Division by Zero Calculus For Differentiable Functions - l'Hôpital's Theorem Versions -

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Abstract: Based on the preprint survey paper ([25]), we will give a generalization of the division by zero calculus to differentiable functions and its basic properties. Typically, we can obtain l'Hôpital's theorem versions and some deep properties on the division by zero.

Key Words: Division by zero, division by zero calculus, differentiable, analysis, Laurent expansion, l'Hôpital's theorem, $1/0 = 0/0 = z/0 = \tan(\pi/2) = \log 0 = 0, (z^n)/n = \log z$ for $n = 0$, $e^{(1/z)} = 1$ for $z = 0$.

AMS Mathematics Subject Classifications: 00A05, 00A09, 42B20, 30E20.

1 Division by zero calculus

In order to state the new results in a self-contained way, we will recall the simple background on the division by zero calculus for analytic functions.

As the number system containing the division by zero, the Yamada field structure is perfect ([7]). However, for applications of the division by zero to **functions**, we need the concept of the division by zero calculus for the sake of unique determination of the results and for some deep reasons.

We will introduce the division by zero calculus. For any Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{-1} C_n(z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n(z-a)^n, \tag{1.1}$$

we **define** the identity, for $n > -1$

$$f^{(n)}(a) = n!C_n. \tag{1.2}$$

Apart from the motivation, we define the division by zero calculus by (1.2). With this assumption, we can obtain many new results and new ideas. However, for this assumption we have to check the results obtained whether they are reasonable or not. By this idea, we can avoid any logical problems. – At this point, the division by zero calculus may be considered **as a fundamental assumption like an axiom.** We consider that the division by zero calculus is over **convention**. We already have many and many applications.

In addition, we will refer to an interesting viewpoint of the division by zero calculus.

Recall the Cauchy integral formula for an analytic function $f(z)$; for an analytic function $f(z)$ around $z = a$ and for a smooth simple Jordan closed curve γ enclosing one time the point a, we have

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}} dz.$$

Even when the function $f(z)$ has any singularity at the point a, we assume that this formula is valid as the division by zero calculus.

We **define** the value of the function $f^{(n)}(a)$ with the above Cauchy integral.

2 We can divide the numbers and analytic functions by zero

In the division by zero like $1/0, 0/0$ the important problem was on their definitions. We will give our interpretation.

The meaning (definition) of

$$\frac{1}{0} = 0$$

is given by $f(0) = 0$ by means of the division by zero calculus for the function $f(z) = 1/z$. Similarly, the definition

$$\frac{0}{0} = 0$$

is given by $f(0) = 0$ by means of the division by zero calculus for the function $f(z) = 0/z$.

In the division by zero, the essential problem was in the sense of the division by zero (**definition**) $z/0$. Many confusions and simple history of division by zero may be looked in [16].

In order to give the precise meaning of division by zero, we will give a simple and affirmative answer, for a famous rule that we are not permitted to divide the numbers and functions by zero. In our mathematics, **prohibition** is a famous word for the division by zero ([28]).

For any analytic function $f(z)$ around the origin $z = 0$ that is permitted to have any singularity at $z = 0$ (of course, any constant function is permitted), we can consider the value, by the division by zero calculus

$$\frac{f(z)}{z^n} \tag{2.1}$$

at the point $z = 0$, for any positive integer n. This will mean that from **the form** we can consider it as follows:

$$\frac{f(z)}{z^n} \mid_{z=0} . \tag{2.2}$$

For example,

$$\frac{e^x}{x^n} \mid_{x=0} = \frac{1}{n!}.$$

This is the definition of our division by zero (general fraction). In this sense, we can divide the numbers and analytic functions by zero. For $z \neq 0$, $\frac{f(z)}{z^n}$ means the usual division of the function $f(z)$ by z^n.

Surprisingly enough, Brahmāgupta said 0/0=0 in 628, thirteen hundred years ago. However, our world history shows that his result is wrong and we have still in confusions on the division by zero.

However, his result and idea was right.

For many applications, see the original survey paper ([25]) and the references in this paper.

In order to show the importance of our division by zero and division by zero calculus we obtained many applications and many examples over 1000 items from an elementary mathematics. However, with the results stated in the references, we think that the importance of our division by zero calculus may be definitely stated already and clearly.

3 Division by zero calculus for differentiable functions

For a function $y = f(x)$ which is n order differentiable at $x = a$, we will **define** the value of the function, for $n > 0$

$$\frac{f(x)}{(x-a)^n}$$

at the point $x = a$ by the value

$$\frac{f^{(n)}(a)}{n!}.$$

For the important case of $n = 1$,

$$\frac{f(x)}{x-a}|_{x=a} = f'(a). \tag{3.1}$$

We will give its naturality of the definition.

Indeed, we consider the function $F(x) = f(x)-f(a)$ and by the definition, we have

$$\frac{F(x)}{x-a}|_{x=a} = F'(a) = f'(a).$$

Meanwhile, by the definition, we have

$$\lim_{x\to a} \frac{F(x)}{x-a} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a).$$

The identity (3.1) may be regarded as an interpretation of the differential coefficient $f'(a)$ by the concept of the division by zero. Here, we do not use the concept of limitings.

In the expression (3.1), the value $f'(a)$ in the right hand side is represented by the point a, meanwhile the expression

$$\frac{f(x)}{x-a}|_{x=a} \tag{3.2}$$

in the left hand side, is represented by the dummy variable $x-a$ that represents the property of the function around the point $x=a$ with the sense of the division

$$\frac{f(x)}{x-a}.$$

For $x \neq a$, it represents the usual division.

Of course, by our definition

$$\frac{f(x)}{x-a}|_{x=a} = \frac{f(x)-f(a)}{x-a}|_{x=a}, \tag{3.3}$$

however, here $f(a)$ may be replaced by any constant. This fact shows that the function

$$\frac{1}{x-a}$$

is zero at $x=a$.

When we apply the relation (3.1) in the elementary formulas for differentiable functions, we can imagine some deep results. For example, in the simple formula

$$(u+v)' = u' + v',$$

we have the result

$$\frac{u(x)+v(x)}{x-a}|_{x=a} = \frac{u(x)}{x-a}|_{x=a} + \frac{v(x)}{x-a}|_{x=a},$$

that is not trivial in our definition.

In the following well-known formulas, we have some deep meanings on the division by zero calculus.

$$(uv)' = u'v + uv',$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

and the famous laws

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

and

$$\frac{dy}{dx} \cdot \frac{dx}{dy} = 1.$$

Note also the logarithm derivative, for $u > 0$

$$(u^v)' = u^v \left(v' \log u + v\frac{u'}{u}\right).$$

4 l'Hôpital's theorem versions

As a version of l'Hôpital's theorem, we obtain the following idea.

We assume that f and g are differentiable of n orders at $x = a$ and we assume that

$$g(a) = g'(a) = ... = g^{(n-1)}(a) = 0, g^{(n)}(a) \neq 0.$$

Then, we **define**

$$\frac{f(x)}{g(x)}|_{x=a} := \frac{f(x)/(x-a)^n|_{x=a}}{g(x)/(x-a)^n|_{x=a}} = \frac{f^{(n)}(a)}{g^{(n)}(a)}.$$

The function

$$\frac{f(x)}{g(x)}$$

is not defined at $x = a$ in the usual sense, because $g(a) = 0$. The denominator $g(x)$ has not to be zero. The initial non-vanishing value is $g^{(n)}(a)$. Therefore, in order to catch the value we consider

$$\frac{g(x)}{(x-a)^n}|_{x=a}.$$

Therefore, in a natural sense, we can obtain the above desired definition. We gave the natural reason. This is our interpretation for the l'Hôpital's theorem. We gave the definition of $\frac{f(x)}{g(x)}|_{x=a}$ in the case $g(a) = 0$.

In this idea, we obtain

$$\frac{1}{x}|_{x=0} = \frac{1'}{x'}|_{x=0} = \frac{0}{1} = 0.$$

5 Remarks

The representation of the higher order differential coefficients $f^{(n)}(a)$ is very simple and, for example, for the Taylor expansion we have the beautiful representation

$$f(a) = \sum_{n=0}^{\infty} \frac{f(x)}{(x-a)^n}|_{x=a} \cdot (x-a)^n.$$

Further

$$\frac{f(x)}{(x-a)^2}|_{x=a} = \frac{f''(a)}{2}$$
$$= \lim_{x \to 0} \frac{f(a+x) + f(a-x) - 2f(a)}{2x^2}.$$

Of course, our definition of

$$\frac{f(x)}{g(x)}|_{x=a}$$

and the limit

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

are, in general, different.

For example,

$$\frac{1 + (x-1)\log x}{\sin^2(x-1)}|_{x=1} = 1,$$

but

$$\lim_{x \to 1} \frac{1 + (x-1)\log x}{\sin^2(x-1)} = \infty.$$

Furthermore, note that

$$\lim_{x \to 0} \frac{x + 2x^2 \sin(1/x)}{x} = 1,$$

but

$$\frac{x + 2x^2 \sin(1/x)}{x}|_{x=0}$$

is not defined in our new sense. However, with the division by zero calculus for analytic functions in the sense of Section 1, we have the value 3.

However, for some complicated statements of l'Hôpital's theorem, we can apply our definition and method for many cases in order to derive the same results. In our method it is necessary to consider only the point at $x = a$, meanwhile, in the l'Hôpital's theorem, we consider the behavior of the function on a neighborhood of the point $x = a$ except the point $x = a$. They are different concepts.

References

[1] C. B. Boyer, An early reference to division by zero, The Journal of the American Mathematical Monthly, **50** (1943), (8), 487- 491. Retrieved March 6, 2018, from the JSTOR database.

[2] W. W. Däumler, H. Okumura, V. V. Puha and S. Saitoh, Horn Torus Models for the Riemann Sphere and Division by Zero, viXra:1902.0223 submitted on 2019-02-12 18:39:18.

[3] R. Kaplan, THE NOTHING THAT IS A Natural History of Zero, OXFORD UNIVERSITY PRESS (1999).

[4] M. Kuroda, H. Michiwaki, S. Saitoh and M. Yamane, New meanings of the division by zero and interpretations on $100/0 = 0$ and on $0/0 = 0$, Int. J. Appl. Math. **27** (2014), no 2, pp. 191-198, DOI: 10.12732/ijam.v27i2.9.

[5] T. Matsuura and S. Saitoh, Matrices and division by zero $z/0 = 0$, Advances in Linear Algebra & Matrix Theory, **6**(2016), 51-58 Published Online June 2016 in SciRes. http://www.scirp.org/journal/alamt http://dx.doi.org/10.4236/alamt.2016.62007.

[6] T. Matsuura, H. Michiwaki and S. Saitoh, $\log 0 = \log \infty = 0$ and applications, Differential and Difference Equations with Applications, Springer Proceedings in Mathematics & Statistics, **230** (2018), 293-305.

[7] H. Michiwaki, S. Saitoh and M.Yamada, Reality of the division by zero $z/0 = 0$, IJAPM International J. of Applied Physics and Math. **6**(2015), 1–8. http://www.ijapm.org/show-63-504-1.html

[8] H. Michiwaki, H. Okumura and S. Saitoh, Division by Zero $z/0 = 0$ in Euclidean Spaces, International Journal of Mathematics and Computation, **2**8(2017); Issue 1, 1-16.

[9] H. Okumura, To Divide by Zero is to Multiply by Zero, viXra:1811.0132., (2018).

[10] H. Okumura, A Chain of Circles Touching a Circle and Its Tangent and Division by Zero, viXra:2001.0034 submitted on 2020-01-03 01:08:58.

[11] H. Okumura, S. Saitoh and T. Matsuura, Relations of 0 and ∞, Journal of Technology and Social Science (JTSS), **1**(2017), 70-77.

[12] H. Okumura and S. Saitoh, The Descartes circles theorem and division by zero calculus, https://arxiv.org/abs/1711.04961 (2017.11.14).

[13] H. Okumura and S. Saitoh, Harmonic Mean and Division by Zero, Dedicated to Professor Josip Pečarić on the occasion of his 70th birthday, Forum Geometricorum, **18** (2018), 155—159.

[14] H. Okumura and S. Saitoh, Remarks for The Twin Circles of Archimedes in a Skewed Arbelos by H. Okumura and M. Watanabe, Forum Geometricorum, **18**(2018), 97-100.

[15] H. Okumura and S. Saitoh, Applications of the division by zero calculus to Wasan geometry, GLOBAL JOURNAL OF ADVANCED RESEARCH ON CLASSICAL AND MODERN GEOMETRIES" (GJARCMG), **7**(2018), 2, 44–49.

[16] H. Okumura and S. Saitoh, Wasan Geometry and Division by Zero Calculus, Sangaku Journal of Mathematics (SJM), **2** (2018), 57–73.

[17] H. Okumura and S. Saitoh, Values of the Riemann Zeta Function by Means of Division by Zero Calculus viXra:1907.0437 submitted on 2019-07-23 20:48:54.

[18] S. Pinelas and S. Saitoh, Division by zero calculus and differential equations. Differential and Difference Equations with Applications. Springer Proceedings in Mathematics & Statistics, **230** (2018), 399-418.

[19] H. G. Romig, Discussions: Early History of Division by Zero, American Mathematical Monthly, **3**1, No. 8. (Oct., 1924), 387-389.

[20] S. Saitoh, A reproducing kernel theory with some general applications, Qian,T./Rodino,L.(eds.): Mathematical Analysis, Probability and Applications - Plenary Lectures: Isaac 2015, Macau, China, Springer Proceedings in Mathematics and Statistics, **177**(2016), 151-182.

[21] S. Saitoh, Mysterious Properties of the Point at Infinity, arXiv:1712.09467 [math.GM](2017.12.17).

[22] S. Saitoh, We Can Divide the Numbers and Analytic Functions by Zero with a Natural Sense, viXra:1902.0058 submitted on 2019-02-03 22:47:53.

[23] S. Saitoh, Zero and Infinity; Their Interrelation by Means of Division by Zero, viXra:1902.0240 submitted on 2019-02-13 22:57:25.

[24] S. Saitoh, Division by Zero Calculus in Trigonometric Functions, viXra:1903.0566 submitted on 2019-03-31 15:59:03.

[25] S. Saitoh, What Was Division by Zero?; Division by Zero Calculus and New World, viXra:1904.0408 submitted on 2019-04-22 00:32:30.

[26] S. Saitoh, Fundamental of Mathematics; Division by Zero Calculus and a New Axiom, viXra:1908.0100 submitted on 2019-08-06 20:03:01.

[27] S. Saitoh, Essential Problems on the Origins of Mathematics; Division by Zero Calculus and New World Authors, viXra:1912.0300 submitted on 2019-12-16 18:37:53.

[28] S.K.S. Sen and R. P. Agarwal, ZERO A Landmark Discovery, the Dreadful Volid, and the Unitimate Mind, ELSEVIER (2016).

[29] F. Soddy, The Kiss Precise, Nature **137**(1936), 1021. doi:10.1038/1371021a0.

[30] E. Sondheimer and A. Rogerson, NUMBERS AND INFINITY A Historical Account of Mathematical Concepts, Dover (2006) unabridged republication of the published by Cambridge University Press, Cambridge (1981).

[31] S.-E. Takahasi, M. Tsukada and Y. Kobayashi, Classification of continuous fractional binary operations on the real and complex fields, Tokyo Journal of Mathematics, **38**(2015), no. 2, 369-380.