The illuminated area of the Moon exceeds half its surface

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Abstract. We show that the extent of the illuminated area of the Moon is greater than 50% of its surface; that is, a part of the lunar hemisphere opposite the Sun is illuminated, which occurs because the Sun is larger than the Moon.

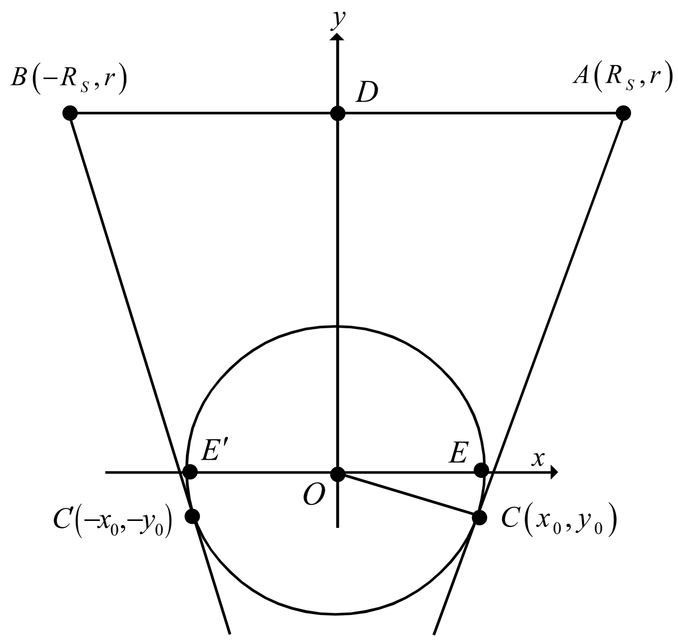
1. Illuminated area of the Moon in the hemisphere opposite the Sun

The size Sun is much larger than of the Moon; therefore, we conclude that it is illuminated more than half of the lunar sphere. The illuminated part of the Moon will change as a result of the rotation on its axis, but the extent of the area where the Sun's rays arrive always exceeds 50% of the lunar surface.

The size of the illuminated area of the Moon is independent of where the observer is. The apparent aspect Moon will change according to where it is observed, but this is independent of the extent of the illuminated area of the Moon, which is what interests, so we disregard the aspect that the Moon has for an observer.

In the reasoning we make below, we will assume that the Moon is a smooth sphere, and therefore we do not consider its surface irregularities, which also affect its illumination by the shadows cast by the lunar mountains.

In figure 1, the line *A-B* represents the diameter of the Sun, of radius R_S, and the circle is the



Drawing 1.-The *A-B* line represents the Sun, whose center is the point *D*. R_S is the solar radius. *r* the distance between the centers of the Sun and the Moon. The center circumference *O* is the Moon, which has a radius R_L. We define a coordinate system with origin in O, with the axis *y* directed towards the Sun. Points *C* and C' are the most extreme points illuminated by the Sun and which are in the lunar hemisphere opposite to the Sun.

Moon that we assume a sphere of radius R_L. The distance from the center of the Moon to the center of the Sun is *r*. Note that the following reasoning, nothing involves the axis rotation of Moon and its inclination with respect to the ecliptic.

As shown in the drawing, the rays that come from the solar edges reach the lunar hemisphere opposite the Sun, which shows that in addition to the hemisphere facing the Sun, a small part of the hemisphere opposite the Sun is also illuminated, which are the points between *C* and *E* or between C' and E'. From the same drawing, it is verified that from these points, only part of the Sun is observed. This phenomenon occurs at all points of the lunar edge and with the same intensity, that is, if the Moon is observed from a position located on the *O-D* line that joins centers of the Sun and the Moon, a luminous disk would be theoretically observed at the edge and of the same thickness in all those points.

Keep in mind that as a result of the variable distance between the Moon and the Sun, the illuminated area will not always have the same size, but will vary slightly. Since the solar disk is partially visible on the lighted edge of which we are talking, these areas correspond to the places where the Sun is rising or setting. As the Moon rotates on its axis, the zone illuminated it changes its place with respect to the Moon but will remain unchanged with respect to the Sun.

2. Extreme points of the Moon illuminated by the Sun

We try to find out the proportion of the Moon illuminated by the Sun, which, as we have said, exceeds 50%. For which it is previously necessary to determine the extreme points of the lunar hemisphere opposite to the Sun where solar rays arrive, that is, points *C* and C' of drawing 1. In this drawing we have drawn a Cartesian coordinate system with origin in *O*, with the axis *y* coinciding with the line that joins the centers of the Sun and the Moon; the rays that come out of the edges of the Sun illuminate the most extreme parts of the Moon points *C* and C'.

The equation of the lunar circumference is

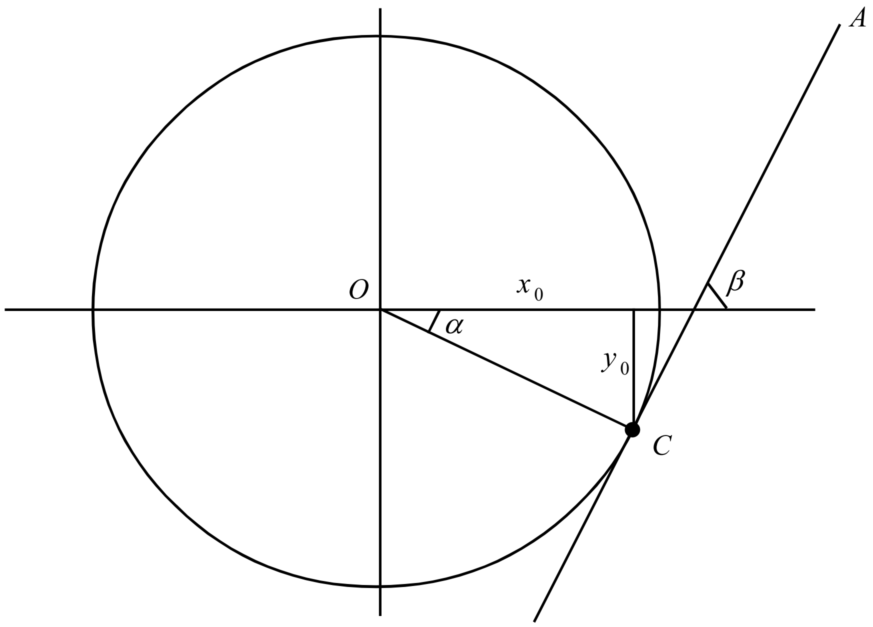
$$x^2 + y^2 = R_L^2 .$$

Drawing 2 shows that

$$\alpha + \beta + 90 = 180 \quad \Rightarrow \quad \alpha = 90 - \beta \quad \Rightarrow \tan\beta = \frac{1}{\tan\alpha},$$

the slope m_0 of the line tangent to the circumference at point *C* is

$$m_0 = \tan\beta = -\frac{x_0}{y_0} = \frac{-x_0}{\sqrt{R_L^2 - x_0^2}}$$



Drawing 2.- The circumference is the Moon, and the line *A-C* is the tangent to the lunar surface at point *C*.

The minus sign is because, according to the drawing, y_0 has a negative sign, but the slope of the line is positive. Since the line passes through the point (R_S, r) the equation of the line that connects *A* with *C* is

$$y = m_0 (x - R_S) + r.$$

This line must pass through point *C* of coordinates (x_0, y_0), therefore

$$y_0 = -\frac{x_0}{y_0}(x_0 - R_S) + r \quad \Rightarrow \quad y_0^2 = -x_0^2 + x_0 R_S + r y_0 \tag{1}$$

as the point (x_0, y_0) belongs to the circle is fulfilled

$$y_0^2 = R_L^2 - x_0^2,$$

putting this result in (1)

$$R_L^2 = x_0 R_S + r\, y_0 \tag{2}$$

which is solved by successive approximations. Initially, we assume $x_{0-1} = R_L$, then

$$y_{0-1} = \frac{R_L^2 - R_L R_S}{r}$$

using the mean numerical values

$$R_L = 1{,}738 \cdot 10^6; \quad R_S = 6{,}96 \cdot 10^8; \quad r = 149{,}6 \cdot 10^9$$

expressed in meters; we find in first approximation

$$x_{0-1} = 1{,}738 \cdot 10^6; \quad y_{0-1} = -8065{,}69$$

also in meters. For the second approach, we use

$$x_{0-2} = \sqrt{R_L^2 - y_{0-1}^2} = 1{,}737\ 981 \cdot 10^6\ m \tag{3}$$

and from (2) we get

$$y_{0-2} = -8065{,}60\ m \tag{4}$$

results that we consider useful.

3. Extension of the illuminated area of the lunar hemisphere opposite the Sun

The results (3) and (4) mean that in addition to the hemisphere of the Moon facing the Sun, there is a small strip illuminated in the opposite hemisphere, it is a circular ring of 19 meter ($= R_L - x_0$) as seen by an observer located looking at the part of the Moon opposite the Sun. Therefore a circular strip of approximately 8065 meters ($\approx y_0$) measured on the lunar surface is illuminated.

This area is illuminated only by a portion solar disk; that is, it is in the area where the sunrise is beginning or sunset ending. Also, the sun's rays that illuminate this area arrive with high inclination, which further decreases the illumination of this ring, to which it should be added that the irregularities of the lunar surface with its shadows will further diminish the illumination of this area.

Observed this strip from the Earth and on the axis that joins the centers of Sun and Moon, the width of the strip has an angular dimension of 0.01 arc seconds. Together with the low intensity of the reflected light, as we have said before, makes this strip unobservable at a distance considered.

4. The proportion of the illuminated area of the Moon

The surface area of the Moon is $4\pi R^2$, so the illuminated area facing the Sun is $2\pi R^2$; to calculate the total illuminated area, we must add the area of the illuminated area of the hemisphere opposite the Sun.

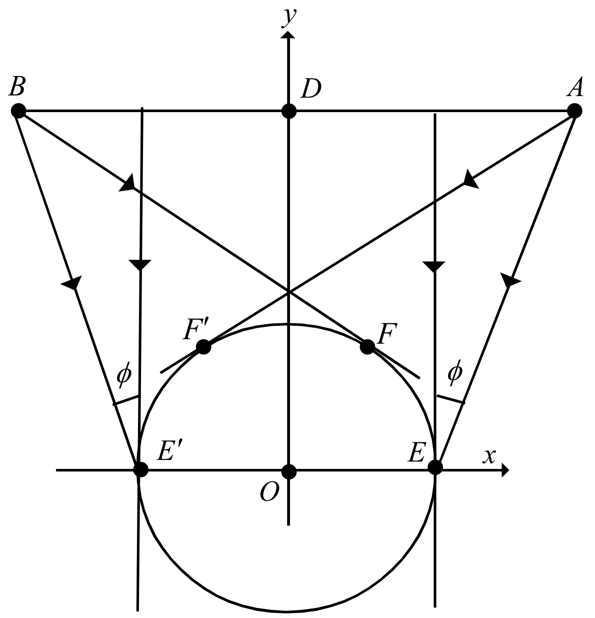
We want to find the lunar area between the lunar circunference E-E' and the parallel circumference that passes through C-C' (see drawing 1). The illuminated spherical zone of the hemisphere opposite the Sun is

$$2\pi R_L y_0,$$

then the total illuminated area of the Moon is

$$A = 2\pi R_L^2 + 2\pi R_L y_0$$

which corresponds to a fraction of the whole of the lunar surface given by the parameter *k*



Drawing 3.- Line *A-B* represents the Sun whose center is *D*. Points *E* and E' are at the junction of the two lunar hemispheres relative to the Sun. The points between *F* and *E* and between F' and E' observe part of the Sun, and the points located on the lunar surface between points F' and *F* are those from which the entire solar disk is visible.

$$k = \frac{2\pi R_L^2 + 2\pi R_L y_0}{4\pi R_L^2} = \frac{1}{2} + \frac{y_0}{2R_L}$$

for the numerical values that we have used in the previous section we obtain

$$k = 50.23\%$$

or 0.46% of the lunar hemisphere opposite the Sun.

5. The inclination of the sun's rays

As we have said, the solar rays that reach the illuminated strip that we are considering, arrive with high inclination. At the extreme points of this zone (that is, points *C* and C') the solar rays arrive tangentially, that is to say with a zero inclination, while the rays that leave the edge of the Sun reach the points E and E' of the Moon with an inclination f (see drawing 3), which has the mean value

$$\phi = \tan^{-1} \frac{R_S - R_L}{r} = 15'.95$$

practically the same as the apparent semi-axis of the Sun seen from the Moon. Therefore, from points *E* and E', almost half of the Sun is observed.

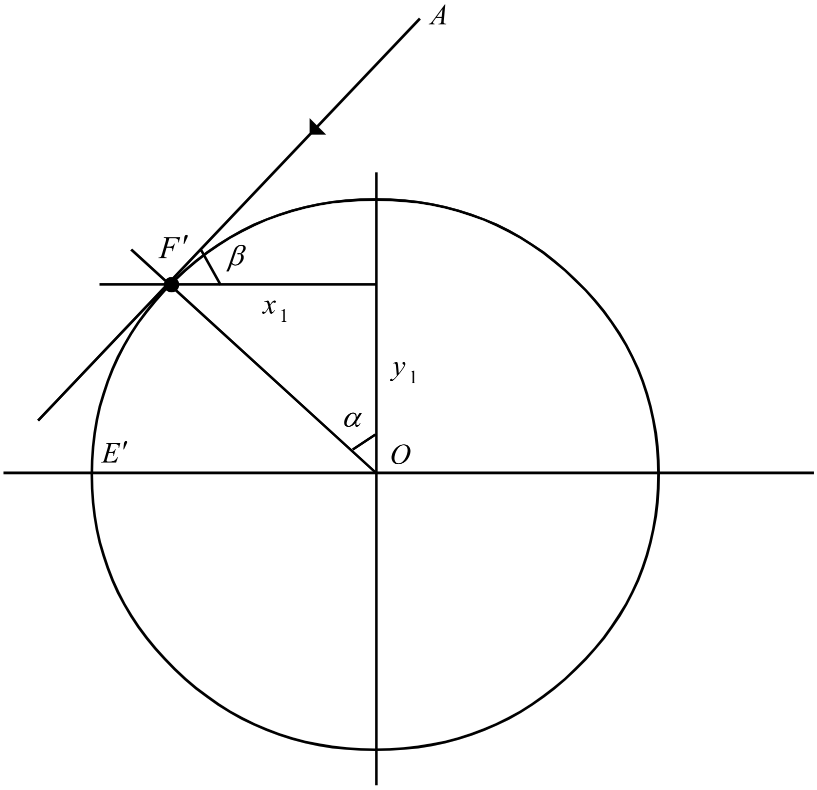
6. Extension of the area of the Moon where the sunrise or solar sunset is being observed

The points between *E* and *F* and between E' and F' of drawing 3, which are in the hemisphere facing the Sun, receive rays from only part of the solar disk. At the points between *F* and F' come rays of the entire Sun. Therefore, the area between points *E* and *F* and between E' and F' is part of the Moon where the sunrise or sunset Sun is being observed; that is, Sun is partially seen.

We want to find the distance between points *E* and *F* (or between E' and F') to know the extent of the area of the Moon where a sunrise or the sunset of the Sun is observed at the same time.

From drawing 4, we obtain that $\alpha = \beta$ and therefore the slope of the line *A*-F' is

$$m_1 = \tan\beta = \tan\alpha = x_1 / y_1$$



Drawing 4.- The inclination of ray that comes from *A* and is tangent to point F' of Moon has an inclination *β*. Its tangent is the slope of the line *A-F* that has a positive value.

x_1, y_1, and the slope of the line are positive. Since the line passes through point *A*, then its equation is

$$y = m_1 (x - R_s) + r$$

as point (x_1, y_1) belongs to the line

$$y_1 = m_1 (x_1 - R_S) + r \quad \Rightarrow \quad 2x_1^2 - x_1 R_S + y_1 r - R_L^2 = 0 \qquad (5)$$

where we have taken into account that the point (x_1, y_1) is on the lunar surface and therefore $y_1 = \sqrt{R_L^2 - x_1^2}$.

We solve equation (5) by successive approximations. First, we take $x_{1-1} = R_L$, then

$$y_{1-1} = \frac{R_L R_S - R_L^2}{r} = 8065,69\, m,$$

for the next approach we take

$$x_{1-2} = \sqrt{R_L^2 - y_{1-1}^2} = 1.737981 \cdot 10^6\, m \qquad (6)$$

and from (5) we get the new approach

$$y_{1-2} = \frac{R_L^2 - 2x_{1-2}^2 + x_{1-2} R_S}{r} = 8065,10\, m, \qquad (7)$$

as a new approach gives us practically the same values, we can consider that the desired point has coordinates (6) and (7).

The distance between points E' and F' is therefore

$$8065.60 + 8065.60 = 16131.20\ m$$

Note that this distance is the same on the entire edge of the Moon. However, the duration of sunset and sunrise (that is, the time elapsed between the passage through the lunar horizon of the two solar limbs), will be different for each point of the Moon and will depend on the selenocentric geographical latitude and the inclination of the axis of Moon rotation relative to the ecliptic.

7. Width of the illuminated area as seen from Earth

Observed from Earth, the Moon is a circle, which is partially illuminated. The lunar phase f is defined as the ratio between the illuminated area seen from the Earth and the total area of the Moon facing the Earth, understanding the Moon as a flat surface, that is

$$f = \frac{A}{\pi R_L^2}.$$

The phase angle χ is the selenocentric angle between the centers of the Sun and the Earth. Depending on this angle the phase is

$$f = \frac{1}{2}\left(1 + \cos\chi\right)$$

and the maximum width of the illuminated area of the Moon is

$$\omega_{\max} = R_L\left(1 + \cos\chi\right) = 2R_L f. \tag{8}$$

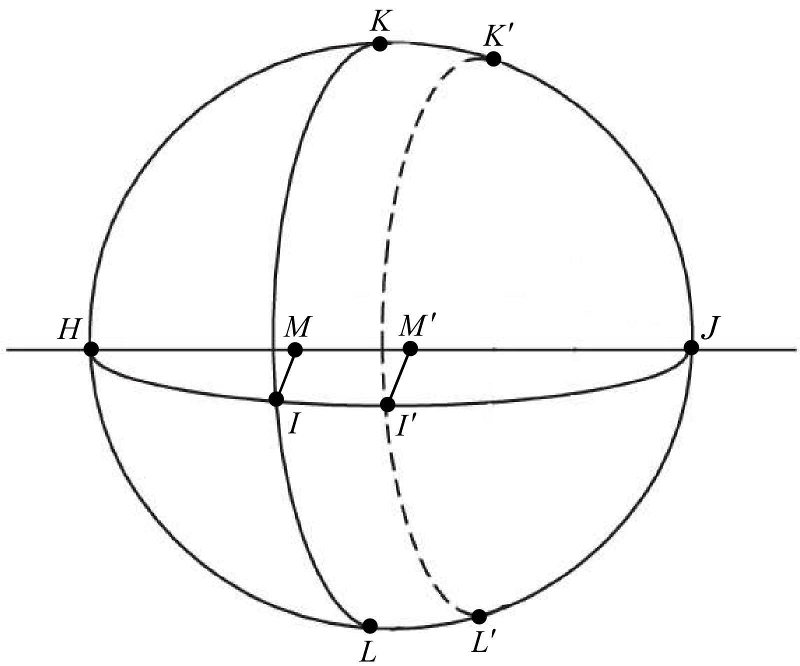
The above formulas are correct in the assumption that the Sun only illuminates half of the lunar sphere. But as we have seen, an extension somewhat larger than half of the Moon is illuminated. Now we determine the increase in the Moon phase f' and the maximum width of the illuminated area $\omega'_{\max}$, as a result of the increase in the illuminated area

Drawing 5 shows the Moon as seen from Earth. If only half of the Moon is illuminated by the Sun, then the part *HKIL* would be seen from Earth. But if we assume that part of the Sun's rays reaches the opposite hemisphere, then the zone $HK'I'L'$ will be seen.

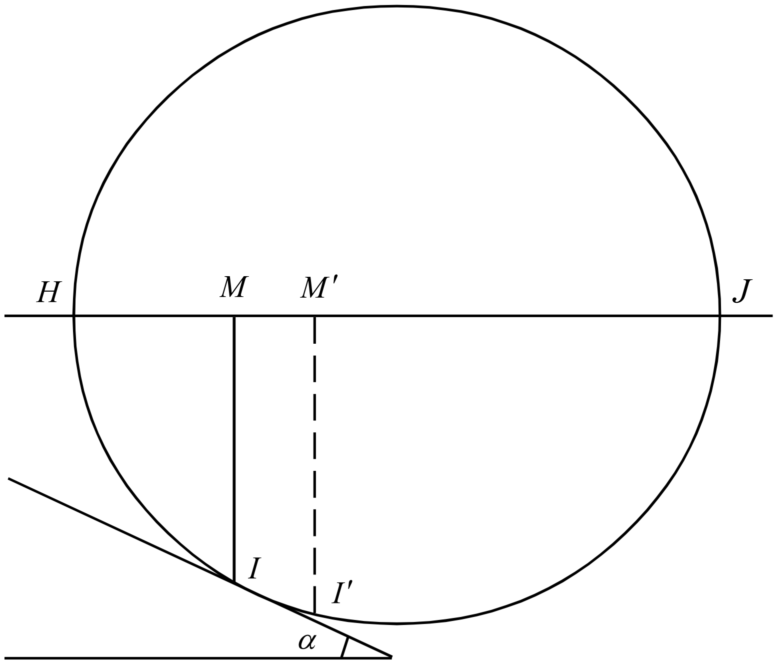
The distance *HM* is the maximum width of the illuminated area $\omega_{\max}$ calculated by (8). To calculate $\omega'_{\max}$, we have to find the distance MM'.

In drawing 6, we represent the circumference $HII'J$. The distance II' is 8065.60 meters calculated in (4). From this data, the length MM' must be estimated, which is the increase in the maximum lunar width of the decreasing or increasing phase of the Moon.

Since distance II' is small in comparison with the lunar radius, we can consider it as a straight line, then (see drawing 6)



Drawing 5.- The drawing represents the Moon seen from Earth. If only half of the Moon is illuminated, *HKL* is the bright area of the Moon. But if we assume that the sun's rays reach the opposite hemisphere of the Moon, then $HK'I'L'$ 'is the illuminated area seen from Earth. Point *M* is the projection of point I on the lunar diameter. HM' is the largest width of the Moon seen from Earth. The theory of the lunar phases calculates *HM*, and to this value, we add the distance MM'.



Drawing 6.- The circle of the drawing is the equatorial plane of the Moon. HM' is the maximum lunar width, as observed from Earth. To determine the distance between *M* and M' we use a right triangle with II' as the hypotenuse and MM' as one of the legs. The angle a is the slope of the tangent to the circumference at the point *I*.

$$\cos\alpha = \frac{MM'}{II'}$$

α is the slope of the tangent to the circumference at the point *I*, therefore

$$\tan\alpha = y'(I) = \left(\frac{x}{\sqrt{R_L^2 - x^2}}\right)_I = \frac{R_L - \omega_{\max}}{\sqrt{2R_L\omega_{\max} - \omega_{\max}^2}}$$

we have used the equation of the circumference $y = \sqrt{R_L^2 - x^2}$ to find the slope, that is the tangent of α, then

$$\cos\alpha = \frac{1}{\sqrt{1+\tan^2\alpha}} = \frac{\sqrt{2R_L\omega_{\max} - \omega_{\max}^2}}{R_L} \quad \Rightarrow \quad MM' = II'\frac{\sqrt{2R_L\omega_{\max} - \omega_{\max}^2}}{R_L},$$

and finally, we calculate

$$\omega'_{\max} = \omega_{\max} + II'\frac{\sqrt{2R_L\omega_{\max} - \omega_{\max}^2}}{R_L} \tag{9}$$

by (8) we can express (9) in function on the phase angle.

The minimum increase occurs on the new Moon, when $\omega_{\max} = 0$, then $\omega'_{\max} = \omega_{\max}$ While the maximum increase in the width of the visible area of the Moon occurs in the quarters, when $\omega_{\max} = R_L$ and therefore $\omega'_{\max} = \omega_{\max} + II'$.With the numerical data, we find that in the most advantageous situation, the increase in width corresponds to 0.46% of the lunar diameter, which corresponds to about eight arc seconds seen from Earth. This is an angle, in theory, measurable from Earth. By lunar orography, the internal terminator of the Moon is very irregular; in addition, the solar rays that reach the area closest to the internal terminator have a high inclination, reason because they scarcely illuminate the lunar surface.