Letter N°5: Integrals

Edgar Valdebenito

September 2 , 2019

ABSTRACT. We give some elementary integrals.

I. Introduction.

A. Recall that

$$\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 3.14159265 \ldots$$

II. Integrals 1

A. Entry 1.

$$\frac{\pi}{2} = \int_0^1 \left(\frac{35 - 232x^2 + 744x^4 - 1024x^6 + 512x^8}{49 - 416x^2 + 1440x^4 - 2048x^6 + 1024x^8} \right) \frac{1}{\sqrt{1-x^2}} dx$$

B. Entry 2.

$$\frac{\pi}{6} = \int_0^1 \left(\frac{7 - 16x^2 + 16x^4}{49 - 416x^2 + 1440x^4 - 2048x^6 + 1024x^8} \right) \frac{1}{\sqrt{1-x^2}} dx$$

C. Entry 3.

$$0 = \int_0^1 \frac{52\,x - 360\,x^3 + 768\,x^5 - 512\,x^7}{49 - 416\,x^2 + 1440\,x^4 - 2048\,x^6 + 1024\,x^8}\,dx$$

III. Integrals 2

A. Entry 4.

$$0 = \int_0^1 \frac{52 - 360\,x + 768\,x^2 - 512\,x^3}{49 - 416\,x + 1440\,x^2 - 2048\,x^3 + 1024\,x^4}\,dx$$

B. Entry 5.

$$\int_0^{\frac{1}{4}\sqrt{8-2\sqrt{3}}} \frac{52\,x - 360\,x^3 + 768\,x^5 - 512\,x^7}{49 - 416\,x^2 + 1440\,x^4 - 2048\,x^6 + 1024\,x^8}\,dx = \frac{\ln 7}{8} - \frac{3\ln 3}{16} + \frac{\ln 2}{8}$$

C. Entry 6.

$$\int_{\frac{1}{4}\sqrt{8-2\sqrt{3}}}^{\frac{1}{2}\sqrt{2}} \frac{52\,x - 360\,x^3 + 768\,x^5 - 512\,x^7}{49 - 416\,x^2 + 1440\,x^4 - 2048\,x^6 + 1024\,x^8}\,dx = \frac{\ln 3}{16} - \frac{\ln 2}{8}$$

D. Entry 7.

$$\int_{\frac{1}{2}\sqrt{2}}^{\frac{1}{4}\sqrt{8+2\sqrt{3}}} \frac{52\,x - 360\,x^3 + 768\,x^5 - 512\,x^7}{49 - 416\,x^2 + 1440\,x^4 - 2048\,x^6 + 1024\,x^8}\,dx = \frac{\ln 2}{8} - \frac{\ln 3}{16}$$

E. Entry 8.

$$\int_{\frac{1}{4}\sqrt{8+2\sqrt{3}}}^{1} \frac{52\,x - 360\,x^3 + 768\,x^5 - 512\,x^7}{49 - 416\,x^2 + 1440\,x^4 - 2048\,x^6 + 1024\,x^8}\,dx = \frac{3\ln 3}{16} - \frac{\ln 2}{8} - \frac{\ln 7}{8}$$

IV. Integrals 3

A. Entry 9.

$$\frac{\pi}{4} = \int_0^{\frac{1}{\sqrt{2}}} \left(\frac{35 - 232\,x^2 + 744\,x^4 - 1024\,x^6 + 512\,x^8}{49 - 416\,x^2 + 1440\,x^4 - 2048\,x^6 + 1024\,x^8} \right) \frac{1}{\sqrt{1 - x^2}}\, dx$$

B. Entry 10.

$$\frac{7\,\pi}{24} + \frac{1}{8} \tan^{-1}\left(3\,\sqrt{3}\right) =$$

$$\int_0^{\frac{\sqrt{3}}{2}} \left(\frac{35 - 232\,x^2 + 744\,x^4 - 1024\,x^6 + 512\,x^8}{49 - 416\,x^2 + 1440\,x^4 - 2048\,x^6 + 1024\,x^8} \right) \frac{1}{\sqrt{1 - x^2}}\, dx$$

C. Entry 11.

$$\frac{\pi}{8} + \frac{1}{2} \sin^{-1}\left(\frac{1}{\sqrt{3}} \right) - \frac{1}{8} \tan^{-1}\left(\frac{12\left(40\,\sqrt{3} - 49\,\sqrt{2}\right)}{17\left(20\,\sqrt{6} - 49\right)} \right) =$$

$$\int_0^{\frac{1}{\sqrt{3}}} \left(\frac{35 - 232\,x^2 + 744\,x^4 - 1024\,x^6 + 512\,x^8}{49 - 416\,x^2 + 1440\,x^4 - 2048\,x^6 + 1024\,x^8} \right) \frac{1}{\sqrt{1 - x^2}}\, dx$$

V. Integrals 4

A. Entry 12.

$$\frac{\pi}{12} = \int_0^{\frac{1}{\sqrt{2}}} \left(\frac{7 - 16\,x^2 + 16\,x^4}{49 - 416\,x^2 + 1440\,x^4 - 2048\,x^6 + 1024\,x^8} \right) \frac{1}{\sqrt{1 - x^2}}\, dx$$

B. Entry 13.

$$\frac{\pi}{12} + \frac{1}{12}\tan^{-1}\left(3\sqrt{3}\right) =$$

$$\int_0^{\frac{\sqrt{3}}{2}} \left(\frac{7 - 16x^2 + 16x^4}{49 - 416x^2 + 1440x^4 - 2048x^6 + 1024x^8}\right)\frac{1}{\sqrt{1 - x^2}}\, dx$$

VI. Integrals 5

A. Entry 14.

$$I = \int_0^1 \frac{52 + 768x^2}{49 - 416x + 1440x^2 - 2048x^3 + 1024x^4}\, dx$$

$$I = \int_0^1 \frac{360x + 512x^3}{49 - 416x + 1440x^2 - 2048x^3 + 1024x^4}\, dx$$

$$I = \frac{79\sqrt{6}}{24}\pi + \frac{79\sqrt{6}}{18}\tan^{-1}\left(\left(5 - 2\sqrt{6}\right)^2\right) + \frac{43\sqrt{2}}{36}\ln\left(\frac{193 + 132\sqrt{2}}{49}\right)$$

VII. References

A. Arndt, J., and Haenel, C.: π unleashed. Springer - Verlag, 2001.

B. Boros, G., and Moll, V.H.: Irresistible Integrals. Cambridge University Press, 2004.

C. Gradshteyn, I.S., and Ryzhik, I.M.: Table of Integrals, Series and Products. 7th ed. edited by Alan Jeffrey and Daniel Zwillinger, Academic Press, 2007.