ON THE ERDÓS-ULAM PROBLEM

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Abstract. In this paper we introduce and develop the topology of compression of points in space. We then use this Topology to solve the Erdós-Ulam problem. We provide a positive solution in this paper.

1. **Introduction**

The Erdós-Ulam problem is a question about the possible existence of dense set of points in the plane at rational distances from each other. More formally, the problem states

Conjecture 1.1. Is there a dense set of points in a plane at rational distances from each other?

Eventhough the Erdós-Ulam problem remained unsolved until now, there has been various studies concerning the rational distances between pairs of points in a plane. An important observation has been made in [1], which shows that the only algebraic curves containing dense set of points at rational distances from each other are **circles** and **lines**. In this paper however, we provide a positive solution to the problem. We start by introducing and developing the **topology** of compression of points in space. Consequently, we managed to prove the following theorem:

Theorem 1.1. *There exist a dense set of points in* $\mathbb{R}^n$ *at rational distances from each other.*

2. **Compression**

Definition 2.1. By the compression of scale $m \geq 1$ on $\mathbb{R}^n$ we mean the map $\mathbb{V} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that

$$\mathbb{V}_m[(x_1, x_2, \ldots, x_n)] = \left(\frac{m}{x_1}, \frac{m}{x_2}, \ldots, \frac{m}{x_n}\right)$$

for $n \geq 2$ and with $x_i \neq 0$ for all $i = 1, \ldots, n$.

Remark 2.2. The notion of compression is in some way the process of re scaling points in $\mathbb{R}^n$ for $n \geq 2$. Thus it is important to notice that a compression pushes points very close to the origin away from the origin by certain scale and similarly draws points away from the origin close to the origin.

Proposition 2.1. *A compression of scale* $m \geq 1$ *with* $\mathbb{V}_m : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *is a bijective map.*

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Proof. Suppose $\mathbb{V}_m[(x_1, x_2, \dots, x_n)] = \mathbb{V}_m[(y_1, y_2, \dots, y_n)]$, then it follows that

$$\left(\frac{m}{x_1}, \frac{m}{x_2}, \dots, \frac{m}{x_n}\right) = \left(\frac{m}{y_1}, \frac{m}{y_2}, \dots, \frac{m}{y_n}\right).$$

It follows that $x_i = y_i$ for each $i = 1, 2, \dots, n$. Surjectivity follows by definition of the map. Thus the map is bijective. □

2.1. **The mass of compression.**

Definition 2.3. By the mass of a compression of scale $m \geq 1$ we mean the map $\mathcal{M} : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \dots, x_n)]) = \sum_{i=1}^{n} \frac{m}{x_i}.$$

Remark 2.4. Next we prove upper and lower bounding the mass of the compression of scale $m \geq 1$.

Proposition 2.2. *Let $(x_1, x_2, \dots, x_n) \in \mathbb{N}^n$, then the estimates holds*

$$m \log\left(1 - \frac{n-1}{\sup(x_j)}\right)^{-1} \ll \mathcal{M}(\mathbb{V}_m[(x_1, x_2, \dots, x_n)]) \ll m \log\left(1 + \frac{n-1}{\mathrm{Inf}(x_j)}\right)$$

for $n \geq 2$.

Proof. Let $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ for $n \geq 2$ with $x_j \geq 1$. Then it follows that

$$\begin{aligned}\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \dots, x_n)]) &= m \sum_{j=1}^{n} \frac{1}{x_j} \\ &\leq m \sum_{k=0}^{n-1} \frac{1}{\mathrm{Inf}(x_j) + k}\end{aligned}$$

and the upper estimate follows by the estimate for this sum. The lower estimate also follows by noting the lower bound

$$\begin{aligned}\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \dots, x_n)]) &= m \sum_{j=1}^{n} \frac{1}{x_j} \\ &\geq m \sum_{k=0}^{n-1} \frac{1}{\sup(x_j) - k}.\end{aligned}$$

□

Definition 2.5. Let $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ with $x_i \neq 0$ for all $i = 1, 2 \dots, n$. Then by the gap of compression of scale m $\mathbb{V}_m$, denoted $\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \dots, x_n)]$, we mean the expression

$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \dots, x_n)] = \left|\left|\left(x_1 - \frac{m}{x_1}, x_2 - \frac{m}{x_2}, \dots, x_n - \frac{m}{x_n}\right)\right|\right|$$

3. The ball induced by compression

In this section we introduce the notion of the ball induced by a point $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ under compression of a given scale. We launch more formally the following language.

Definition 3.1. Let $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ with $x_i \neq x_j$ for all $1 \leq i < j \leq n$. Then by the ball induced by $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ under compression of scale m, denoted $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]}[(x_1, x_2, \ldots, x_n)]$ we mean the inequality

$$\left|\left|\vec{y} - \frac{1}{2}\left(x_1 + \frac{m}{x_1}, x_2 + \frac{m}{x_2}, \ldots, x_n + \frac{m}{x_n}\right)\right|\right| \leq \frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)].$$

A point $\vec{z} = (z_1, z_2, \ldots, z_n) \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]}[(x_1, x_2, \ldots, x_n)]$ if it satisfies the inequality.

Remark 3.2. Next we prove that smaller balls induced by points should essentially be covered by the bigger balls in which they are embedded. We state and prove this statement in the following result.

For simplicity we will on occasion choose to write the ball induced by the point $\vec{x} = (x_1, x_2, \ldots, x_n)$ under compression as

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

We adopt this notation to save enough work space in many circumstances. We first prove a preparatory result in the following sequel. We find the following estimates for the compression gap useful.

Proposition 3.1. *Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ for $n \geq 2$ with $x_j \neq 0$ for $j = 1, \ldots, n$, then we have*

$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]^2 = \mathcal{M} \circ \mathbb{V}_1\left[\left(\frac{1}{x_1^2}, \ldots, \frac{1}{x_n^2}\right)\right] + m^2\mathcal{M} \circ \mathbb{V}_1[(x_1^2, \ldots, x_n^2)] - 2mn.$$

In particular, we have the estimate

$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]^2 = \mathcal{M} \circ \mathbb{V}_1\left[\left(\frac{1}{x_1^2}, \ldots, \frac{1}{x_n^2}\right)\right] - 2mn + O\left(m^2\mathcal{M} \circ \mathbb{V}_1[(x_1^2, \ldots, x_n^2)]\right).$$

Proposition 3.1 offers us an extremely useful identity. It allows us to pass from the gap of compression on points to the relative distance to the origin. It tells us that points under compression with a large gap must be far away from the origin than points with a relatively smaller gap under compression. That is to say, the inequality

$$\mathcal{G} \circ \mathbb{V}_m[\vec{x}] \leq \mathcal{G} \circ \mathbb{V}_m[\vec{y}]$$

if and only if $||\vec{x}|| \leq ||\vec{y}||$ for $\vec{x}, \vec{y} \in \mathbb{N}^n$. This important transference principle will be mostly put to use in obtaining our results.

Lemma 3.3 (Compression estimate)**.** *Let $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ for $n \geq 2$, then we have*

$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]^2 \ll n\mathrm{sup}(x_j^2) + m^2 \log\left(1 + \frac{n-1}{\mathrm{Inf}(x_j)^2}\right) - 2mn$$

and

$$\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]^2 \gg n\mathrm{Inf}(x_j^2)+m^2\log\left(1-\frac{n-1}{\sup(x_j^2)}\right)^{-1}-2mn.$$

Theorem 3.4. *Let $\vec{z}=(z_1,z_2,\ldots,z_n)\in\mathbb{N}^n$ with $z_i\neq z_j$ for all $1\leq i<j\leq n$. Then $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ if and only if*

$$\mathcal{G}\circ\mathbb{V}_m[\vec{z}]\leq\mathcal{G}\circ\mathbb{V}_m[\vec{y}].$$

Proof. Let $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ for $\vec{z}=(z_1,z_2,\ldots,z_n)\in\mathbb{N}^n$ with $z_i\neq z_j$ for all $1\leq i<j\leq n$, then it follows that $||\vec{y}||>||\vec{z}||$. Suppose on the contrary that

$$\mathcal{G}\circ\mathbb{V}_m[\vec{z}]>\mathcal{G}\circ\mathbb{V}_m[\vec{y}],$$

then it follows that $||\vec{y}||<||\vec{z}||$, which is absurd. Conversely, suppose

$$\mathcal{G}\circ\mathbb{V}_m[\vec{z}]\leq\mathcal{G}\circ\mathbb{V}_m[\vec{y}]$$

then it follows from Proposition 3.1 that $||\vec{z}||\leq||\vec{y}||$ and $\sup(z_j)\leq\sup(y_j)$ by Lemma 3.3. It follows that

$$\begin{aligned}\left|\left|\vec{z}-\frac{1}{2}\left(y_1+\frac{m}{y_1},\ldots,y_n+\frac{m}{y_n}\right)\right|\right| &\leq \left|\left|\vec{y}-\frac{1}{2}\left(y_1+\frac{m}{y_1},\ldots,y_n+\frac{m}{y_n}\right)\right|\right|\\ &\leq\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}].\end{aligned}$$

This certainly implies $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ and the proof of the theorem is complete. □

Theorem 3.5. *Let $\vec{x}=(x_1,x_2,\ldots,x_n)\in\mathbb{N}^n$ with $x_i\neq x_j$ for all $1\leq i<j\leq n$. If $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ then*

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]\subseteq\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

Proof. First let $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ and suppose for the sake of contradiction that

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]\not\subseteq\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

Then there must exist some $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ such that $\vec{z}\notin\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$. It follows from Theorem 3.4 that

$$\mathcal{G}\circ\mathbb{V}_m[\vec{z}]>\mathcal{G}\circ\mathbb{V}_m[\vec{x}].$$

It follows that

$$\begin{aligned}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]&\geq\mathcal{G}\circ\mathbb{V}_m[\vec{z}]\\&>\mathcal{G}\circ\mathbb{V}_m[\vec{x}]\\&\geq\mathcal{G}\circ\mathbb{V}_m[\vec{y}]\end{aligned}$$

which is absurd, thereby ending the proof. □

Remark 3.6. Theorem 3.5 tells us that points confined in certain balls induced under compression should by necessity have their induced ball under compression covered by these balls in which they are contained.

3.1. **Interior points and the limit points of balls induced under compression.** In this section we launch the notion of an interior and the limit point of balls induced under compression. We study this notion in depth and explore some connections.

Definition 3.7. Let $\vec{y}=(y_1,y_2,\ldots,y_n)\in\mathbb{N}^n$ with $y_i\neq y_j$ for all $1\leq i<j\leq n$. Then a point $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ is an interior point if

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{z}]}[\vec{z}]\subseteq\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

for most $\vec{x}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$. An interior point $\vec{z}$ is then said to be a limit point if

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{z}]}[\vec{z}]\subseteq\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

for all $\vec{x}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$

Remark 3.8. Next we prove that there must exist an interior and limit point in any ball induced by points under compression of any scale in any dimension.

Theorem 3.9. *Let $\vec{x}=(x_1,x_2,\ldots,x_n)\in\mathbb{N}^n$ with $x_i\neq x_j$ for all $1\leq i<j\leq n$. Then the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ contains an interior point and a limit point.*

Proof. Let $\vec{x}=(x_1,x_2,\ldots,x_n)\in\mathbb{N}^n$ with $x_i\neq x_j$ for all $1\leq i<j\leq n$ and suppose on the contrary that $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ contains no limit point. Then pick

$$\vec{z}_1\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

for $\vec{z}_1\neq\vec{x}$. Then by Theorem 3.5 and Theorem 3.4 It follows that

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{z}_1]}[\vec{z}_1]\subset\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

with $\mathcal{G}\circ\mathbb{V}_m[\vec{z}_1]<\mathcal{G}\circ\mathbb{V}_m[\vec{x}]$. Again pick $\vec{z}_2\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{z}_1]}[\vec{z}_1]$ for $\vec{z}_2\neq\vec{z}_1$. Then by employing Theorem 3.5 and Theorem 3.4, we have

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{z}_2]}[\vec{z}_2]\subset\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{z}_1]}[\vec{z}_1]$$

with $\mathcal{G}\circ\mathbb{V}_m[\vec{z}_2]<\mathcal{G}\circ\mathbb{V}_m[\vec{z}_1]$. By continuing the argument in this manner we obtain the infinite descending sequence of the gap of compression

$$\mathcal{G}\circ\mathbb{V}_m[\vec{x}]>\mathcal{G}\circ\mathbb{V}_m[\vec{z}_1]>\mathcal{G}\circ\mathbb{V}_m[\vec{z}_2]>\cdots>\mathcal{G}\circ\mathbb{V}_m[\vec{z}_n]>\cdots$$

which follows from Theorem 3.4 that

$$||\vec{x}||>||\vec{z}_1||>\cdots>||\vec{z}_n||>\cdots>\cdots$$

thereby ending the proof of the theorem. □

3.2. **Admissible points of balls induced under compression.** We launch the notion of admissible points of balls induced by points under compression. We study this notion in depth and explore some possible connections.

Definition 3.10. Let $\vec{y}=(y_1,y_2,\ldots,y_n)\in\mathbb{N}^n$ with $y_i\neq y_j$ for all $1\leq i<j\leq n$. Then $\vec{y}$ is said to be an admissible point of the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ if

$$\left|\left|\vec{y}-\frac{1}{2}\left(x_1+\frac{m}{x_1},\ldots,x_n+\frac{m}{x_n}\right)\right|\right|=\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}].$$

Remark 3.11. It is important to notice that the notion of admissible points of balls induced by points under compression encompasses points on the ball. These points in geometrical terms basically sit on the outer of the induced ball. Next we show that all balls can in principle be generated by their admissible points.

Theorem 3.12. *The point* $\vec{y} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ *is admissible if and only if*

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}] = \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

and $\mathcal{G} \circ \mathbb{V}_m[\vec{y}] = \mathcal{G} \circ \mathbb{V}_m[\vec{x}]$.

Proof. First let $\vec{y} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ be admissible and suppose on the contrary that

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}] \neq \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

Then there exist some $\vec{z} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ such that

$$\vec{z} \notin \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}].$$

Applying Theorem 3.4, we obtain the inequality

$$\mathcal{G} \circ \mathbb{V}_m[\vec{y}] < \mathcal{G} \circ \mathbb{V}_m[\vec{z}] \leq \mathcal{G} \circ \mathbb{V}_m[\vec{x}].$$

It follows from Proposition 3.1 that $||\vec{y}|| < ||\vec{x}||$. This contradicts the fact that the point $\vec{y} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ is an admissible point. Now we notice that $\vec{y} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ certainly implies $\mathcal{G} \circ \mathbb{V}_m[\vec{y}] \leq \mathcal{G} \circ \mathbb{V}_m[\vec{x}]$. Conversely we notice as well that $\vec{x} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$, which certainly implies $\mathcal{G} \circ \mathbb{V}_m[\vec{x}] \leq \mathcal{G} \circ \mathbb{V}_m[\vec{y}]$ by Theorem 3.4. Thus the conclusion follows. Conversely, suppose

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}] = \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

and $\mathcal{G} \circ \mathbb{V}_m[\vec{y}] = \mathcal{G} \circ \mathbb{V}_m[\vec{x}]$. Then it follows that the point $\vec{y}$ must satisfy the inequality

$$\begin{aligned}\left|\left|\vec{z} - \frac{1}{2}\left(y_1 + \frac{m}{y_1}, \ldots, y_n + \frac{m}{y_n}\right)\right|\right| &= \left|\left|\vec{z} - \frac{1}{2}\left(x_1 + \frac{m}{x_1}, \ldots, x_n + \frac{m}{x_n}\right)\right|\right| \\ &\leq \frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[\vec{x}].\end{aligned}$$

It follows that

$$\begin{aligned}\frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[\vec{x}] &= \left|\left|\vec{y} - \frac{1}{2}\left(x_1 + \frac{m}{x_1}, \ldots, x_n + \frac{m}{x_n}\right)\right|\right| \\ &\leq \frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[\vec{x}]\end{aligned}$$

and $\vec{y}$ is indeed admissible, thereby ending the proof. □

3.3. The dilation of the ball induced by compression.

In this section we introduce the notion of the dilation of balls induced by points under compression. We study this in relation to other concepts of compression.

Definition 3.13. Let $\vec{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ and $\mathbb{V}_m : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a compression of scale m. Then by the dilation of the induced ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ by a scale factor of $t > 0$, we mean the map

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}] \longrightarrow \mathcal{B}^t_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}] = \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[t\vec{x}]}[t\vec{x}].$$

Remark 3.14. Next we show that we can in practice embed all balls in their positive dilation.

Proposition 3.2. *Let $\vec{x}=(x_1,x_2,\ldots,x_n)\in\mathbb{N}^n$. For all $t>1$, we have*

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]\subset\mathcal{B}^t_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]=\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[t\vec{x}]}[t\vec{x}].$$

Proof. First let $\vec{x}=(x_1,x_2,\ldots,x_n)\in\mathbb{N}^n$ and take $t>1$. Suppose

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]\not\subset\mathcal{B}^t_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]=\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[t\vec{x}]}[t\vec{x}].$$

Then it follows that there exist some $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ such that $\vec{z}\notin\mathcal{B}^t_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]=\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[t\vec{x}]}[t\vec{x}]$. By Theorem 3.4, It follows that

$$\begin{aligned}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]&>\mathcal{G}\circ\mathbb{V}_m[\vec{z}]\\&>\mathcal{G}\circ\mathbb{V}_m[t\vec{x}]\\&>t\mathcal{G}\circ\mathbb{V}_m[\vec{x}].\end{aligned}$$

This is absurd since $t>1$, and the proof is complete. □

The result in Proposition 3.2 can be thought of as an analogue of most embedding theorems. It tells us for the most part we can in principle cover all balls of various sizes by their dilates. Next we show that dilation of balls and their sub-balls still preserves an embedding in the ball. We formalize this assertion in the following proposition.

Proposition 3.3. *Let $\vec{y}\in\mathbb{N}^n$ with $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$. Then for any $t>1$, we have*

$$\mathcal{B}^t_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]\subseteq\mathcal{B}^t_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

Proof. First suppose $\vec{y}\in\mathbb{N}^n$ with $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$. Then by Theorem 3.5 it follows that

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]\subseteq\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

and it follows from Proposition 3.1 that $||\vec{y}||\leq||\vec{x}||$. Now suppose on the contrary that

$$\mathcal{B}^t_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]\not\subseteq\mathcal{B}^t_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

Then it follows that there exist some $\vec{z}\in\mathbb{N}^n$ with $\vec{z}\in\mathcal{B}^t_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ such that $\vec{z}\notin\mathcal{B}^t_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$. By appealing to Theorem 3.4, it follows that

$$\begin{aligned}\mathcal{G}\circ\mathbb{V}_m[t\vec{y}]&\geq\mathcal{G}\circ\mathbb{V}_m[\vec{z}]\\&>\mathcal{G}\circ\mathbb{V}_m[t\vec{x}].\end{aligned}$$

This certainly implies $||t\vec{x}||<||t\vec{y}||$ for $t>1$ by appealing to Proposition 3.1. This is a contradiction, and the proof of the Proposition is complete. □

3.4. **The order of points in the ball induced under compression.** In this section we introduce the notion of the order of points contained in balls induced under compression on points in $\mathbb{N}^n$. We launch the following formal language.

Definition 3.15. Let $\vec{y}=(y_1,y_2,\ldots,y_n)\in\mathbb{N}^n$ with $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$. Then we say the point $\vec{y}$ is of order $t>0$ in the ball if $\vec{x}\parallel\vec{y}$ and there exist some $t>0$ such that

$$\mathcal{B}^t_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]=\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

Otherwise we say the point $\vec{y}$ is free in the ball.

Remark 3.16. Next we show that the existence of order of points in a ball induced by points under compression is mostly in continuum. We formalize this claim in the following proposition.

Proposition 3.4. *Let $\vec{x},\vec{y},\vec{z}\in\mathbb{N}^n$ with $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ and $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]\subset\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{z}]}[\vec{z}]$. If the point $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ is of order $t>1$ and the point $\vec{x}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{z}]}[\vec{z}]$ is of order $s>1$. Then the point*

$$\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{z}]}[\vec{z}]$$

is of order $st>1$.

Proof. First suppose $\vec{x},\vec{y},\vec{z}\in\mathbb{N}^n$ with $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ and $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]\subset\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{z}]}[\vec{z}]$. Then by Theorem 3.5, we have the following chains of ball embedding

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]\subset\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]\subset\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{z}]}[\vec{z}].$$

Since $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ is of order $t>1$, It follows that

$$\begin{aligned}\mathcal{B}^t_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]&=\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[t\vec{y}]}[t\vec{y}]\\&=\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]\end{aligned}$$

and by appealing to Theorem 3.4, $\mathcal{G}\circ\mathbb{V}_m[t\vec{y}]=\mathcal{G}\circ\mathbb{V}_m[\vec{x}]$ and it follows that $||t\vec{y}||=||\vec{x}||$, by Proposition 3.1. Again the point $\vec{x}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{z}]}[\vec{z}]$ is of order $s>1$ and it follows that

$$\begin{aligned}\mathcal{B}^s_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]&=\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[s\vec{x}]}[s\vec{x}]\\&=\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{z}]}[\vec{z}].\end{aligned}$$

By appealing to Theorem 3.4, It follows that $\mathcal{G}\circ\mathbb{V}_m[s\vec{x}]=\mathcal{G}\circ\mathbb{V}_m[\vec{z}]$ and $||s\vec{x}||=||\vec{z}||$. By combining the two relations, we have

$$st||\vec{y}||=||\vec{z}||$$

It follows that $st\vec{y}=\vec{z}$ and the result follows immediately. □

4. **Application to the Erdós-Ulam problem**

In this section we apply the topology to the Erdós-Ulam problem in the following sequel. We first launch the following preparatory results.

Lemma 4.1. *Let $\vec{x}\in\mathbb{N}^n$ with $m\in\mathbb{N}$. Then $\mathcal{G}\circ\mathbb{V}_m[\vec{x}]\times\mathcal{G}\circ\mathbb{V}_m[\vec{x}]\in\mathbb{Q}$. That is, the square of compression gap induced on the point $\vec{y}\in\mathbb{N}^n$ is always rational.*

Proof. Suppose $\vec{x}=(x_1,x_2,\ldots,x_n)\in\mathbb{N}^n$ and let $m\in\mathbb{N}$, then by invoking Proposition 3.1, we have

$$\mathcal{G}\circ\mathbb{V}_m[\vec{x}]\times\mathcal{G}\circ\mathbb{V}_m[\vec{x}]=\mathcal{M}\circ\mathbb{V}_1\left[\left(\frac{1}{x_1^2},\ldots,\frac{1}{x_n^2}\right)\right]+m^2\mathcal{M}\circ\mathbb{V}_1[(x_1^2,\ldots,x_n^2)]-2mn.$$

The result follows since

$$\mathcal{M}\circ\mathbb{V}_1\left[\left(\frac{1}{x_1^2},\ldots,\frac{1}{x_n^2}\right)\right],\ m^2\mathcal{M}\circ\mathbb{V}_1[(x_1^2,\ldots,x_n^2)],2mn\in\mathbb{Q}$$

thereby proving the Lemma. ☐

Lemma 4.2. *Let $\vec{x}\in\mathbb{N}^n$ with $\mathcal{G}\circ\mathbb{V}_m[\vec{x}]>1$, then*

$$\mathcal{B}_{\frac{1}{2}(\mathcal{G}\circ\mathbb{V}_m[\vec{x}])^2}[\mathcal{G}\circ\mathbb{V}_m[\vec{x}]\vec{x}]\subset\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\mathcal{G}\circ\mathbb{V}_m[\vec{x}]\vec{x}]}[\mathcal{G}\circ\mathbb{V}_m[\vec{x}]\vec{x}].$$

Proof. Suppose $\vec{x}\in\mathbb{N}^n$ and let $\mathcal{G}\circ\mathbb{V}_m[\vec{x}]>1$. First, we notice that the two balls so constructed

$$\mathcal{B}_{\frac{1}{2}(\mathcal{G}\circ\mathbb{V}_m[\vec{x}])^2}[\mathcal{G}\circ\mathbb{V}_m[\vec{x}]\vec{x}]\quad\text{and}\quad\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\mathcal{G}\circ\mathbb{V}_m[\vec{x}]\vec{x}]}[\mathcal{G}\circ\mathbb{V}_m[\vec{x}]\vec{x}]$$

are centered at the same point. Thus it suffices to show that

$$(\mathcal{G}\circ\mathbb{V}_m[\vec{x}])^2\le\mathcal{G}\circ\mathbb{V}_m[\mathcal{G}\circ\mathbb{V}_m[\vec{x}]\vec{x}].$$

Now let us set $t=\mathcal{G}\circ\mathbb{V}_m[\vec{x}]>1$. Then we obtain

$$\mathcal{G}\circ\mathbb{V}_m[t\vec{x}]>t\mathcal{G}\circ\mathbb{V}_m[\vec{x}]$$

and the result follows by substitution. ☐

Remark 4.3. We are now ready to prove the Erdós-Ulam conjecture. We assemble the tools we have developed thus far to solve the problem.

4.1. **Proof of the Erdós-Ulam conjecture.** In this section we assemble the tools we have developed thus far to solve the Erdós-Ulam problem. We provide a positive solution to the problem as espoused in the following result.

Theorem 4.4. *There exist a dense set of points in $\mathbb{R}^n$ at rational distances from each other.*

Proof. Pick arbitrarily $\vec{x}\in\mathbb{N}^n$ and apply the compression $\mathbb{V}_m[\vec{x}]$ for $m\in\mathbb{N}$. Consider the ball induced under compression $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$. Now dilate the ball with the scale factor $t=\mathcal{G}\circ\mathbb{V}_m[\vec{x}]>1$, then by Lemma 4.2 we obtain the embedding of balls

$$\mathcal{B}_{\frac{1}{2}(\mathcal{G}\circ\mathbb{V}_m[\vec{x}])^2}[\mathcal{G}\circ\mathbb{V}_m[\vec{x}]\vec{x}]\subset\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\mathcal{G}\circ\mathbb{V}_m[\vec{x}]\vec{x}]}[\mathcal{G}\circ\mathbb{V}_m[\vec{x}]\vec{x}].$$

Let us now consider the inner ball, centered at the same point as the outer ball, but of rational radius by Lemma 4.1

$$\frac{1}{2}(\mathcal{G}\circ\mathbb{V}_m[\vec{x}])^2.$$

For each admissible point $\vec{z}$ of $\mathcal{B}_{\frac{1}{2}(\mathcal{G}\circ\mathbb{V}_m[\vec{x}])^2}[\mathcal{G}\circ\mathbb{V}_m[\vec{x}]\vec{x}]$ we join with a line to the admissible point exactly opposite. These two points are at rational distances

$$\frac{1}{2}(\mathcal{G}\circ\mathbb{V}_m[\vec{x}])^2+\frac{1}{2}(\mathcal{G}\circ\mathbb{V}_m[\vec{x}])^2=(\mathcal{G}\circ\mathbb{V}_m[\vec{x}])^2$$

from each other. We remark that the point $\vec{z} \in \mathbb{R}^n$ is an arbitrary admissible point and are dense on the ball. Thus there are infinitely many lines of rational distances joining admissible points of the ball

$$\mathcal{B}_{\frac{1}{2}(\mathcal{G}\circ\mathbb{V}_m[\vec{x}])^2}[\mathcal{G}\circ\mathbb{V}_m[\vec{x}]\vec{x}].$$

We construct sequence of embedding of balls in the following manner

$$\mathcal{B}_{\frac{1}{2n}(\mathcal{G}\circ\mathbb{V}_m[\vec{x}])^2}[\mathcal{G}\circ\mathbb{V}_m[\vec{x}]\vec{x}] \subset \cdots \subset \mathcal{B}_{\frac{1}{2}(\mathcal{G}\circ\mathbb{V}_m[\vec{x}])^2}[\mathcal{G}\circ\mathbb{V}_m[\vec{x}]\vec{x}]$$

for $n \geq 2$. The upshot is concentric balls all centered at the same point with successively smaller radius

$$\frac{1}{2n}(\mathcal{G}\circ\mathbb{V}_m[\vec{x}])^2$$

for $n \geq 2$. We remark that the lines drawn joining points on the bigger ball will also join points on the smaller balls at rational distance. The distance of points on different balls on the same line are also at rational distance from each other. That is, if $\vec{s_1} \in \mathcal{B}_{\frac{1}{2}(\mathcal{G}\circ\mathbb{V}_m[\vec{x}])^2}[\mathcal{G}\circ\mathbb{V}_m[\vec{x}]\vec{x}]$ and $\vec{s_2} \in \mathcal{B}_{\frac{1}{4}(\mathcal{G}\circ\mathbb{V}_m[\vec{x}])^2}[\mathcal{G}\circ\mathbb{V}_m[\vec{x}]\vec{x}]$ and $\vec{s_1}$ and $\vec{s_2}$ sit on the same line, then they must be of rational distance

$$\frac{1}{2}(\mathcal{G}\circ\mathbb{V}_m[\vec{x}])^2 - \frac{1}{4}(\mathcal{G}\circ\mathbb{V}_m[\vec{x}])^2 = \frac{1}{2}(\mathcal{G}\circ\mathbb{V}_m[\vec{x}])^2$$

by Lemma 4.1. In general, the radius of the annular region of successive balls so constructed is rational given by

$$\frac{1}{2n}(\mathcal{G}\circ\mathbb{V}_m[\vec{x}])^2 - \frac{1}{2(n+1)}(\mathcal{G}\circ\mathbb{V}_m[\vec{x}])^2 = \frac{1}{2n(n+1)}(\mathcal{G}\circ\mathbb{V}_m[\vec{x}])^2$$

for $n \in \mathbb{N}$ for all $n \geq 1$. Again we constructs sequence of embedding of balls centered at the same point as before below

$$\mathcal{B}_{\frac{1+2n}{4n(n+1)}(\mathcal{G}\circ\mathbb{V}_m[\vec{x}])^2}[\mathcal{G}\circ\mathbb{V}_m[\vec{x}]\vec{x}] \subset \cdots \subset \mathcal{B}_{\frac{3}{8}(\mathcal{G}\circ\mathbb{V}_m[\vec{x}])^2}[\mathcal{G}\circ\mathbb{V}_m[\vec{x}]\vec{x}]$$

for $n \in \mathbb{N}$ with $n \geq 2$. Admissible points of each of these balls are at rational distances away from the admissible point exactly opposite. That is, they are

$$\frac{1+2n}{2n(n+1)}$$

for $n \geq 1$. It is not difficult to see that we can embed this sequence of ball embedding into the a priori sequence of ball embedding. By carrying out the argument in this manner repeatedly, we then generate a dense set of points $\vec{s_n} \in \mathbb{R}^n$ that are at rational distance from each other. This completes the proof of the theorem, since the radius of the ball is determined by the point $\vec{x} \in \mathbb{N}^n$ under compression and this point can be chosen arbitrarily in space. That is, we can cover the entire space with this construction by arbitrarily taking points far away from the origin. □

5. Further remarks

It is important to have noticed that Theorem 4.4 is a general Theorem that holds in any euclidean space of any dimension. The Erdós-Ulam problem is a special case of Theorem 4.4, by taking $n = 2$. Also it needs to be said that this method is constructive in nature as opposed to analytic methods that might require the use of various exotic estimates. [1].

[1]

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References

1. Solymosi, Jozsef and De Zeeuw, Frank *On a question of Erdős and Ulam*, Discrete & Computational Geometry, vol. 43:2, 2010, Springer, pp 393–401.

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