A remark on an Archimedan square of a triangle associated with an arbelos

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Abstract. We consider squares with Archimedean incircle arising from a triangle associated with an arbelos.

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1. Introduction

We consider an arbelos formed by three semicircles α, β and γ with diameters AO, BO and AB, respectively for a point O on the segment AB, where $|AO| = 2a$ and $|BO| = 2b$ (see Figure 1). Circles of radius $r_{\rm A} = ab/(a+b)$ are said to be Archimedean. The radical axis of α and β is called the axis, and the incircle of the arbelos is denoted by δ.

For a triangle EFG, let $PQRS$ be a square such that P and Q lies on the side GE and EF, respectively, and RS lies on the line FG (see Figure 2). We call $PQRS$ the square of EFG on FG. If $|PQ| = 2r_{\rm A}$, the square $PQRS$ is said to be Archimedean. The incircle of an Archimedean square is Archimedean. In this article we construct several triangles whose squares on the base are Archimedean.

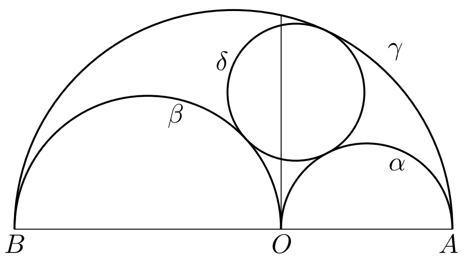

Figure 1.

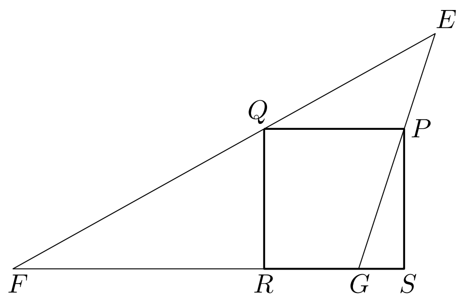

Figure 2.

2. Result

Theorem 1. *For a triangle EFG, let x and y be the length of the base FG and the height, respectively. Then the square of EFG on FG has side length $xy/(x+y)$. Therefore the square is Archimedean if and only if*

$$\frac{xy}{x+y} = 2r_{\rm A}. \tag{1}$$

Proof. If s is the side length of the square of EFG on FG, we get $(y-s)/s = y/x$ by the similar triangles EFG and EQP. This implies $s = xy/(x+y)$. □

Table 1 shows several pairs of x and y satisfying (1). We now construct triangles with base length x and height y for x and y in the table, where the case 1 is easy since $|AO| = 2a$ and $|BO| = 2b$. We use a rectangular coordinate system with origin O such that the farthest point on α from AB has coordinates (a, a).

Case	x	y
1	$2a$	$2b$
2	$2(a+b)$	$\dfrac{2ab(a+b)}{a^2+ab+b^2}$
3	$a+b$	$\dfrac{2ab(a+b)}{a^2+b^2}$
4	$a,\ (a>b)$	$\dfrac{2ab}{a-b}$
5	$4r_{\mathrm{A}}$	$4r_{\mathrm{A}}$
6	$2\sqrt{ab}$	$\dfrac{2ab}{a+b-\sqrt{ab}}$

Table 1. Pairs satisfying (1).

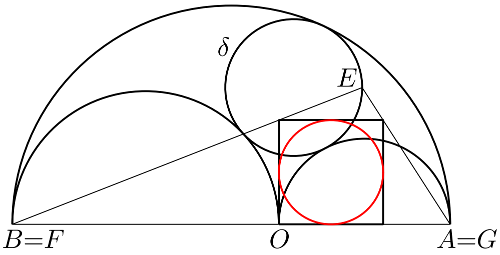

Figure 3.

2.1. **Case 2** $x = 2(a+b)$**,** $y = 2ab(a+b)/(a^2+ab+b^2)$**.** The circle δ has radius $r = ab(a+b)/d$ and center of coordinates $(ab(b-a)/d, 2r)$, where $d = a^2+ab+b^2$. Therefore if E lies on the diameter of δ parallel to AB, and $F = B$ and $G = A$, the square of EFG on FG is Archimedean. The case in which E coincides with the center of δ can be found in [2]. If E coincides with one of the endpoints of the diameter of δ parallel to AB, one of the sides of the square of EFG on FG lies on the axis (see Figure 3).

2.2. **Case 3** $x = a+b$**,** $y = 2ab(a+b)/(a^2+b^2)$**.** If E is the point of tangency of γ and δ, it has coordinates $(2j(b-a), 2j(a+b))$, where $j = ab/(a^2+b^2)$ [3].

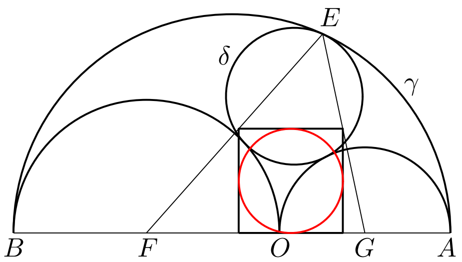

Figure 4.

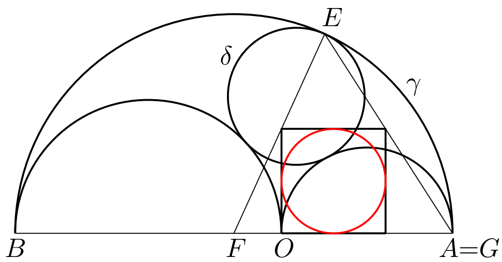

Figure 5.

Therefore if G and F are the centers of α and β, the square of EFG on FG is Archimedean (see Figure 4). Also if F is the center of γ and $G = A$, the square of EFG on FG is Archimedean, and one of the sides of the square lies on the axis (see Figure 5).

2.3. **Case 4** $x = a > b$**,** $y = 2ab/(a-b)$**.** Assume $a > b$. Let E be the point of intersection of AB and the external common tangent of α and β, which has an $(a-b)x - 2\sqrt{ab}y + 2ab = 0$ [4]. If F is the orthogonal projection of the farthest point on α from AB to the axis, and $G = O$, then $|FG| = a$ and $|GE| = 2ab/(a-b)$. Therefore the square of EFG on GE (or FG) is Archimedean (see Figure 6).

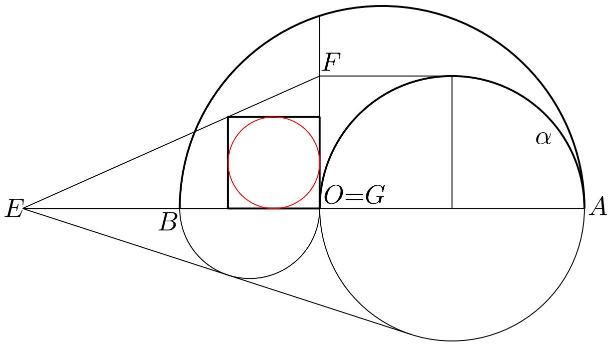

Figure 6.

2.4. **Case 5** $x = 4r_{\mathrm{A}}$**,** $y = 4r_{\mathrm{A}}$**.** Let ε_1 be the circle touching γ internally and AB at O. Then ε has radius $2r_{\mathrm{A}}$ [4]. Therefore if FG is the orthogonal projection of ε_1 to AB and E is the farthest point on ε from AB, then $|FG| = |EO| = 4r_{\mathrm{A}}$. Hence the square of EFG on FG is Archimedean (see Figure 7). The incircle of the square coincides with Bankoff circle.

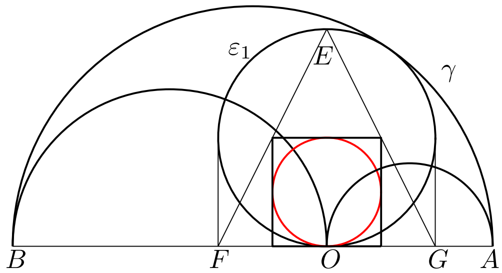

Figure 7.

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