B))$.

Theorem 12. *Let $f:(X,\tau_1,\tau_2,\mathscr{I})\to(Y,\sigma_1,\sigma_2)$ be a function. If $g:(X,\tau_1,\tau_2,\mathscr{I})\to(X\times Y,\sigma_1\times\sigma_2)$ defined by $g(x)=(x,f(x))$ is an $(i,j)-\beta-\mathscr{I}$-continuous function, then f is $(i,j)-\beta-\mathscr{I}$-continuous.*

Proof. Let V be a σ_i-open set of Y. Then $f^{-1}(V)=X\cap f^{-1}(V)=g^{-1}(X\times V)$. Since g is an $(i,j)-\beta-\mathscr{I}$-continuous function and $X\times V$ is a $\tau_i\times\sigma_i$-open set of $X\times Y$, $f^{-1}(V)$ is an $(i,j)-\beta-\mathscr{I}$-open set of X. Hence f is $(i,j)-\beta-\mathscr{I}$-continuous.

Definition 10. *A bitopological space (X,τ_1,τ_2) is said to be pairwise connected [8] if it cannot be expressed as the union of two nonempty disjoint sets U and V such that U is τ_i-open and V is τ_j-open, where $i,j=\{1,2\}$.*

Definition 11. *An ideal bitopological space* $(X,\tau_1,\tau_2,\mathscr{I})$ *is said to be* $(i,j)-\beta-\mathscr{I}$*-connected if it cannot be expressed as the union of two nonempty disjoint sets* U *and* V *such that* U *is* $(i,j)-\beta-\mathscr{I}$*-open and* V *is* $(i,j)-\beta-\mathscr{I}$*-open.*

Theorem 13. *Let* $f:(X,\tau_1,\tau_2,\mathscr{I})\to(Y,\sigma_1,\sigma_2)$ *is* $(i,j)-\beta-\mathscr{I}$*-continuous surjection and* $(X,\tau_1,\tau_2,\mathscr{I})$ *is* $(i,j)-\beta-\mathscr{I}$*-connected, then* (Y,σ_1,σ_2) *is pairwise connected.*

Proof. Suppose Y is not pairwise connected, Then $Y=A\cup B$ where $A\cap B=\emptyset$, $A\neq\emptyset$, $B\neq\emptyset$ and $A\in\sigma_i$, $B\in\sigma_j$. Since f is $(i,j)-\beta-\mathscr{I}$-continuous $f^{-1}(A)\in(i,j)-\beta\mathscr{I}O(X)$ and $f^{-1}(B)\in(i,j)-\beta\mathscr{I}O(X)$, such that $f^{-1}(A)\neq\emptyset$, $f^{-1}(B)\neq\emptyset$. $f^{-1}(A)\cap f^{-1}(B)=\emptyset$ and $f^{-1}(A)\cup f^{-1}(B)=X$, which implies that X is not $(i,j)-\beta-\mathscr{I}$-connected.

References

[1] G. Balasubramanian. *Extremally disconnected bitopological spaces*, Bulletin of the Calcutta Mathematical Society, 83, 247-252. 1991.

[2] M. Caldas, S. Jafari and N. Rajesh. *Preopen sets in ideal bitopological spaces*, Bulletin of Parana's Mathematical Society, 29(2), 61-68. 2011.

[3] M. Caldas, S. Jafari and N. Rajesh. *Semiopen sets in ideal bitopological spaces* (to appear in CUBO Mathematics Journal).

[4] D. Jankovic and T. R. Hamlett. *New topologies from old via ideals*, American Mathematical Monthly, 97, 295-310. 1990.

[5] M. Jelic, *Feeble P-continuous mappings*. Rendicondi del Circolo Matematico di Palermo, **24**, 387-395. 1990

[6] F. Khedr, S. Al-Areefi and T. Noiri. *Precontinuity and semi-precontinuity in bitopological spaces*, Indian Journal of Pure and Applied Mathematics **23**(9), 624-633. 1992.

[7] K. Kuratowski. Topology, *Academic press, New York*. 1966.

[8] W. J. Pervine. *Connectedness in bitopological spaces*. Indagationes Mathematicae, **29**, 369-372. 1967.

[9] R. Vaidyanathaswamy. *The localisation theory in set topology*, Proceedings of the Indian Acadamic of Sciences, **20**, 51-61. 1945.

[10] S. Yuksel, A. H. Kocaman and A. Acıkgoz. *On* $\beta-\mathscr{I}$*-irresolute functions*, Far East Journal of Mathematical Sciences. **26**(3), 673-684. 2007.