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New Types of Permuting n-Derivations with Their Applications on Associative Rings

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Abstract: In this article, we introduce new generators of a permuting n-derivations to improve and increase the action of usual derivation. We produce a permuting n-generalized semiderivation, a permuting n-semigeneralized semiderivation, a permuting n-antisemigeneralized semiderivation and a permuting skew n-antisemigeneralized semiderivation of non-empty rings with their applications. Actually, we study the behaviour of those types and present their results of semiprime ring R. Examples of various results have also been included. That is, many of the branches of science such as business, engineering and quantum physics, which used a derivation, have the opportunity to invest them in solving their problems.

Keywords: permuting semiderivation; permuting skew n-antisemigeneralized semiderivation; semiprime ring; weak zero-divisor; anticommutative ring; semicommutative ring

1. Introduction

Through the 20th century, noncommutative rings have only been issues of systematic study quite recently. Commutative rings, on the contrary, have seemed, though in a covered way, much before, and as with countless theories, it all comes back up to Fermat's Last Theorem. In 1847, the mathematician Lamé stated an optimal solution of Fermat's Last Theorem. In dissimilarity to commutative ring theory, which increases from quantity theory, non-commutative ring theory progresses from a notion of Hamilton. He attempted to release the complex numbers as a two-dimensional algebra over the real to a tri-dimensional algebra. Other natural noncommutative objects that grow are matrices. In 1850, they were presented by Cayley, together with their rules of addition and multiplication and, in 1870, Pierce noted that the now commonplace ring axioms held for square matrices.

However, the origin of commutativity theorems for rings could be traced to the paper of Wedderburn (1905) which was under-titled "a finite division ring is necessarily a field" in theTransaction American Mathematical society. The study of derivation was initiated during the 1950s and 1960s. Despite the concept of derivation in rings being quite old and playing a significant role in various branches of mathematics, it developed tremendously when, in 1957, Posner [1] founded two very striking results on derivations in prime rings. Additionally, there has been substantial interest in examining commutativity of rings, generally that of prime ring sand semiprime rings admitting suitably constrained the additive mappings a derivations. Over and above, Vukman [2,3] extended the above result for bi-derivations. Derivations in rings have been studied by several algebraists in various directions. It is very enjoyable and it is important that the analogous properties of derivation which is one of the requisite theory in analysis and applied mathematics are also satisfied in the ring theory.

Derivations of prime and semiprime rings were studied by several researchers? near-rings, BCI-algebras, lattices and various algebraic structures [4–9]. Multiderivations which are covering (e.g., biderivation, 3-derivation, or n-derivation, semiderivation and anti derivation in general) have been examined in (semi-) rings [2,10–14]. Some researchers have studied n-derivations, (n,m)-derivations and higher derivations on various algebraic structures, such as triangular rings, von Neumann algebras, lattice ordered rings and J-subspace lattice algebras [15–21].

In 1976, I. N. Herstein [22] depended on the composition of rings to find fundamental properties, where he established that, letting R be a ring in which, given $a, b \in R$, there exist integers $m = m(a, b), n = n(a, b)$ greater than or equal to 1 such that $a^m b^n = b^n a^m$. Then, the commutator ideal of R is nil. Particularly, if R has no non-zero nil ideals, then R must be commutative. As a matter of fact, the theorems, especially the commutativity case for rings and near-rings with their applications, have been discussed by a lot of researchers. The core of that research is to encourage the pursuit of research on applications of ring theory in diverse areas, such as to emphasise the interdisciplinary efforts involved in the pursuit of information technology and coding theory. All types of rings collected so far contribute to their application in diverse sections of mathematics as well as in data communications, computer science, digital computing and so forth.

During the years, a lot of work has been finished in this context by a several of authors in different aspects. In 1980, G.Maksa [23] pointed out to the concept of a symmetric biderivation on a ring R. The concept of additive commuting mappings is closely connected to the concept of biderivations. Every commuting additive mapping $d\colon R \to R$ gives rise to a biderivation on R. Linearizing $[d(x), x] = 0$, for all $x \in R$, we get $[d(x), y] = [x, d(y)]$, for all $x, y \in R$ and hence we note that the mapping $(x, y) \to [d(x), y]$ is a biderivation on R. Furthermore, all derivations appearing are inner. More details about biderivations and their applications can be found in Reference [24].

Indeed, in Reference [25] it was shown that every biderivation D of a noncommutative prime ring R is of the form $D(x, y) = \lambda[x, y]$,$x, y \in R$, where λ is a fixed element from the extended centroid of the ring R. Using certain functional identities, Brešar [24] extended this result to semiprime rings. Later, several authors have studied permuting 3-derivations in rings (see References [26–28], where several references can be found). Nevertheless, some authors have done a great deal of work concerning commutativity of prime and semiprime rings admitting various types of maps which are centralizing (resp.commuting) on some appropriate subsets of a ring R (see References [29–33]).

The concept of a permuting tri-derivation has been introduced Öztürk in Reference [34], while Ajda Fošner [35] presented the notion of symmetric skew 3-derivations and made some basic observations. Taking into account the definitions of skew derivations, we would like to point out that in Reference [35] Ajda Fošner introduced the notion of permuting skew 3-derivations in rings and extended the results given by Jung and Park [10] for $\triangle$ is a permuting skew 3-derivations and proved the commutativity of R under certain identities, where R is a 3!-torsion free prime ring and $\triangle \neq 0$. Meanwhile, Ajda Fošner [36] also extended the notion of permuting skew 3-derivation to permuting skew n-derivations in rings and proved several other results. In another contribution, the authors of Reference [37] have obtained the commutativity of a ring satisfying certain identities involving the trace of permuting n-derivation. Further, Mohammad Ashraf and Nazia Parveen [38] introduced the notion of permuting generalized $(\alpha, \beta) - n$-derivations and permuting α-left n-centralizers in rings and generalized the above results given by Ajda Fošner [36] in a different setting under some suitable torsion restrictions imposed on the underlying ring. Notwithstanding, several authors have done a great deal of work concerning commutativity of prime and semiprime rings admitting different kinds of maps which are skew derivations on some appropriate subsets of R, then Xiaowei Xu, Yang Liu and Wei Zhang [4] considered a skew n-derivation ($n \geq 3$) on a semiprime ring R must map into the center of R.

On the other hand, Badr Nejjar et al. [39] proved that n is a fixed positive integer and R is a $(n+1)$!-torsion free prime ring and J a non-zero Jordan ideal of R. If R admits a non-zero permuting generalized n-derivation Ω with associated n-derivation $\triangle$ such that the trace of Ω is centralizing on J. Then R is commutative, where an n-additive mapping $\Omega : R^n \longrightarrow R$ is called a generalized n-derivation of R with associated n-derivation $\triangle$ if $\Omega(x_1, x_2, ..., \grave{x}_i x_i, ..., x_n) = \Omega(x_1, x_2, ..., x_i, ..., x_n)\grave{x}_i +$

$x_i \triangle (x_1, x_2, ..., \grave{x}_i, ..., x_n)$ for all $\grave{x}_i, x_i \in R$, and additive subgroup J of R is said to be a Jordan ideal of R if $u \circ r \in J$, for all $u \in J$ and $r \in R$. In the near ring the subject studied by some authors like A. Ali et al. [40] assumed N to be a 3!-torsion free 3-prime near ring and U be a non-zero additive subgroup and a semigroup ideal of N. If $\triangle$ is a permuting 3-derivation with trace δ and $x \in N$ such that $x\delta(y) = 0$ for all $y \in U$, then either $x = 0$ or $\triangle = 0$ on U. In addition to that, Mohammad Ashraf et al. [41] came out with the notion of $(\sigma, \tau) - n$-derivation in near-ring N and investigated some properties involving $(\sigma, \tau) - n$-derivations of a prime near-ring N which force N to be a commutative ring. Also, Mohammad Ashraf and Mohammad Aslam Siddeeque [42] produced let N be a 3-prime near-ring admitting a non-zero generalized n-derivation F with associated n-derivation D of N. Then $F(U_1, U_2, ..., U_n) \neq \{0\}$, where $U_1, U_2, ..., U_n$ are non-zero semigroup left ideals or non-zero semigroup right ideals of N.

Other authors had tried of a permuting n-derivation of algebraic structure, for example D. Eremita [43] who discussed that if functional identities of degree 2 in triangular rings and obtained some descriptions of commuting maps and generalized inner biderivations of triangular rings. Yao Wang et al. [21] showed that if $A = Tri(A, M, B)$ be a triangular algebra. Suppose that there exists $m \in M$ such that $[m, [A, A]] = 0$. Set $\Psi_n(x_1, x_2, ..., x_n) = [x_1, [x_2, ..., [x_n, m]...]]$ for all $x_1, x_2, ..., x_n \in A$. Then Ψ_n is a permuting n-derivation of A.

However, Skosyrskii [44] who treated biderivations for different reasons, namely, in connection with noncommutative Jordan algebras.

K. H. Park [12] initiated the notion of an n-derivation and symmetric n-derivation, where n is any positive integer in rings and extended several known results, earlier in the setting of derivations in prime rings and semiprime rings as follows?suppose $n \geq 2$ be a fixed positive integer and $R^n = R \times R \times ... \times R$. A map $\Delta\colon R^n \to R$ is said to be symmetric (or permuting) if the equation $\Delta(x_1, x_2, ..., x_n) = \Delta(x_{\pi(1)}, x_{\pi(2)}, \cdots, x_{\pi(n)})$ holds for all $x_i \in R$ and for every permutation $\{\pi(1), \pi(2), ..., \pi(n)\}$. that is, for every permutation $\pi \in S_n$ (permutation on n symbol), where $R^n = R \times R \times R \times ... \times R$.

Let us consider the following map: let $n \geq 2$ be a fixed positive integer. An n-additive map $\Delta\colon R^n \to R$ (that is,, additive in each argument) will be called an n-derivation if the relations

$$\Delta(x_1\grave{x}_1, x_2, ..., x_n) = \Delta(x_1, x_2, ..., x_n)\grave{x}_1 + x_1\Delta(x_1\grave{x}_1, x_2, ..., x_n),$$

$$\Delta(x_1, x_2\grave{x}_2, ..., x_n) = \Delta(x_1, x_2, ..., x_n)\grave{x}_2 + x_2\Delta(x_1, \grave{x}_2, ..., x_n), ...,$$

$$\Delta(x_1, x_2, ..., x_n\grave{x}_n) = \Delta(x_1, x_2, ..., x_n)\grave{x}_n + x_n\Delta(x_1, x_2, ..., \grave{x}_n)$$

are valid for all $x_i, \grave{x}_i \in R$.

Also, in the same Reference [12], a 1-derivation is a derivation and a 2-derivation is called a bi-derivation. As in the case of $n = 3$ we get the concept of tri-derivation. If Δ is symmetric, then the above equalities are equivalent to each other. Let $n \geq 2$ be a fixed positive integer and let a map $\delta\colon R \to R$ defined by $\delta(x) = \Delta(x_1, x_2, ..., x_n)$ for all $x \in R$, where $\Delta\colon R^n \to R$ is a symmetric map, be the trace of Δ. It is clear that, in the case when $\Delta\colon R^n \to R$ is a symmetric map which is also n-additive, the trace δ of Δ satiates the identity $\delta(x + y) = \delta(x) + \delta(y) + \sum_{r=1}^{n-1} \binom{n}{r} \Delta(x, x, ..., x, y, y..., y)$ for all $x, y \in R$ where y appears r times and x appears $n - r$ times.

Since we have $\Delta(0, x_2, ..., x_n) = \Delta(0 + 0, x_2, ..., x_n) = \Delta(0, x_2, ..., x_n) + \Delta(0, x_2, ..., x_n)$ for all $x_i \in R$, $i = 2, 3, ..., n$, we obtain $\Delta(0, x_2, ..., x_n) = 0$ for all $x_i \in R$, $i = 2, 3, ..., n$.

Hence, we get $0 = \Delta(0, x_2, ..., x_n) = \Delta(x_1 - x_1, x_2, ..., x_n) = \Delta(x_1, x_2, ..., x_n) + \Delta(-x_1, x_2, ..., x_n)$ and so we see that $\Delta(-x_1, x_2, ..., x_n) = -\Delta(x_1, x_2, ..., x_n)$ for all $x_i \in R$, $i = 1, 2, ..., n$. This tells us that δ is an odd function if n is odd and δ is an even function if n is even.

Yilmaz Çeven [45] issued the definition which generalizes the notions of derivation, biderivation and 3-derivation on lattices, where the map $\Delta\colon L^n \to L$ will be called an n-derivation if Δ is a derivation according to all components; that is,

$$\Delta(x_1 \wedge a, x_2, ..., x_n) = (\Delta(x_1, x_2, ..., x_n) \wedge a) \vee (x_1 \wedge \Delta(a, x_2, ..., x_n)$$
$$\Delta(x_1, x_2 \wedge a, x_3, ..., x_n) = (\Delta(x_1, ..., x_n) \wedge a) \vee (x_2 \wedge \Delta(x_1, x_2, x_3, ..., a))$$
$$\vdots$$
$$\Delta(x_1, ..., x_{n-1}, x_n \wedge a) = (\Delta(x_1, ..., x_n) \wedge a) \vee (x_n \wedge \Delta(x_1, ..., x_{n-1}, a))$$

are valid for all x_i and $a \in L$.

In Reference [46], Bell and Martindale have stated the following results. Specify $f \neq 0$ be a semiderivation of a prime ring R of characteristic not 2 with associated endomorphism g of R and $U \neq 0$ be an ideal of R. Suppose that $a \in R$ such that $af(U) = 0$. Then $a = 0$.

Recently, Emine Koç and Nadeem ur Rehman [32] studied symmetric n-derivations on prime or semiprime rings with non-zero ideals. They proved that if a symmetric skew n-derivation $\triangle\colon R^n \to R$ associated with an automorphis T satisfies any one of the conditions

(i) $\delta(x) = 0$,
(ii) $[\delta(x), T(x)] = 0$ for all $x \in U$,

where δ is the trace of Δ, then $\Delta = 0$.

Furthermore, Basudeb Dhara and Faiza Shujat [47], have obtained that let R be a $n!$-torsion free prime ring, I a nonzero ideal of R, α an automorphism of R and $D : R^n \longrightarrow R$ be a symmetric skew n-derivation associated with the automorphism α. If τ is the trace of D such that $\tau(I) = 0$, then $D(x_1, x_2, ..., x_n) = 0$ for all $x_1, x_2, ..., x_n \in R$.

Throughout this paper, R represents an associative ring always. Denote by $Z(R)$ the center of R. Let $x, y, z \in R$. We write the notation $[y, x]$ for the commutator $yx - xy$ (the Lie product) and $x \circ y$ for anti-commutator $xy + yx$ (the Jordan product) also make use of the identities $[xy, z] = [x, z]y + x[y, z]$ and $[x, yz] = [x, y]z + y[x, z]$. The ring R is called semiprime if R satisfies the relation $aRa = 0$ implies $a = 0$ and R is called prime if R satisfies $aRb = 0$ implies $a = 0$ or $b = 0$. The relation between the prime ring and semiprime ring said every prime ring is semiprime ring, but the converse is not true always. A map $d\colon R \to R$ is said to be commuting on R if d satisfies $[d(x), x] = 0$ holds for all $x \in R$. If $[d(x), x] \in Z(R)$ is fulfilled for all $x \in R$ then a map $d\colon R \to R$ is said centralizing on R. If the Leibniz's formula $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in R$ then an additive map $D : R \longrightarrow R$ is called a derivation.

The concept of a generalized derivation was introduced in Reference [48] as follows. An additive mapping $D : R \longrightarrow R$ is called a generalized derivation if there exists an additive mapping d on R such that $D(xy) = D(x)y + xd(y)$ for all $x, y \in R$. Besides derivations and generalized inner derivations this also generalizes the concept of left multipliers, that is, additive mappings satisfying $D(xy) = D(x)y$, for all $x, y \in R$.

The inner derivation is fundamental example of derivation, that is, mappings of the form $\delta_a(x) = ax - xa$ where a is a fixed element in R. Generally, the mappings of the form $D(x) = ax + xb$ (with $a, b \in R$ fixed elements) are called generalized inner derivations. The additive map $d : R \longrightarrow R$ into itself which satisfies the rule $d(xy) = d(x)y + \alpha(x)d(y)$ for all $x, y \in R$ named a skew derivation of R. If $\alpha = 1$ is the identity automorphism of R, then d is known as a derivation of R. If there exists a skew derivation d of R with associated automorphism α such that $D(xy) = D(x)y + \alpha(x)d(y)$ holds for all $x, y \in R$ then an additive mapping $D : R \longrightarrow R$ is said to be a (right) generalized skew-derivation of R.

In Reference [49], J. Bergen introduced the concept of semiderivation of a ring R as. An additive mapping d of a ring R into itself is called a semiderivation if there exists a function $g : R \longrightarrow R$ such that $d(xy) = d(x)g(y) + xd(y) = d(x)y + g(x)d(y)$ and $d(g(x)) = g(d(x))$ for all $x, y \in R$. For $g = 1$ a semiderivation is of course a derivation. In Reference [13], Mohammad Ashraf and Muzibur Rahman Mozumder generalized the concept of multiplicative (generalized)- derivation to multiplicative (generalized)- skew derivation. A mapping $D\colon R \to R$ (not necessarily additive) is called a multiplicative (generalized)-skew derivation if $D(xy) = D(x)y + \alpha(x)g(y) = D(x)\alpha(y) + xg(y))$ for all $x, y \in R$, where $g\colon R \to R$ is any mapping (not necessarily a skew derivation nor an additive map) and $\alpha\colon R \to R$ is an automorphism of R. Since the sum of two generalized derivations is a generalized derivation, every map of the form $D(x) = cx + d(x)$ is a generalized derivation, where c is a fixed element of R and d is a derivation of R. Furthermore, Brešar and Vukman [14] have introduced the notion of a reverse derivation (anti-derivation) as an additive mapping d from a ring R into itself satisfying $d(xy) = d(y)x + yd(x)$,for all $x, y \in R$. Obviously, if R is commutative, then both derivation and reverse derivation are the same. The generalized reverse(anti) derivations were defined by [50] Let R be a ring and let d be a reverse derivation of R. An additive mapping $F\colon R \to R$ is said to be a left generalized reverse derivation of R associated with d if $D(xy) = D(y)x + yd(x)$ for all $x, y \in R$. Also, the additive mapping D is said to be a right generalized reverse derivation associated with d if $D(xy) = d(y)x + yD(x)$ for all $x, y \in R$. However, there exists a controversial question here, which is whether we can find new generators of a permuting n-derivations. So, the answer to this question is affirmative where the aim of this paper is to introduce the new types of a permuting n-derivations for associative rings and to make some basic observations.

2. The Main Definitions

This section contains the main definitions which represent keystone of the sequel. Consider $n \geq 2$ be a fixed positive integer of the following definitions:

Definition 1. *An n-additive permuting mapping $\triangle\colon R^n \to R$ is said to be a permuting n-semiderivation associated with a function λ if*

$$\triangle(x_1, x_2, ..., \grave{x}_i x_i, ..., x_n) = \triangle(x_1, x_2, ..., \grave{x}_i, ..., x_n)\lambda(x_i) + \grave{x}_i d(x_1, x_2, ..., x_i, ..., x_n)$$

$$= \lambda(x_1, x_2, ..., \grave{x}_i, ..., x_n) \triangle (x_i) + d(x_i)(x_1, x_2, ..., x_i, ..., x_n),$$

for all $\grave{x}_i, x_i \in R$, $i = 1, 2, .., n$ such that d is a permuting n-derivation of R, $\triangle(\lambda) = \lambda(\triangle)$ and $\triangle(d) = d(\triangle)$.

Additive map $\delta : R \longrightarrow R$ defined by $\delta(x) = \triangle(x_1, x_2, ..., x_n)$ for all $x \in R$ is called the trace of $\triangle$.

Definition 2. *An n-additive permuting mapping $D\colon R^n \to R$ is said to be a permuting n-generalized semiderivation associated with a function ψ if*

$$D(x_1, x_2, ..., \grave{x}_i x_i, ..., x_n) = D(x_1, x_2, ..., \grave{x}_i, ..., x_n)x_i + \psi(\grave{x}_i) \triangle (x_1, x_2, ..., x_i, ..., x_n)$$

$$= (x_1, x_2, ..., \grave{x}_i, ..., x_n)D(x_i) + \triangle(\grave{x}_i)\psi(x_1, x_2, ..., x_i, ..., x_n),$$

for all $\grave{x}_i, x_i \in R$, $i = 1, 2, .., n$ such that $\triangle$ acts as a permuting n-semiderivation of R, $D(\psi) = \psi(D)$ and $D(\triangle) = \triangle(D)$.

Additive map $\mu : R \longrightarrow R$ defined by $\mu(x) = D(x_1, x_2, ..., x_n)$ for all $x \in R$ is called the trace of D.

In Reference [22], Ajda and Mehsin introduced the definition of semigeneralized semiderivation of a ring R with some results about it as following:

An additive mapping $D\colon R \to R$ is said a semigeneralized semiderivation associated with a semiderivation $d : R \longrightarrow R$ and the functions $h, g : R \longrightarrow R$ if for all $x, y \in R$, then

(i) $D(xy) = D(x)h(y) + g(x)d(y) = d(x)g(y) + h(x)D(y)$,
(ii) $D(d(x)) = d(D(x))$,
(iii) $D(g(x)) = g(D(x))$,
(iv) $D(h(x)) = h(D(x))$,
(v) $g(h(x)) = h(g(x))$.

In the following definition of a permuting n-semigeneralized semiderivation associated with functions g, h and a permuting n-generalized semiderivation alternation of a permuting n-semiderivation which used in Reference [22].

Definition 3. *An n-additive permuting mapping $\Omega\colon R^n \to R$ is said to be a permuting n-semigeneralized semiderivation associated with a functions g and h if*

$$\Omega(x_1, x_2, ..., \grave{x}_i x_i, ..., x_n) = \Omega(x_1, x_2, ..., \grave{x}_i, ..., x_n)g(x_i) + h(\grave{x}_i)D(x_1, x_2, ..., x_i, ..., x_n)$$

$$= g(x_1, x_2, ..., \grave{x}_i, ..., x_n)\Omega(x_i) + D(\grave{x}_i)h(x_1, x_2, ..., x_i, ..., x_n),$$

for all $\grave{x}_i, x_i \in R$, $i = 1, 2, .., n$ such that D acts as a permuting n-generalized semiderivation of R with the following properties

(i) $\Omega(D) = D(\Omega)$,
(ii) $\Omega(g) = g(\Omega)$,
(iii) $\Omega(h) = h(\Omega)$,
(iv) $h(g) = g(h)$.

Additive map $\kappa : R \longrightarrow R$ defined by $\kappa(x) = \Omega(x_1, x_2, ..., x_n)$ for all $x \in R$ is called the trace of Ω. Obviously, we can gain from the formula of the Definition 3 the definition of n-antisemigeneralized semiderivation associated with identity functions g and h is defined as

Definition 4. *An n-additive permuting mapping $\varrho\colon R^n \to R$ is said to be a permuting n-antisemigeneralized semiderivation associated with identity functions g and h if*

$$\varrho(x_1, x_2, ..., \grave{x}_i x_i, ..., x_n) = x_i \varrho(x_1, x_2, ..., \grave{x}_i, ..., x_n) + D(x_1, x_2, ..., x_i, ..., x_n)\grave{x}_i$$

$$= \varrho(x_i)(x_1, x_2, ..., \grave{x}_i, ..., x_n) + (x_1, x_2, ..., x_i, x_n)D(\grave{x}_i)$$

for all $\grave{x}_i, x_i \in R$, $i = 1, 2, .., n$ such that D acts as a permuting n-generalized semiderivation of R. Additive map $\varphi : R \longrightarrow R$ defined by $\varphi(x) = \varrho(x_1, x_2, ..., x_n)$ for all $x \in R$ is called the trace of ϱ. In fact, the definition of n-antisemigeneralized semiderivation associated with identity functions g and h. However, it has the property $\varrho(D) = D(\varrho)$.

Hence, from the definition of a permuting n-antisemigeneralized semiderivation, we achieve the new concept which is skew n-antisemigeneralized semiderivation as follows.

Definition 5. *An n-additive permuting mapping $\gamma\colon R^n \to R$ is said to be a permuting skew n-antisemigeneralized semiderivation associated with a functions g and h such that h acts as an identity and g acts as automorphism if*

$$\gamma(x_1, x_2, ..., \grave{x}_i x_i, ..., x_n) = \gamma(x_1, x_2, ..., \grave{x}_i, ..., x_n)x_i + g(\grave{x}_i)D(x_1, x_2, ..., x_i, ..., x_n)$$

$$= (x_1, x_2, ..., \grave{x}_i, ..., x_n)\gamma(x_i) + D(\grave{x}_i)g(x_1, x_2, ..., x_i, ..., x_n)$$

for all $\grave{x}_i, x_i \in R$, $i = 1, 2, ..., n$ such that $\gamma(D) = D(\gamma), \gamma(g) = g(\gamma)$ and $D(g) = g(D)$. Additive map $\eta : R \longrightarrow R$ defined by $\eta(x) = \gamma(x_1, x_2, ..., x_n)$ for all $x \in R$ is called the trace of $\gamma(R)$.

The following example demonstrates the previous definitions.

Example 1. *Let $R = M_n(\mathbb{F})$ be a ring of $n \times n$ matrices over a field $\mathbb{F}$, $n > 1$ that is:*

$$R = \begin{pmatrix} x_j & x_{j+1} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \setminus x_j^2 = x_j,$$

for all $x_j \in \mathbb{F}$, $j = 1, 2, ..., n$. Let $\triangle$ be the n-additive mapping of R, defined by

$$\triangle(s_1, s_2, ..., \grave{s}_i s_i, ..., s_n) = \begin{pmatrix} 0 & (x_1 x_2 x_j \cdots x_{n-1})(x_1 x_2 x_j \cdots , x_n) & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

for all $\grave{x}_j, x_{j+1} \in \mathbb{F}, \grave{s}_i, s_i \in R$, $i, j = 1, 2, ..., n$

Now, let us determine whether $\triangle$ is a permuting n-semiderivation of R.

Therefore, suppose d is the n-additive mapping of R, defined by

$$d([s,a]) = s \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} s = \begin{pmatrix} 0 & x_1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

where $s \in R$.

Obviously, d is n-derivation of R. In other words, d satisfies the relation

$$d(s_1, s_2, ..., \grave{s}_i s_i, ...s_n) = d(s_1, s_2, ..., \grave{s}_i, ...s_n)s_i + \grave{s}_i d(s_1, s_2, ..., s_i, ...s_n),$$

for all $\grave{s}_i, s_i \in R$, $i = 1, 2, .., n$.

The function λ which associated with $\triangle$ defined by

$$\lambda(s) = \begin{pmatrix} x_j x_{j+1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

for all $x_j, x_{j+1} \in \mathbb{F}, s \in R$.

Now, we detect $\triangle$ from a permuting n-semiderivation of R.

Then, for all $\grave{x}_j, x_{j+1} \in R$, $i = 1, 2, ..., n$. We notice that

$$\triangle(s_1, s_2, ..., \grave{s}_i s_i, ..., s_n) = \begin{pmatrix} 0 & (x_1 x_2 x_j \cdots x_{n-1})(x_1 x_2 x_j \cdots x_n) & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Meanwhile, using the property $x_j^2 = x_j$ from the relation

$$\triangle(s_1, s_2, ..., \grave{s}_i, ..., s_n)\lambda(s_i) + \grave{s}_i d(s_1, s_2, ..., s_i, ..., s_n),$$

for all $\grave{s}_i, s_i \in R$, $i = 1, 2, ..., n$. *We achieve*

$$= \begin{pmatrix} 0 & (x_1x_2x_j\cdots x_{n-1})(x_1x_2x_j\cdots x_n) & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0. \end{pmatrix}.$$

We employ the same technique to satisfy the relation

$$\triangle(s_1, s_2, ..., \grave{s}_i s_i, ..., s_n) = \lambda(s_1, s_2, ..., \grave{s}_j, ..., x_n) \triangle (s_i) + d(\grave{s}_i)(s_1, s_2, ..., s_i, ..., s_n),$$

for all $\grave{s}_i, s_i \in R$ *and* $\grave{s}_i, s_i \in R$, $i = 1, 2, ..., n$.

In addition to that, we obtain the relation

$$\triangle(\lambda(s_1, s_2, ..., \grave{s}_i s_i, ..., s_n)) = \triangle(\begin{pmatrix} (x_1x_2x_j\cdots x_{n-1})(x_1x_2x_j\cdots x_n) & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix})$$

$$= \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Furthermore, in the same way we possess the relation

$$\lambda(\triangle(s_1, s_2, ..., \grave{s}_i s_i, ..., s_n)) = \begin{pmatrix} 0 & (x_1x_2x_j\cdots x_{n-1})(x_1x_2x_j\cdots x_n) & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Thus, we obtain $\triangle$ *is a permuting n-semiderivation of R. There is a similar result for*

$$D(d) = d(D).$$

To illustrate the concept of a permuting n-generalized semiderivation D of R which is associated with a function ψ, *we define the n-additive mapping D as follows*

$$D(s_1, s_2, ..., s_i s_i, ..., s_n) = \begin{pmatrix} 0 & (x_1x_3x_j\grave{x}_j\cdots x_{n-1})(x_1x_3x_3x_j\grave{x}_j\cdots x_n) & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

and the function $\psi(s)$ defined by

$$= \begin{pmatrix} x_j & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

for all $\grave{s}_i, s_i \in R$, $\grave{x}_j \in \mathbb{F}$, $i, j = 1, 2, ..., n$.

Now we have sufficient information adjudicate that D is a permuting n-generalized semiderivation of R. That means a permuting n-generalized semiderivation D satisfies the following relation

$$D(s_1, s_2, ..., \grave{s}_i s_i, ..., s_n) = D(s_1, s_2, ..., \grave{s}_i, ...s_n)s_i + \psi(\grave{s}_i) \triangle (s_1, s_2, ..., s_i, ..., s_n).$$

Immediately, the left side produces the following value

$$D(s_1, s_2, ..., \grave{s}_i s_i, ..., s_n) = \begin{pmatrix} 0 & (x_1 x_3 x_3 x_j \grave{x}_j \cdots x_{n-1})(x_1 x_3 x_3 x_j \grave{x}_j \cdots x_n) & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Moreover, the right side gives us the following

$$D(s_1, s_2, ..., \grave{s}_i, ...s_n)s_i + \psi(\grave{s}_i) \triangle (s_1, s_2, ..., s_i, ...s_n) = \begin{pmatrix} 0 & (x_1 x_3 \cdots x_{n-1})(x_1 x_3 \cdots x_n) & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 & x_2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} + \begin{pmatrix} x_j & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 0 & (x_1 x_3 \cdots x_{n-1})(x_1 x_3 \cdots x_n) & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & (x_1 x_3 \grave{x}_j x_j \cdots x_{n-1})(x_1 x_3 \grave{x}_j x_j \cdots x_n) & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Applying the property $x_j^2 = x_j$, we find that D is a permuting n-generalized semiderivation of R and the two side of previous relation equal to each other. Also, we see that.

$$D(\psi(s)) = D \begin{pmatrix} x_j & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} = \psi(D).$$

Similar result for

$$D(\triangle) = \triangle(D).$$

Pay attention to applying in the same way to obtain the same result for the formula

$$D(s_1, s_2, ..., \grave{s}_i s_i, ..., s_n) = (s_1, s_2, ..., \grave{s}_i, ..., s_n)D(s_i) + \triangle(\grave{s}_i)\psi(s_1, s_2, ..., s_i, ..., s_n),$$

for all $\hat{s}_i, s_i \in R$, $i = 1, 2, ..., n$.

In the penultimate stepl, we assume that the n-additive mapping Ω *appear by*

$$\Omega(s_1, s_2, ..., \hat{s}_i s_i, ..., s_n) = \begin{pmatrix} 0 & (x_1x_3\grave{x}_jx_j\cdots x_{n-1})(x_1x_3\grave{x}_j\cdots x_n) & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

and the functions g and h defined as follows:

$$g(s) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & x_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

and

$$h(s) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & x_1x_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

We investigate Ω *as a permuting n-semigeneralized semiderivation of R. In other words, we check whether* Ω *satisfies the following formula or not.*

$$\Omega(s_1, s_2, ..., \hat{s}_i s_i, ...s_n) = \Omega(s_1, s_2, ..., \hat{s}_i, ...s_n)g(s_i) + h(\hat{s}_i)D(s_1, s_2, ..., s_i, ...s_n)$$

for all $\hat{s}_i, s_i \in R$, $i = 1, 2, ..., n$ *such that D acts as a permuting n-generalized semiderivation of R.*

Directly, the left side supplies the following value of $\Omega(s_1, s_2, ..., \hat{s}_i s_i, ..., s_n)$ *as*

$$\Omega(s_1, s_2, ..., \hat{s}_i s_i, ...s_n) = \begin{pmatrix} 0 & (x_1x_3\grave{x}_jx_j\cdots x_{n-1})(x_1x_3\grave{x}_jx_j\cdots x_n) & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

On the other hand, the right side acquires

$$= \begin{pmatrix} 0 & (x_1x_3\grave{x}_jx_j\cdots x_{n-1})(x_1x_3\grave{x}_jx_j\cdots x_n) & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} g \begin{pmatrix} x_j & x_{j+1} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} +$$

$$h \begin{pmatrix} x_j & x_{j+1} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 0 & (x_1x_2x_j\cdots x_{n-1})(x_1x_2x_j\cdots x_n) & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

After substituting the values of the functions g and h with using the property $x^2 = x$, *the above relation becomes*

$$= \begin{pmatrix} 0 & (x_1x_2x_j\cdots x_{n-1})(x_1x_2x_j\cdots x_n) & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Straightforwardly, the two sides of the above equation are equal, therefore, arriving at Ω from a permuting n-semigeneralized semiderivation of R, we must check the properties. So, we assume

$$\Omega(D(s_1, s_2, ..., \grave{s}_i s_i, ..., s_n)) = \Omega(\begin{pmatrix} 0 & (x_1 x_2 x_j \cdots x_{n-1})(x_1 x_2 x_j \cdots x_n) & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}) =$$

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

By the same way we retain

$$D(\Omega(s_1, s_2, ..., \grave{s}_i s_i, ..., s_n)) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

which implies $\Omega(D) = D(\Omega)$. Also, we obtain

$$g(h(s)) = g(\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & x_j x_{j+1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} = (\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}) = h(g(s)).$$

For the other cases we apply the same technique. For illustration, the definition of n-antisemigeneralized semiderivation is associated with identity functions g and h, using the same technique of the case of a permuting n-semigeneralized semiderivation, we conclude that

$$\varrho(x_1, x_2, ..., \grave{x}_i x_i, ..., x_n) = x_i \varrho(x_1, x_2, ..., \grave{x}_i, ..., x_n) + D(x_1, x_2, ..., x_i, ..., x_n)\grave{x}_i,$$

with take discreet of the simple differences between them.

The final definition of a skew n-antisemigeneralized semiderivation has the following formula

$$\gamma(s_1, s_2, ..., \grave{s}_i s_i, ..., s_n) = \gamma(s_1, s_2, ..., \grave{s}_i, ..., s_n)s_i + g(\grave{s}_i)D(s_1, s_2, ..., s_i, ...s_n)$$

for all $\grave{s}_i, s_i \in R$, $i = 1, 2, ..., n$.

For the sake of satisfying this relation, we give permission to the terms

$$\gamma(s_1, s_2, ..., \grave{s}_i s_i, ..., s_n) = \begin{pmatrix} 0 & (x_1 x_2 x_j \cdots x_{n-1})(x_1 x_2 x_j \cdots x_n) & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

and the n-additive mapping $D(s_1, s_2, ..., \grave{s}_i s_i, ..., s_n)$ *by*

$$\begin{pmatrix} 0 & (x_1 x_2 x_j \cdots x_{n-1})(x_1 x_2 x_j \cdots x_n) & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

and the function $g(s)$ *defineD by*

$$\begin{pmatrix} \grave{x}_j \grave{x}_{j+1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

for all $\grave{s}_i, s_i \in R$, $\grave{x}_j, x_{j+1} \in \mathbb{F}$, $j = 1, 2$ *and* $i = 1, 2, ..., n$.

Over and above that, we apply the property $x_j^2 = x_j$ *too, as well as satisfying the relation* $\gamma(D) = D(\gamma), \gamma(g) = g(\gamma)$ *and* $D(g) = g(D)$. *Naturally, the previous tools are sufficient to achieve our aim.*

We begin with the list of results which are crucial for developing the proof of our results.

Lemma 1 ([51] Proposition 8.5.3)**.** *Let R be a ring. Then every intersection of prime ideals is semiprime; conversely every semiprime ideal is an intersection of prime ideals.*

Lemma 2 ([52] Lemma 1)**.** *The center of semiprime ring contains no non-zero nilpotent elements .*

Lemma 3 ([53] Lemma 2.4)**.** *Let n be a fixed positive integer and let R be a n!-torsion free ring. Suppose that* $y_1, y_2, ..., y_n \in R$ *satisfy* $\lambda y_1 + \lambda^2 y_2 + ... + \lambda^n y_n = 0$ *for* $\lambda = 1, 2, ..., n$. *Then* $y_i = 0$ *for all i, where* $i = 1, 2, ..., n$.

Lemma 4 ([54] Lemma 2.4)**.** *Let R be a semiprime ring and let* $a \in R$. *Then* $[a, [a, x]] = 0$ *holds for all* $x \in R$ *if and only if* $a^2, 2a \in Z(R)$.

Lemma 5 ([55] Sublemma P.5)**.** *Let R be a 2-torsion free semiprime ring. Suppose* $a \in R$ *such that a commute with every* $[x, a]$, $x \in R$, *then* $a \in Z(R)$.

3. Δ Acts as a Permuting *n*-Semiderivations of Semiprime Ring

In this section, we want to study semiprime ring R with a permuting n-semiderivations Δ. In Reference [56], W.D. Burgess, A. Lashgari and A. Mojiri introduced the concept of a weak zero-divisor of a ring R. An element $a \in R$ is called a weak zero-divisor if there are $r, s \in R$ with $ras = 0$ and $rs \neq 0$. The set of elements of R which are not weak zero divisors is denoted by S_{nw}. In the following theorem, we obtain a semiprime ring R has a weak zero-divisor.

For convenience, we suppose all the results of this section satisfy the identity $aRb \subset Z(R), a, b \in R$.

Theorem 1. *Let R be a semiprime ring and* Δ *be a permuting non-zero n-semiderivation with a trace* δ *such that* Δ *acts as right-multiplier. If R admits* Δ *satisfying the identity* $[\Delta(R_1), \delta(R_2)] \subseteq Z(R)$ *then R has a weak zero-divisor.*

Proof. From the main relation $[\Delta(R_1), \delta(R_2)] \subseteq Z(R)$, after $(x_1, x_2, ..., \grave{x}_i x_i, ..., x_n) \in R$ take place of R_1 with applying the Definition 1, we gain the following relation

$$[\Delta(x_1, x_2, ..., \grave{x}_i, ..., x_n)\lambda(x_i), \delta(R_2)] + [\grave{x}_i d(x_1, x_2, ..., x_i, ..., x_n), \delta(R_2)] \in Z(R)$$

for all $\grave{x}_i x_i \in R$.

By reason of Δ is a right-multiplier mapping, then for all $\grave{x}_i, x_i, r \in R$ this relation becomes

$$[[\Delta((x_1, x_2, ..., \grave{x}_i, ..., x_n)\lambda(x_i)), \delta(R_2)], r] + [[\grave{x}_i d(x_1, x_2, ..., \grave{x}_i, ..., x_n), \delta(R_2)], r] = 0. \quad (1)$$

In agreement with the main relation with using the fact that Δ is a right-multiplier and replacing R_1 by $((x_1, x_2, ..., \grave{x}_i, ..., x_n)\lambda(x_i))$. That means Relation (1) reduces to

$$[[\grave{x}_i d(x_1, x_2, ..., \grave{x}_i, ..., x_n), \delta(R_2)], r] = 0$$

for all $\grave{x}_i, x_i, r \in R$.

Obviously, when we substitute this value of (1) with using the fact that Δ is a right-multiplier and the main relation, we find that

$$\Delta(x_1, x_2, ..., \grave{x}_i, ..., x_n)[[\lambda(x_i), \delta(R_2)], r] + [\Delta(x_1, x_2, ..., \grave{x}_i, ..., x_n), r][\lambda(x_i), \delta(R_2)]$$

$$+[\Delta(x_1, x_2, ..., \grave{x}_i, ..., x_n), \delta(R_2)][\lambda(x_i), r] = 0.$$

Replacing r by $\delta(R_2)$, we achieve that

$$\Delta(x_1, x_2, ..., \grave{x}_i, ..., x_n)[[\lambda(x_i), \delta(R_2)], \delta(R_2)] + 2[\Delta(x_1, x_2, ..., \grave{x}_i, ..., x_n), \delta(R_2)][\lambda(x_i), \delta(R_2)] = 0. \quad (2)$$

Now replacing x_i by $D(R_1)$ of the above relation with the property $D(\lambda) = \lambda(D)$, we show that

$$\Delta(x_1, x_2, ..., \grave{x}_i, ..., x_n)[[D(\lambda(R_1)), \delta(R_2)], \delta(R_2)] + 2[\Delta(x_1, x_2, ..., \grave{x}_i, ..., x_n), \delta(R_2][D(\lambda(R_1)), \delta(R_2)] = 0. \quad (3)$$

Without doubt, applying the main relation on this equation gives us

$$2[\Delta(x_1, x_2, ..., \grave{x}_i, ..., x_n), \delta(R_2)][D(\lambda(R_1)), \delta(R_2)] = 0.$$

Substituting this relation in the Equation (3), we find that

$$\Delta(x_1, x_2, ..., \grave{x}_i, ..., x_n)[[D(\lambda(R_1)), \delta(R_2)], \delta(R_2)] = 0. \quad (4)$$

Moreover, for any arbitrary element $t \in R$, left-multiplying by $[[\lambda(x_i), \delta(R_2)], \delta(R_2)]t$ and right-multiplying by $t\Delta(x_1, x_2, ..., \grave{x}_i, x_{i+1} ..., x_n)$ with employing the fact from Lemma 2 yields that

$$\Delta(x_1, x_2, ..., \grave{x}_i, x_{i+1} ..., x_n)t[[\lambda(x_i), \delta(R_2)], \delta(R_2)] = 0.$$

Due to R being a semiprime, we acknowledge the set $\{P_\alpha\}$ of prime ideals of R such that $\cap P_\alpha = \{0\}$. In agreement with Lemma 1, we gain the set $\{P_\alpha\}$ of prime ideals of R is semiprime ideal. Let $\cap P_\alpha = U$. We achieve that either $t\Delta(x_1, x_2, ..., \grave{x}_i, x_{i+1} ..., x_n) \in U$ that is,

$$t\Delta(x_1, x_2, ..., \grave{x}_i, x_{i+1} ..., x_n) = 0$$

which implies to a contradiction where $\Delta \neq 0$. Actually, the above result is enough to achieve our proof after right-multiplying the Equation (4) by any non-zero element of R. $\square$

Now let us introduce the definition of non-zero elements set of a semiprime ring R which is a nest in the collection of prime ideals of R. It is denoted by M-set respect to the name of author.

Definition 6. *A set of a non-zero elements which are located in the intersection of prime ideals $\cap P_\alpha$ of a semiprime ring R is said to be M-set if has the following property: For $a \in R$, then $a^2 \in$ M-set while $a \notin$ M-set. that is, $a^2 \in$ M-set $\subseteq \cap P_\alpha$ whilst $a \notin$ M-set $\subseteq \cap P_\alpha$.*

To make the previous definition closer for the readers, we list the following example.

Example 2. *Let $R = M_n(\mathbb{F})$ be a ring of $n \times n$ matrices over a field $\mathbb{F}$, $n > 1$ that is:*

$$R = \begin{pmatrix} 0 & x_j & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

for all $x_j \in \mathbb{F}$, $j = 1, 2..., n$.

Then it is clear to be seen that

$$R = \begin{pmatrix} 0 & x_j & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Since M-set $\subseteq \cap P_\alpha$, we find that

$$\begin{pmatrix} 0 & x_j & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}^2 \in M\text{-}set$$

while

$$\begin{pmatrix} 0 & x_j & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \notin \cap P_\alpha.$$

Clearly, any non-zero element of R has nilpotency index 2 belong to M-set.

Corollary 1. *Let R be a semiprime ring and Δ be a permuting non-zero n-semiderivation with a trace δ such that Δ and δ act as right-multiplier and surjective function respectively. If R satisfies the identity $[\Delta(R_1), \delta(R_2)] \subseteq Z(R)$ then $\Delta^2 \in$ M-set.*

Proof. Employing the same technique which is applied in the proof of Theorem 1 specific to the Equation (4).

Hence, in the Equation (4) replacing t by $t\Delta(x_1, x_2, ..., \grave{x}_i, x_i..., x_n)$ with consideration that λ acts as surjective function, we find that

$$[[D(\lambda(R_1)), \delta(R_2)], \delta(R_2)]R\Delta(x_1, x_2, ..., \grave{x}_i, x_i..., x_n)^2 = (0).$$

Since R is semiprime that is, $\{P_\alpha\}$ of prime ideals of R such that $\cap P_\alpha = \{0\}$. In agreement with Lemma 1, we obtain the set $\{P_\alpha\}$ of prime ideals of R is semiprime ideal.

Employing the same proceeding in the proof of Theorem 1, we conclude that either $[[D(\lambda(R_1)), \delta(R_2)], \delta(R_2)] \in U$ for all $x_i \in R$ or $\Delta(x_1, x_2, ..., \grave{x}_i, x_i..., x_n)^2 \in U$.

Basically, from the main relation the first case yields $[[D(\lambda(R_1)), \delta(y)], \delta(y] = 0$ for all $x_i, y \in R$. Notwithstanding the second case proving $\Delta(x_1, x_2, ..., \grave{x}_i, x_i..., x_n)^2 \in U = \cap P_\alpha = \{0\}$ yields that $\Delta(x_1, x_2, ..., \grave{x}_i, x_i..., x_n)^2 = 0$ for all $\grave{x}_i, x_i \in R$.

We utilize that Δ is non-zero permuting n-semiderivation of R which means $\Delta(x_1, x_2, ..., \grave{x}_i, x_i..., x_n)^2 \in$ *M-set* and coinciding with the relation $\Delta(x_1, x_2, ..., \grave{x}_i, x_i..., x_n) \notin U = \cap P_\alpha = \{0\}$

In fact, this result is meaningful, in which the *M-set* collect a non-zero nilpotent element having degree 2 of semiprime ideal. □

Theorem 2. *Let R be a n-torsion free semiprime ring and Δ be a permuting non-zero n-semiderivation with a trace δ which is a homomorphism mapping. If Δ satisfies $\Delta(R_1) \circ \delta(R_2) \subseteq Z(R)$ then*

(*i*) $\delta^2(R) \subseteq Z(R)$.
(*ii*) *either $\delta^2(R) \subseteq$ M-set or $\Delta(R) \subseteq Z(R)$.*

Proof. Basically, from our hypothesis we have $\Delta(R_1) \circ \delta(R_2) \subseteq Z(R)$.

Replacing R_1 by $(x_1, x_2, ..., \grave{x}_i, x_i..., x_n)$ for all $\grave{x}_i, x_i \in R, i = 1, 2, ..., n$, we obtain

$$(\Delta(x_1, x_2, ..., \grave{x}_i, ..., x_n)\lambda(x_i) + \grave{x}_i d(x_1, x_2, ..., x_i, x_{i+1}..., x_n)) \circ \delta(R_2) \in Z(R).$$

Obviously, we see that

$$[(\Delta(x_1, x_2, ..., \grave{x}_i, ..., x_n)\lambda(x_i) + \grave{x}_i d(x_1, x_2, ..., x_i, x_{i+1}..., x_n))\delta(y) + \delta(y)$$

$$(\Delta(x_1, x_2, ..., \grave{x}_i, ..., x_n)\lambda(x_i) + \grave{x}_i d(x_1, x_2, ..., x_i, ..., x_n))), r] = 0$$

for all $\grave{x}_i, x_i, y, r \in R$.

Taking $r = \delta(y)$ this relation modifies into

$$[\Delta(x_1, x_2, ..., \grave{x}_i, ..., x_n)\lambda(x_i), \delta(y)]\delta(y) + [\grave{x}_i d(x_1, x_2, ..., x_i, ..., x_n), \delta(y)]\delta(y) +$$

$$\delta(y)[\Delta(x_1, x_2, ..., \grave{x}_i, x_i..., x_n)\lambda(x_i), \delta(y)] + \delta(y)[\grave{x}_i d(x_1, x_2, ..., x_i, x_{i+1}..., x_n), \delta(y)] = 0.$$

After simple calculation, we find that

$$[\Delta(x_1, x_2, ..., \grave{x}_i, ..., x_n), \delta(y)]\delta(y) + \delta(y)[\Delta(x_1, x_2, ..., \grave{x}_i, x_i..., x_n), \delta(y)] = 0.$$

Consequently, we see that

$$[\Delta(x_1, x_2, ..., \grave{x}_i, ..., x_n), \delta(y)^2] \in Z(R)$$

According to our hypothesis that δ is a homomorphism, this relation modifies to

$$[\Delta(x_1, x_2, ..., \grave{x}_i, ..., x_n), \delta(y^2)] \in Z(R) \tag{5}$$

In the main relation, we replace R_1 by $(x_1, x_2, ..., \grave{x}_i, ..., x_n)$ and R_2 by y^2 showing that

$$\Delta(x_1, x_2, ..., \grave{x}_i, ..., x_n) \circ \delta(y^2) \in Z(R) \tag{6}$$

Combining the Equations (5) and (6), we immediately obtain

$$2\Delta(x_1, x_2, ..., \grave{x}_i, ..., x_n)\delta(y^2) \in Z(R)$$

For any arbitrary element such as $t \in R$ and agreement with Lemma 4, we arrive to

$$[[t, \Delta(x_1, x_2, ..., \grave{x}_i, ..., x_n)\delta(y^2)], \Delta(x_1, x_2, ..., \grave{x}_i, ..., x_n)\delta(y^2)] = 0$$

for all $\grave{x}_i, x_i, y, t \in R$. In agreement with Lemma 5, we obtain $\Delta(x_1, x_2, ..., \grave{x}_i, ..., x_n)\delta(y^2) \in Z(R)$.

Immediately, we obtain the result

$$[t, \Delta(x_1, x_2, ..., \grave{x}_i, ..., x_n)\delta(y^2)] = 0 \tag{7}$$

Let $y = y_1, y_2, ..., \grave{y}_i y_i, ..., y_n$ and $z = z_1, z_2, ..., \grave{z}_i z_i, ..., z_n$ for all $\grave{x}_i x_i, \grave{z}_i z_i \in R$. Thus, in the Equation (5), taking $y = y + kz$, where we consider a positive integer k,$1 \leq k \leq n-1$ and $z \in R$, we deduce that $[t, \Delta(x_1, x_2, ..., \grave{x}_i, ..., x_n)\delta(y + kz)^2] = 0$.

Of course, by reason of $\Delta : R^n \longrightarrow R$ is permuting and n-additive mapping, then the trace δ of Δ satisfies the following relation $\delta(x+y) = \delta(x) + \delta(y) + \sum_{i=1}^{n-1} \binom{n}{1} \Delta(x, x, ..., x, y..., y)$ where x appears $n-i$-times and y appears i-times, with a consequence being that the above relation can be rewritten as follows:

$$\begin{aligned}[t, \Delta(x_1, x_2, ..., \grave{x}_i, ..., x_n)(\delta(y^2+z^2) + \delta(k(yz+zy)) + \sum_{i=1}^{n-1} \Delta((y^2+z^2), (y^2+z^2), ..., (y^2+z^2), \\ (k(yz+zy)), ..., (k(yz+zy)))] = 0.\end{aligned} \tag{8}$$

According to (7), the Equation (8) reduces to $[t, \Delta(x_1, x_2, ..., \grave{x}_i, ..., x_n)(\delta(k(yz+zy)) + \sum_{i=1}^{n-1} \Delta((y^2+z^2), (y^2+z^2), ..., (y^2+z^2), (k(yz+zy)), ..., (k(yz+zy)))] = 0$ for all $\grave{x}_i, x_i, t \in R$.

The element y is used as a substitute for z of this relation and, applying the Equation (7), we achieve that $[t, \Delta(x_1, x_2, ..., \grave{x}_i, ..., x_n) \sum_{i=1}^{n-1} \Delta((y^2+z^2), (y^2+y^2), ..., (y^2+y^2), (k(y^2+y^2)), ..., (k(y^2+y^2))] = 0$.

Applying Lemma 3, we see that

$$2n[t, \Delta(x_1, x_2, ..., \grave{x}_i, ..., x_n)\Delta(y^2, y^2, ..., y^2)] = 0.$$

Utilization of our hypothesis that R is n-torsion free and putting $x_1 = x_2 = ... = \grave{x}_i = ... = x_n = x^2$, we find that $[t, \Delta(x^2)\Delta(y^2)] = [t, \delta(y^2)^2] = 0$.

Clearly, we have that $\delta(y^2)^2 \in Z(R)$.

Particularly, Lemma 4 can change this identity to $[[t, \delta(y)^2], \delta(y)^2] = 0$. Applying the same technique and using Lemma 4, we conclude that $2\delta(y)^2 \in Z(R)$.

Due to R being n-torsion free, we see that $[t, \delta(y)^2] = 0$ yields $\delta(y)^2 \in Z(R)$.

(ii) From the first branch, we have the Equation (5) which is $[t, \Delta(x_1, x_2, ..., \grave{x}_i, ..., x_n)\delta(y^2)] = 0$.

From this relation and using the result of the fist branch, we find that

$$[t, \Delta(x_1, x_2, ..., \grave{x}_i, ..., x_n)]\delta(y^2) = 0.$$

Putting $t = tr, r \in R$, the last expression can be written as $[t, \Delta(x_1, x_2, ..., \grave{x}_i, ..., x_n)]r\delta(y^2) = 0$.

The last expression is the same as the proof of Theorem 1, therefore applies the similar arguments as used in the proof of Theorem 1. Hence, we obtain two options, either $[t, \Delta(x_1, x_2, ..., \grave{x}_i, ..., x_n)] \in U$ or $\delta(y^2) \in U$. The first case proved $\Delta(x_1, x_2, ..., \grave{x}_i, ..., x_n) \in Z(R)$ while the second $\delta(y^2) = 0$. Since $\delta \neq 0$ then $\delta(y^2) \in$ *M-set*. □

Theorem 3. *Let R be a 2-torsion free semiprime ring and Δ be a permuting non-zero n-semiderivation. If Δ satisfies the identity $[\Delta(R_1), \Delta(R_2)] \subseteq Z(R)$ then $\Delta(d)$ and $\Delta(R_2)$ commute with R.*

Proof. Putting $y\Delta(x_1, x_2, ..., \grave{x}_i, x_i..., x_n)$ instead of for R_1 in the main relation, we arrive to

$$[\Delta(y)\lambda(\Delta(x_1, x_2, ..., \grave{x}_i, x_i..., x_n)), \Delta(R_2)] + [yd(\Delta(x_1, x_2, ..., \grave{x}_i, x_i..., x_n)), \Delta(R_2)] \in Z(R)$$

for all $\grave{x}_i, x_i, y \in R$

For any arbitrary element of R and using the property $\Delta(\lambda) = \lambda(\Delta)$, we find that

$$[[\Delta(y)\Delta(x_1, x_2, ..., \grave{x}_i, x_i..., x_n), \Delta(R_2)], r] + [[yd(\Delta(x_1, x_2, ..., \grave{x}_i, x_i..., x_n)), \Delta(R_2)], r] = 0.$$

Replacing R_1 by y in the main relation, we conclude that $[\Delta(y), \Delta(R_2)] \in Z(R)$.

Putting $\lambda(x_1, x_2, ..., \grave{x}_i, x_i..., x_n)$ instead of R_1 give us $[\Delta(x_1, x_2, ..., \grave{x}_i, x_i..., x_n)), \Delta(R_2)] \in Z(R)$.

Using these results for $[[\Delta(y)\Delta(x_1, x_2, ..., \grave{x}_i, x_i..., x_n), \Delta(R_2)], r] + [[yd(\Delta(x_1, x_2, ..., \grave{x}_i, x_i..., x_n)), \Delta(R_2)], r] = 0$. It reduces to

$[[yd(\Delta(x_1, x_2, ..., \grave{x}_i, x_i..., x_n)), \Delta(R_2)], r] = 0$. Again, applying the property $\Delta(d) = d(\Delta)$ yields

$$y[[\Delta(d(x_1, x_2, ..., \grave{x}_i, x_i..., x_n)), \Delta(R_2)], r] + [y, r][\Delta(d(x_1, x_2, ..., \grave{x}_i, x_i..., x_n)), \Delta(R_2)] +$$

$$[y, \Delta(R_2)][\Delta(d(x_1, x_2, ..., \grave{x}_i, x_i..., x_n)), r] + [[y, \Delta(R_2)], r]\Delta(d(x_1, x_2, ..., \grave{x}_i, x_i..., x_n)) = 0.$$

In the main relation, replacing R_1 by $d(x_1, x_2, ..., \grave{x}_i, x_i..., x_n)$ and applying the result of this relation, it follows that

$$[y, r][\Delta(d(x_1, x_2, ..., \grave{x}_i, x_i..., x_n)), \Delta(R_2)] + [y, \Delta(R_2)][\Delta(d(x_1, x_2, ..., \grave{x}_i, x_i..., x_n)), r]$$

$$+[[y, \Delta(R_2)], r]\Delta(d(x_1, x_2, ..., \grave{x}_i, x_i..., x_n)) = 0.$$

Taking $\Delta(R_2)$ for r this relation modifies to

$$2[y, \Delta(R_2)][\Delta(d(x_1, x_2, ..., \grave{x}_i, x_i..., x_n)), \Delta(R_2)] + [[y, \Delta(R_2)], r]\Delta(d(x_1, x_2, ..., \grave{x}_i, x_i..., x_n)) = 0$$

for all $\grave{x}_i, x_i, y \in R$.

Putting $\Delta(R_1)$ instead of y, with applying the main relation and employing the fact that R is 2-torsion fee. In addition to that, replacing R_1 by $d(x_1, x_2, ..., \grave{x}_i, x_i..., x_n)$ based on Lemma 2. From this equation, we arrive tot $\Delta(d)$ and $\Delta(R_2)$ commute with R. We have completed the proof. □

We now state the consequence of Theorem 3.

Corollary 2. *Let R be a semiprime ring and Δ be a non-zero permuting n-semiderivation. If R admits Δ satisfying the identity $[\Delta(R_1), \Delta(R_2)] \subseteq Z(R)$ then $\Delta(d(R))$ is central of R.*

Proof. We begin with the identity $[\Delta(R_1), \Delta(R_2)] \subseteq Z(R)$. According to Theorem 4, we find that $[\Delta(d(x_1, x_2, ..., \grave{x}_i, x_i..., x_n)), \Delta(R_2)] = 0$.

Replacing R_2 by $y\Delta(x_1, x_2, ..., \grave{x}_i, x_i..., x_n)$ and applying the property $\Delta(d) = d(\Delta)$, it follows that

$$[\Delta(d(x_1, x_2, ..., \grave{x}_i, x_i..., x_n)), \Delta(y)\Delta(\lambda(x_1, x_2, ..., \grave{x}_i, x_i..., x_n)) + y\Delta(d(x_1, x_2, ..., \grave{x}_i, x_i..., x_n))] = 0$$

for all $\grave{x}_i, x_i, y \in R$.

Furthermore, using the main relation with simple calculation, we see that

$$[\Delta(d(x_1, x_2, ..., \grave{x}_i, x_i..., x_n)), y]\Delta(d(x_1, x_2, ..., \grave{x}_i, x_i..., x_n)) = 0$$

for all $\grave{x}_i, x_i, y \in R$.

Replacing $ty, t \in R$ with y in this relation and using that result, we see that

$$[\Delta(d(x_1, x_2, ..., \grave{x}_i, x_i..., x_n)), t]y\Delta(d(x_1, x_2, ..., \grave{x}_i, x_i..., x_n))) = 0. \tag{9}$$

Replacing y by Ry in the Relation (9). This implies that

$$[\Delta(d(x_1, x_2, ..., \grave{x}_i, x_i..., x_n)), t]Ry\Delta(d(x_1, x_2, ..., \grave{x}_i, x_i..., x_n)) = (0). \tag{10}$$

Again, in (9), substituting R for y and right-multiplying by y. Subtracting this result from (10), we arrive to $[\Delta(d(x_1, x_2, ..., \grave{x}_i, x_i..., x_n)), y] = 0$. This implies to $\Delta(d(R))$ is central of R.

We obtain the required result. $\square$

Theorem 4. *Let R be a semiprime ring and Δ be a permuting n-semiderivation with a trace δ such that Δ acts as a homomorphism. Suppose Δ satisfies the identity $\Delta(R_1) \circ \delta(R_2) \mp [R_1, R_2] \subseteq Z(R)$. Then*

(*i*) *either $\Delta(R_1)^2 \subseteq$ M-set or $\delta(R_2)$ and $\Delta(R_1^2)$ commute with R.*
(*ii*) *R is commutative if $\delta(R) = 0$ or $\Delta(R) = 0$.*

Proof. (i) As above, we have the relation $\Delta(R_1) \circ \delta(R_2) \mp [R_1, R_2] \subseteq Z(R)$. Putting $R_1 = R_2$, in the main relation yields that $\Delta(R_1) \circ \delta(R_2) \subseteq Z(R)$.

Then the last expression of this relation can be written as $[\Delta(R_1)\delta(R_1), r] + [\delta(R_1)\Delta(R_1), r] = 0$ for all $r \in R$.

Replacing r by $\Delta(R_1)$ in this relation, we notice that

$$[\Delta(R_1)^2, \delta(R_1)] = 0. \tag{11}$$

which implies that $[\Delta(R_1)^2, \delta(R_1)] \subseteq Z(R)$. Moreover, in the relation $\Delta(R_1) \circ \delta(R_1) \subseteq Z(R)$ putting R_1^2 instead of R_1 with using the fact that Δ acts as homomorphism yields

$$\Delta(R_1^2) \circ \delta(R_1) \subseteq Z(R).$$

Subtracting this result with Equation (11), gives us $2\Delta(R_1^2)\delta(R_1) \subseteq Z(R)$.

Application of the fact that R has torsion restriction gives the following

$$[\Delta(R_1^2)\delta(R_1), r] = 0$$

for all $r \in R$. Hence, replacing r by $\Delta(R_1^2)$ of this relation becomes

$$\Delta(R_1^2)[\delta(R_1), \Delta(R_1^2)] = 0.$$

Left-multiplying by $[\delta(R_1), \Delta(R_1^2)]t$ and right-multiplying by $t\Delta(R_1^2)$, t is any arbitrary element of R with employed Lemma 2. Then it is easy to see that

$$[\delta(R_1), \Delta(R_1^2)]R\Delta(R_1^2) = (0).$$

Now repeating similar technique to those we applied in the final part of the proof of Theorem 1, we conclude that: either $\Delta(R_1^2) \in U = \cap P_\alpha = \{0\}$ or $[\delta(R_1), \Delta(R_1^2)] \in U = \cap P_\alpha = \{0\}$.

Actually, our hypothesis points out that $\Delta \neq 0$, in addition to the fact that Δ acts as a homomorphism mapping which means the first case implies $\Delta(R_1)^2 \in$ *M-set*. While the second case supplies that $\delta(R_1)$ and $\Delta(R_1^2)$ commute with R. (ii) If $\delta = 0$ then the main identity reduces into $[R_1, R_2] \subseteq Z(R)$ which implies that R is commutative ring. Hence, we get the required result. $\square$

Theorem 5. *Let R be a 2-torsion free semiprime ring and Δ be a permuting n-semiderivation with a trace δ which acts as left-multiplier. Suppose Δ satisfies $\Delta(R_1 \circ R_2) \mp \delta([R_1, R_2]) \mp [R_1, R_2] \subseteq Z(R)$. Then R is commutative.*

Proof. First, we discuss the case Δ and δ are not equal to zero, so the main identity still $\Delta(R_1 \circ R_2) \mp \delta([R_1, R_2]) \mp [R_1, R_2] \subseteq Z(R)$.

For any arbitrary element $t \in R$, we conclude that

$$[\Delta(R_1 \circ R_2), t] \mp [\delta([R_1, R_2]), t] \mp [[R_1, R_2], t] = 0. \tag{12}$$

Putting R_1 instead of for R_2 in this relation and using the fact R is 2-torsion free, we find that

$$[\Delta(R_1^2), t] = 0. \tag{13}$$

Linearizing Equation (13) with depending on the fact R is 2-torsion free and using Equation (13), we show that

$$[\Delta(R_1 R_2), t] = 0. \tag{14}$$

Again, in (13), substituting $R_2 + R_1$ in place of R_1 and using R has a 2-torsion free, we observe that $[\Delta(R_2 R_1), t] = 0$.

Combining Relation (14) with this relation, it follows that $[\Delta(R_1 \circ R_2), t] = 0$.

Substituting this Equation of (12), we gain that

$$[\delta([R_1, R_2]), t] + [[R_1, R_2], t] = 0.$$

Replacing $[R_1, R_2]$ with t, we find that

$$[\delta([R_1, R_2]), [R_1, R_2]] = 0. \tag{15}$$

Particularly, for $R_1 = R_1 R_3$ this relation extends to

$$\begin{aligned} & R_1[\delta([R_1 R_3, R_2]), [R_3, R_2]] \\ & +[\delta([R_1 R_3, R_2]), R_1][R_3, R_2] \\ & +[R_1, R_2][\delta([R_1 R_3, R_2]), R_3] \\ & +[\delta([R_1 R_3, R_2]), [R_1, R_2]]R_3 = 0. \end{aligned}$$

Replacing R_3 by R_2. Last expression implies that

$$[R_1, R_2][\delta([R_1, R_2]R_2), R_2] + [\delta([R_1, R_2]R_2), [R_1, R_2]]R_2 = 0.$$

By reason of that δ is a left-multiplier and applying of Equation (15), this relation arrives to

$$([R_1, R_2][\delta([R_1, R_2]), R_2] + \delta([R_1, R_2])[R_2, [R_1, R_2]])R_2 = 0.$$

Furthermore, simplify this relation and using (15), this relation reduces to

$$(\delta([R_1, R_2])R_2[R_1, R_2] - [R_1, R_2]R_2\delta([R_1, R_2]))R_2 = 0. \tag{16}$$

Multiplying (16) from the right by $R_2 t(\delta([R_1, R_2])R_2[R_1, R_2] - [R_1, R_2]R_2\delta([R_1, R_2]))$ and the left by $R_2^2 t$, t is any arbitrary element of R with applying Lemma 2 and $aRb \subset Z(R)$, we obtain

$$R_2^2 t(\delta([R_1, R_2])R_2[R_1, R_2] - [R_1, R_2]R_2\delta([R_1, R_2])) = 0.$$

Now repeating similar technique we used in the final part of proof of Theorem 1, we arrive to two cases.

In first case, we have $R_2^2 \in U = \cap P_\alpha = \{0\}$. This implies $R_2^2 = 0$.

Where $R_2 \neq 0$, we see that $R_2^2 \in M\text{-}set$. From the second case, we find that

$$\delta([R_1, R_2])R_2[R_1, R_2] = [R_1, R_2]R_2\delta([R_1, R_2].$$

Right-multiplying by $[R_1, R_2]$ this relation becomes

$$\delta([R_1, R_2])R_2[R_1, R_2]^2 = [R_1, R_2]R_2\delta([R_1, R_2])[R_1, R_2] \tag{17}$$

According to Relation (15) this equation modifies to

$$\delta([R_1, R_2])R_2[R_1, R_2]^2 = [R_1, R_2]R_2[R_1, R_2]\delta([R_1, R_2]) \tag{18}$$

Subtracting Relations (17) and (18). Then, it is easy to see that

$$[R_1, R_2]R_2[\delta([R_1, R_2]), [R_1, R_2]] = 0.$$

Left-multiplying by $[\delta([R_1, R_2]), [R_1, R_2]]t$ and right-multiplying by $t[R_1, R_2]R_2$, where t is any arbitrary element of R. In addition to that, applying similar method as we used in the proof of Theorem 1, we observe that; either $[R_1, R_2]R_2 \in U = \cap P_\alpha = \{0\}$ or $[\delta([R_1, R_2]), [R_1, R_2]] \in U = \cap P_\alpha = \{0\}$.

Of course, the second case satisfied by work. That means we have the first case which $[R_1, R_2]R_2 \in U = \cap P_\alpha = \{0\}$ implies to $[R_1, R_2]R_2 = 0$.

Replacing R_1 by $R_1 t$, where t is any arbitrary element of R, we note that

$$[R_1, R_2]tR_2 = 0. \tag{19}$$

Now in Equation (19) putting tR_1 for t, we achieve that

$$[R_1, R_2]tR_1R_2 = 0. \tag{20}$$

Right-multiplying (19) by R_1 and subtracting this result from Relation (20) with using the semiprimeness of R, the R satisfy that
$[R_1, R_2] = 0$. Obviously, R is commutative.

Now we take $\Delta = \delta = 0$ yields $[R_1, R_2] \subseteq Z(R)$. Without doubt, R is commutative. This finishes the proof. □

4. Permuting *n*-Generalized Semiderivation of Semiprime Rings

In this section, we study the behaviour of a permuting n-generalized semiderivation on semiprime rings R with $Z(R)$.

For more convenience, we suppose all the results of this section satisfies the relation $aRb \subset Z(R), a, b \in R$. Except Theorem 7.

Theorem 6. *Let R be a 2-torsion free semiprime ring, U be a non-zero ideal of R and D be a permuting n-generalized semiderivation. Suppose D satisfies the identity $D(R \circ U) \mp [R, U] \subseteq Z(R)$. If*

- (i) *$D(R) \neq 0$ with the property $x_i^2 = x_i$ for all $x \in R$ then either $D(R)$ is central of R or $D(R) \subseteq M$-set or $D(R^2)$ is commuting of R such that $i = 1, 2, ..., n$.*
- (ii) *$D = 0$ then R contains a non-zero central ideal.*

Proof. First, we observe that R satisfies the identity $D(R \circ U) \mp [R, U] \subseteq Z(R)$. Clearly, this relation implies $D(r \circ x) \mp [r, x] \in Z(R)$ for all $x \in U, r \in R$.

Moreover, we note that $D(rx) + D(xr) + [r, x] \in Z(R)$ for all $x \in U, r \in R$.
The last relation can be rewritten as

$$D((r_1, r_2, ..., \grave{r}_i, r_i..., r_n)(x_1, x_2, ..., \grave{x}_i, x_i..., x_n))$$

$$+D((x_1, x_2, ..., \grave{x}_i, x_i..., x_n)(r_1, r_2, ..., \grave{r}_i, r_i..., r_n))$$

$$+[(r_1, r_2, ..., \grave{r}_i, r_i..., r_n), (x_1, x_2, ..., \grave{x}_i, x_i..., x_n)] \in Z(R)$$

for all , $\grave{x}_i, x_i \in U, \grave{r}_i, r_i \in R$ such that $i = 1, 2, ..., n$.
Applying Definition 2 to the first term of this relation, we achieve that

$$D((r_1, r_2, ..., \grave{r}_i, r_i..., r_n)(x_2, x_3, ..., \grave{x}_i, x_i..., x_n))x_1$$

$$+\psi(r_n)\Delta((r_1, r_2, ..., \grave{r}_i, r_i..., r_{n-1})(x_1, x_2, ..., \grave{x}_i, x_i..., x_n))$$

$$+D((x_1, x_2, ..., \grave{x}_i, x_i..., x_n)(r_1, r_2, ..., \grave{r}_i, r_i..., r_n))$$

$$+[(r_1, r_2, ..., \grave{r}_i, r_i..., r_n), (x_1, x_2, ..., \grave{x}_i, x_i..., x_n)] \in Z(R).$$

Let

$$\omega(r_i, x_i) = \psi(r_n)\Delta((r_1, r_2, ..., \grave{r}_i, r_i..., r_{n-1})(x_1, x_2, ..., \grave{x}_i, x_i..., x_n))$$

$$+D((x_1, x_2, ..., \grave{x}_i, x_i..., x_n)(r_1, r_2, ..., \grave{r}_i, r_i..., r_n))$$

$$+[(r_1, r_2, ..., \grave{r}_i, r_i..., r_n), (x_1, x_2, ..., \grave{x}_i, x_i..., x_n)],$$

$$i = 1, 2, ..., n.$$

Hence this relation can be rewritten as

$$D((r_1, r_2, ..., \grave{r}_i, r_i..., r_n)(x_2, x_3, ..., \grave{x}_i, x_i..., x_n))x_1 + \omega(r_i, x_i) \in Z(R)$$

For any arbitrary element of R, we find that

$$D((r_1, r_2, ..., \grave{r}_i, r_i..., r_n)(x_2, x_3, ..., \grave{x}_i, x_i..., x_n))[x_1, t]$$

$$+[D(r_1, r_2, ..., \grave{r}_i, r_i..., r_n)(x_2, x_3, ..., \grave{x}_i, x_i..., x_n)), t]x_1 + [\omega(r_i, x_i), t] = 0.$$

Putting $t = x_1$ of this relation yields

$$[D((r_1, r_2, ..., \grave{r}_i, r_i..., r_n)(x_2, x_3, ..., \grave{x}_i, x_i..., x_n)), x_1]x_1 + [\omega(r_i, x_i), x_1] = 0.$$

In this relation replacing x_1 by $\omega(r_i, x_i)$, we find that

$$[D((r_1, r_2, ..., \grave{r}_i, r_i..., r_n)(x_2, x_3, ..., \grave{x}_i, x_i..., x_n)), \omega(r_i, x_i)]\omega(r_i, x_i) = 0.$$

Now left-multiplying by $\omega(r_i, x_i)R$ and right-multiplying by $R[D((r_1, r_2, ..., \grave{r}_i, r_i..., r_n)(x_2, x_3, ...$ $, \grave{x}_i, x_i, ..., x_n)), \omega(r_i, x_i)]$ with applying Lemma 2 and $aRb \subset Z(R), a, b \in R$, we arrive to

$$\omega(r_i, x_i)R[D((r_1, r_2, ..., \grave{r}_i, r_i..., r_n), (x_2, x_3, ..., \grave{x}_i, x_i, ..., x_n)), \omega(r_i, x_i)] = 0.$$

Due to R is a semiprime, we consider the set $\{P_\alpha\}$ of prime ideals of R such that $\cap P_\alpha = \{0\}$. According to Lemma 1, we obtain $\{P_\alpha\}$ the set of prime ideals of R is semiprime ideal.
Let $\cap P_\alpha = U$. Hence, we have either $\omega(r_i, x_i) \in U$ that is,

$$\omega(r_i, x_i) = \psi(r_n)\Delta((r_1, r_2, ..., \hat{r}_i, r_i..., r_{n-1})(x_1, x_2, ..., \hat{x}_i, x_i..., x_n))$$
$$+D(x_1, x_2, ..., \hat{x}_i, x_i..., x_n)(r_1, r_2, ..., \hat{r}_i, r_i..., r_n)$$
$$+[(r_1, r_2, ..., \hat{r}_i, r_i..., r_n), (x_1, x_2, ..., \hat{x}_i, x_i..., x_n)] = 0.$$

We add the term $-D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_1, x_2, ..., \hat{x}_i, x_i..., x_n))$ to both sided of this relation, we obtain that

$$\psi(r_n)\Delta((r_1, r_2, ..., \hat{r}_i, r_i..., r_{n-1})(x_1, x_2, ..., \hat{x}_i, x_i..., x_n)) - D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_1, x_2, ..., \hat{x}_i, x_i..., x_n))$$
$$= -D((x_1, x_2, ..., \hat{x}_i, x_i..., x_n)(r_1, r_2, ..., \hat{r}_i, r_i..., r_n)) - D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_1, x_2, ..., \hat{x}_i, x_i..., x_n))$$
$$-[(r_1, r_2, ..., \hat{r}_i, r_i..., r_n), (x_1, x_2, ..., \hat{x}_i, x_i..., x_n)].$$

Rewriting this relation as follows

$$\psi(r_n)\Delta((r_1, r_2, ..., \hat{r}_i, r_i..., r_{n-1})(x_1, x_2, ..., \hat{x}_i, x_i..., x_n)) - D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_1, x_2, ..., \hat{x}_i, x_i..., x_n)$$
$$= -D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n) \circ (x_1, x_2, ..., \hat{x}_i, x_i..., x_n))$$
$$-[(r_1, r_2, ..., \hat{r}_i, r_i..., r_n), (x_1, x_2, ..., \hat{x}_i, x_i..., x_n)].$$

Corresponding to the main identity $D(R \circ U) \mp [R, U] \subseteq Z(R)$, from the left side of this relation, we arrive to

$$\psi(r_n)\Delta(r_1, r_2, ..., \hat{r}_i, r_i..., r_{n-1}, x_1, x_2, ..., \hat{x}_i, x_i..., x_n))$$
$$-D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_1, x_2, ..., \hat{x}_i, x_i..., x_n) \in Z(R).$$

Simplifying this expression, we find that

$$-D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_2, ..., \hat{x}_i, x_i..., x_n))x_1 \in Z(R)$$

For any arbitrary element of R from this relation, we show that

$$D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_2, ..., \hat{x}_i, x_i..., x_n))[x_1, t] + [D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_2, ..., \hat{x}_i, x_i..., x_n)), t]x_1 = 0$$

for all $\hat{x}_i, x_i \in U, r, t \in R$

Substituting t for $D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_2, ..., \hat{x}_i, x_i..., x_n))$ and x_1 by yx_1, we notice that

$$D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_2, ..., \hat{x}_i, x_i..., x_n))y[x_1, D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_2, ..., \hat{x}_i, x_i..., x_n))]$$
$$+D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_2, ..., \hat{x}_i, x_i..., x_n))[y, D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_2, ..., \hat{x}_i, x_i..., x_n))]x_1 = 0.$$

Putting $D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_2, ..., \hat{x}_i, x_i..., x_n))$ for y of this relation, we see that

$$D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_2, ..., \hat{x}_i, x_i..., x_n))^2[x_1, D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_2, ..., \hat{x}_i, x_i..., x_n))] = 0.$$

Left-multiplying by $D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_2, ..., \hat{x}_i, x_i..., x_n))[x_1, D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_2, ..., \hat{x}_i, x_i ..., x_n))]$ and right-multiplying by $D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_2, ..., \hat{x}_i, x_i..., x_n))$ with applying Lemma 2 and $aRb \subset Z(R), a, b \in R$, we find that

$$D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_2, ..., \hat{x}_i, x_i..., x_n))[x_1, D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_2, ..., \hat{x}_i, x_i..., x_n))]$$
$$D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_2, ..., \hat{x}_i, x_i..., x_n)) = 0.$$

Now if we continue to carry out the same method as above, after right-multiplying this relation by $[x_1, D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_2, ..., \hat{x}_i, x_i..., x_n))]$ and employing Lemma 2 and $aRb \subset Z(R), a, b \in R$, this relation reduces to

$$[x_1, D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_2, ..., \hat{x}_i, x_i..., x_n))]D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_2, ..., \hat{x}_i, x_i..., x_n)) = 0.$$

Using the action of Lemma 2 with right-multiplying $y[x_1, D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_2, ..., \hat{x}_i, x_i, ..., x_n))]$ and left-multiplying by $D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_2, ..., \hat{x}_i, x_i..., x_n))y$, $y \in R$, we satisfy that

$$D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_2, ..., \hat{x}_i, x_i..., x_n))y[x_1, D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_2, ..., \hat{x}_i, x_i..., x_n))] = 0. \quad (21)$$

Putting $x_1 y$ for y in this relation yields

$$D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_2, ..., \hat{x}_i, x_i..., x_n))x_1 y[x_1, D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_2, ..., \hat{x}_i, x_i..., x_n))] = 0. \quad (22)$$

Left-multiplying (21) by x_1 and subtracting this result from (22) with using R is semiprime and $D \neq 0$, we observe that

$$[x_1, D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_2, ..., \hat{x}_i, x_i..., x_n))] = 0.$$

Hence, from the last equation, we see that $D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_2, ..., \hat{x}_i, x_i..., x_n)) \in Z(R)$.
Replacing x_2 by $x_1 x_2$, we achieve that $D(R) \subseteq Z(R)$.
Or $[D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_2, x_3, ..., \hat{x}_i, x_i, ..., x_n)), \omega(r_i, x_i)] \in U$ which implies to the relation

$$[D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_2, x_3, ..., \hat{x}_i, x_i..., x_n)), \omega(r_i, x_i)] = 0.$$

Now substituting the value of $\omega(r_i, x_i)$ for this relation with simple calculation yields

$$[D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_2, x_3, ..., \hat{x}_i, x_i..., x_n)), \psi(r_n)\Delta((r_1, r_2, ..., \hat{r}_i, r_i..., r_{n-1})(x_1, x_2, ..., \hat{x}_i, x_i..., x_n))]$$
$$+[D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_2, x_3, ..., \hat{x}_i, x_i..., x_n)), D((x_1, x_2, ..., \hat{x}_i, x_i..., x_n))(r_1, r_2, ..., \hat{r}_i, r_i..., r_n)]$$
$$+[D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_2, x_3, ..., \hat{x}_i, x_i..., x_n)), [(r_1, r_2, ..., \hat{r}_i, r_i..., r_n), (x_1, x_2, ..., \hat{x}_i, x_i..., x_n)]] = 0.$$

We add the term $[D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_2, x_3, ..., \hat{x}_i, x_i..., x_n)), D(r_1, r_2, ..., \hat{r}_i, r_i..., r_n, x_2, x_3, ..., \hat{x}_i, x_i..., x_n))x_1]$ to both sided and using D is symmetric mapping, we conclude that

$$2[D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_2, x_3, ..., \hat{x}_i, x_i..., x_n)), D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_1, x_2, x_3, ..., \hat{x}_i, x_i..., x_n))]$$
$$+[(r_1, r_2, ..., \hat{r}_i, r_i..., r_n), (x_1, x_2, ..., \hat{x}_i, x_i..., x_n)]$$
$$= D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_2, x_3, ..., \hat{x}_i, x_i..., x_n))[D((r_1, r_2, ..., \hat{r}_i, r_i..., r_n)(x_2, x_3, ..., \hat{x}_i, x_i..., x_n)), x_1].$$

Replacing x_2 by $x_1 x_2$ and using $x_i^2 = x_1$ with putting x_i for r_i of this relation, we find that

$$D(x_1, x_2, x_3, ..., \hat{x}_i, x_i..., x_n))^2[D(x_1, x_2, x_3, ..., \hat{x}_i, x_i..., x_n))^2, x_1] = 0.$$

Left-multiplying by $[D(x_1, x_2, x_3, ..., \hat{x}_i, x_i..., x_n))^2, x_1]R$ and right-multiplying by $RD(x_1, x_2, x_3, ..., \hat{x}_i, x_i..., x_n))^2$. Also, using Lemma 2 and $aRb \subset Z(R), a, b \in R$ with applying the same previous technique which used of the above part of our proof, we satisfy two options either $D(x_1, x_2, x_3, ..., \hat{x}_i, x_i..., x_n))^2 \in U$ or $[D(x_1, x_2, x_3, ..., \hat{x}_i, x_i..., x_n))^2, x_1] \in U$.
Definitely, the first case and using the condition $D \neq 0$ give us $D(x_1, x_2, x_3, ..., \hat{x}_i, x_i..., x_n)) \in$ *M-set*.
Whereas, the second case implies to $[D(x_1, x_2, x_3, ..., \hat{x}_i, x_i..., x_n))^2, x_1] = 0$. Writing x instead of $x_1 = x_2 = x_3 = ... = \hat{x}_i = x_i... = x_n)$. Hence, we conclude that $[D(x)^2, x] = 0$. Linearization of this equation and using it. Applying that R is 2-torsion free, we conclude that $D(R^2)$ is commuting on R.

(ii) Obviously, if $D = 0$ then the main relation becomes $[R, U] \subseteq Z(R)$. It is clear that R contains a non-zero central ideal. This completes the proof. □

Remark 1. *The condition $x_i^2 = x_i$ which appeared in Branch(i) of the previous theorem is not superfluous. Indeed, the evidence of this fact can be obtained from Example 1.*

Theorem 7. *Let R be a 2-torsion free semiprime ring, U be a non-zero ideal of R and D be a permuting n-generalized semiderivation with a trace μ. Suppose D satisfies the identity $D([R, U]) \mp \mu(R \circ U) \subseteq Z(R)$. If*

(i) $D(R) \neq 0$ *then* $D(U^2) \subseteq Z(R)$.
(ii) $D(R) = 0$ *then* $\mu(U^2) \subseteq Z(R)$.

Proof. (i) Given that $D([R, U]) \mp \mu(R \circ U) \subseteq Z(R)$. Evidently, replacing R by U reduces this relation into $\mp\psi(U^2) \subseteq Z(R)$.

Putting $x + y$ for all $x, y \in U$ instead of U of above relation, for any arbitrary element of R say r, regard that $x = (x_1, x_2, x_3, ..., \grave{x}_i, x_i..., x_n)$ and $y = (y_1, y_2, y_3, ..., \grave{y}_i, y_i..., y_n)$, we conclude that $[\mu(x^2), r] + [\mu(xy + yx), r] + [\mu(y^2), r] = 0$ for all $x, y \in U$,$r \in R$. According to the identity $\mu(U^2) \subseteq Z(R)$, this relation modifies to

$$[\mu(xy + yx), r] = 0. \tag{23}$$

It is clear to be seen that $D : R^n \longrightarrow R$ is permuting and n-additive mapping, then the trace μ of D satisfies the following relation

$$\mu(x + y) = \mu(x) + \mu(y) + \sum_{i=1}^{n-1} \binom{n}{1} D(x, x, ..., x, y..., y)$$

where x appears $n - i$-times and y appears i-times.

Let $\mu \neq 0$. Since $\mu(xy + yx) \in Z(R)$ for all $x, y \in U$, replacing y by $y + kz$ for all $x, y, z \in U$, $1 \leq k \leq n - 1$, in the this relation, we find that

$$\mu(x(y + kz) + (y + kz)x) \in Z(R)$$

for all $x, y \in U$.

Moreover, we observe that

$$[\mu(x \circ y) + \mu(x \circ kz) + \sum_{i=1}^{n-1} \binom{n}{1} D(x \circ y, x \circ y, ..., x \circ y, x \circ kz..., x \circ kz), r] = 0$$

for all $x, y \in U$, $r \in R$. Using Relation (23), we notice that

$$[\sum_{i=1}^{n-1} \binom{n}{1} D(x \circ y, x \circ y, ..., x \circ y, x \circ kz..., x \circ kz), r] = 0$$

for all $x, y \in U$,$r \in R$.

Applying Lemma 3 gives that $n[D(x \circ z, x \circ y, x \circ y, x \circ y..., x \circ y), r] = 0$.

Due to the fact R is 2-torsion free, we receive that

$$[D(x \circ z, x \circ y, x \circ y, x \circ y..., x \circ y), r] = 0.$$

Particularly, we achieve that $D(x \circ y, x \circ y, x \circ y, x \circ y..., x \circ y) \in Z(R)$ for all $x, y \in U$. Putting x instead of y of this relation. This yields $D(x_1^2, x_2^2, ..., \grave{x}_i{}^2, x_i^2, ..., x_n^2) \in Z(R)$ for all $\grave{x}_i, x_i \in U$.

Writing x^2 instead of $x_1^2 = x_2^2 = ... = \grave{x}_i{}^2 = x_i^2 = ... = x_n^2$ in this identity, we conclude that $[D(x^2), r] = 0$ for all $r \in R$ yields $D(x^2) \in Z(R)$.

(ii) Certainly, if $D(R) = 0$ then the main relation reduces to $\mu(R \circ U) \subseteq Z(R)$, which means $2\mu(U^2) \subseteq Z(R)$. By reason of the fact R is 2-torsion free, we find that $\mu(U^2) \subseteq Z(R)$. This finishes the proof. □

Theorem 8. *Let R be a 2-torsion free semiprime ring and D be a non-zero permuting n-generalized semiderivation associated with function ψ such that $\psi(R) = a - bR$, $a, b \in R$. If D satisfies the relation $[\Delta(R), D(R)] \subseteq Z(R)$ then R has a weak zero divisor.*

Proof. For the convenience, let us rewrite the main condition as $[\Delta(R), D((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_1, y_2, ..., \grave{y}_i, y_i, ..., y_n))] \in Z(R)$ for all $\grave{x}_i, x_i, \grave{y}_i, y_i \in R$, $i = 1, 2, ...n$.

Suppose there is an arbitrary element of R with using the main relation such that

$$\begin{aligned} &D((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2, ..., \grave{y}_i, y_i, ..., y_n))[[\Delta(R), y_1], r] \\ &+[D(x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2, ..., \grave{y}_i, y_i, ..., y_n), r][\Delta(R), y_1] \\ &+[\Delta(R), D(x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2, ..., \grave{y}_i, y_i, ..., y_n)][y_1, r] \\ &+[\Delta(R), \psi(x_n)][\Delta(x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_2, ..., \grave{y}_i, y_i, ..., y_n), r] \\ &+[[\Delta(R), \psi(x_n)], r]\Delta((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_2, ..., \grave{y}_i, y_i, ..., y_n)) = 0 \end{aligned}$$

for all $\grave{x}_i, x_i, \grave{y}_i, y_i, r \in R$.

Without loss of generality we replace r by $\Delta(R)$. In this case this relation becomes

$$\begin{aligned} &D((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2, ..., \grave{y}_i, y_i, ..., y_n))[[\Delta(R), y_1], \Delta(R)] \\ &+2[D(x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2, ..., \grave{y}_i, y_i, ..., y_n), \Delta(R)][\Delta(R), y_1] \\ &+[\Delta(R), \psi(x_n)][\Delta(x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_2, ..., \grave{y}_i, y_i, ..., y_n), \Delta(R)] \\ &+[[\Delta(R), \psi(x_n)], \Delta(R)]\Delta((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_2, ..., \grave{y}_i, y_i, ..., y_n)) = 0. \end{aligned}$$

Replacing R by y_1, we find that

$$\begin{aligned} &D((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2, ..., \grave{y}_i, y_i, ..., y_n))[[\Delta(y_1), y_1], \Delta(y_1)] \\ &+2[D(x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2, ..., \grave{y}_i, y_i, ..., y_n), \Delta(y_1)][\Delta(y_1), y_1] \\ &+[[\Delta(y_1), \psi(x_n)], \Delta(y_1)]\Delta((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_2, ..., \grave{y}_i, y_i, ..., y_n)) = 0. \end{aligned}$$

Compatibility with the fact that the associated function ψ acts as $\psi(R) = a - bR$, where a and b are fixed element of R, we conclude that

$$\begin{aligned} &D((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2, ..., \grave{y}_i, y_i, ..., y_n))[[\Delta(y_1), y_1], \Delta(y_1)] \\ &+2[D(x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2, ..., \grave{y}_i, y_i, ..., y_n), \Delta(y_1)][\Delta(y_1), y_1] \\ &+[[\Delta(y_1), a - bx_n], \Delta(y_1)]\Delta((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_2, ..., \grave{y}_i, y_i, ..., y_n)) = 0. \end{aligned}$$

In this relation, putting $-x_n$ for the place of x_n and combining the result with above relation, we see that

$$2[[\Delta(y_1), a], \Delta(y_1)]\Delta((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_2, ..., \grave{y}_i, y_i, ..., y_n)) = 0.$$

Applying the fact that R is 2-torsion free, we see that

$$[[\Delta(y_1), a], \Delta(y_1)]\Delta((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_2, ..., \grave{y}_i, y_i, ..., y_n)) = 0.$$

Replacing y_2 by y_1y_2 of this identity, we arrive to

$$[[\Delta(y_1), a], \Delta(y_1)]\Delta(R) = 0.$$

Arguing in a similar technique as we have done in the proof of Theorem 6, we separate the proof in two cases:

Either $[[\Delta(y_1), a], \Delta(y_1)] \in U$ or $\Delta(R) \in U$. Now if we continue with the process inductively then from the second case, we arrive to $\Delta(R) = 0$.

In agreement with $\Delta \neq 0$, the latter result leads to a contradiction.

If second case holds, that is, $[[\Delta(y_1), a], \Delta(y_1)] = 0$ for all $y_1 \in R$.

Consequently, multiplying this relation by R. This case implies to $R[[\Delta(y_1), a], \Delta(y_1)] = 0$ for all $y_1 \in R$. In other words, this result shows that for each $x \in R \setminus \{0\}$, we satisfy this result. Basically, R has not a zero divisors. Hence, we assume that $R\Delta(y_1) \neq 0$. Therefore, R has a weak zero divisor. This completes the proof. □

Theorem 9. *Let R be a 2-torsion free semiprime ring, U be an ideal and D be a non-zero permuting n-generalized semiderivation associated with function ψ such that $\psi(R) = Ra - bR^2$, $a, b \in R$ and $\lambda(R) = -R^2$ an associated function of Δ. Suppose R satisfies the identity $D([[R_1, R_2], (R_1 \circ R_2)]) \subseteq Z(R)$. Then either $R^2 \subseteq$ M-set or $b^4 \in$ M-set or $b \circ R = 0$ or $b^3 \in Z(R)$.*

Proof. Directly, from the main relation we find that $[D([[R_1R_2, R_2R_1]) - D([R_2R_1, R_1R_2])), t] = 0$ for all $t \in R$. Simplify this relation, we arrive to

$$[D([R_1, R_2]R_1R_2), t] + [D(R_1R_2[R_2, R_1]), t] - [D([R_2, R_1]R_2R_1), t] - [D(R_2R_1[R_2, R_1]), t] = 0$$

for all $t \in R$.

Applying Definition 2 on the last term and putting $(x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)$ instead of R_1 with $y_1, y_2, ..., \grave{y}_i, y_i, ..., y_n$ taking the place of R_2 yields

$$\begin{aligned} &[D([R_1, R_2]R_1R_2), t] + [D(R_1R_2[R_2, R_1]), t] - [D([R_2, R_1]R_2R_1), t] \\ &-[D((y_1, y_2, ..., \grave{y}_i, y_i, ..., y_n)(x_2, ..., \grave{x}_i, x_i, ..., x_n)[(y_1, y_2, ..., \grave{y}_i, y_i, ..., y_n), (x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)])x_1 \\ &+\psi(y_n)\Delta((y_1, y_2, ..., \grave{y}_i, y_i, ..., y_{n-1})(x_2, ..., \grave{x}_i, x_i, ..., x_n)[(y_1, y_2, ..., \grave{y}_i, y_i, ..., y_n), (x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)]), t] \\ &= 0. \end{aligned}$$

Without loss of generality, this relation can be rewritten as the following

$$\begin{aligned} &[D([R_1, R_2]R_1R_2), t] + [D(R_1R_2[R_2, R_1]), t] - [D([R_2, R_1]R_2R_1), t] \\ &-[D((y_1, y_2, ..., \grave{y}_i, y_i, ..., y_n)(x_2, ..., \grave{x}_i, x_i, ..., x_n)[(y_1, y_2, ..., \grave{y}_i, y_i, ..., y_n), (x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)])x_1, t] \\ &-[\psi(y_n)\Delta((y_1, y_2, ..., \grave{y}_i, y_i, ..., y_{n-1})(x_2, ..., \grave{x}_i, x_i, ..., x_n)[(y_1, y_2, ..., \grave{y}_i, y_i, ..., y_n)(x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)]), t] \\ &= 0. \end{aligned}$$

Now in view of our hypothesis we see ψ acts as $\psi(R) = a - bR^2$, a and b are fixed element of R. Hence, this relation becomes

$$[D([R_1, R_2]R_1R_2), t] + [D(R_1R_2[R_2, R_1]), t] - [D([R_2, R_1]R_2R_1), t]$$
$$-[D((y_1, y_2, ..., \grave{y}_i, y_i, ..., y_n)(x_2, ..., \grave{x}_i, x_i, ..., x_n))[(y_1, y_2, ..., \grave{y}_i, y_i, ..., y_n), (x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)])x_1, t]$$
$$-[y_n a\Delta((y_1, y_2, ..., \grave{y}_i, y_i, ..., y_{n-1})(x_2, ..., \grave{x}_i, x_i, ..., x_n)[(y_1, y_2, ..., \grave{y}_i, y_i, ..., y_n), (x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)]), t]$$
$$+[by_n^2\Delta((y_1, y_2, ..., \grave{y}_i, y_i, ..., y_{n-1})(x_2, ..., \grave{x}_i, x_i, ..., x_n)[(y_1, y_2, ..., \grave{y}_i, y_i, ..., y_n), (x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)]), t]$$
$$= 0.$$

In particular for $y_n = -y_n$ of this relation. Combining this result with the relations, we arrive to

$$[D([R_1, R_2]R_1R_2), t] + [D(R_1R_2[R_2, R_1]), t] - [D([R_2, R_1]R_2R_1), t]$$
$$-[D((y_1, y_2, ..., \grave{y}_i, y_i, ..., y_n)(x_2, ..., \grave{x}_i, x_i, ..., x_n)[(y_1, y_2, ..., \grave{y}_i, y_i, ..., y_n), (x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)])x_1, t]$$
$$-[a\Delta((y_1, y_2, ..., \grave{y}_i, y_i, ..., y_{n-1})(x_2, x_3, ..., \grave{x}_i, x_i, ..., x_n)[(y_1, y_2, ..., \grave{y}_i, y_i, ..., y_n), (x_1, x_2, ..., \grave{x}_i x_i, ..., x_n)]), t]$$
$$= 0.$$

Obviously, substituting this relation in the above relation, we conclude that

$$[by_n^2\Delta((y_1, y_2, ..., \grave{y}_i, y_i, ..., y_{n-1})(x_2, ..., \grave{x}_i, x_i, ..., x_n)[(y_1, y_2, ..., \grave{y}_i, y_i, ..., y_n), (x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)], t]$$
$$= 0.$$

Writing y_{n-1}, y_n for y_{n-1} of this relation, we obtain

$$[by_n^2\Delta((y_1, y_2, ..., \grave{y}_i, y_i, ..., y_{n-1}, y_n)(x_2, ..., \grave{x}_i, x_i, ..., x_n)[(y_1, y_2, ..., \grave{y}_i, y_i, ..., y_n), (x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n]), t]$$
$$= 0.$$

Putting $t = \Delta((y_1, y_2, ..., \grave{y}_i, y_i, ..., y_{n-1}, y_n)(x_2, ..., \grave{x}_i, x_i, ..., x_n)[(y_1, y_2, ..., \grave{y}_i, y_i, ..., y_n), (x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n])$ of this relation, we observe that

$$[by_n^2, \Delta((y_1, y_2, ..., \grave{y}_i, y_i, ..., y_{n-1}, y_n)(x_2, ..., \grave{x}_i, x_i, ..., x_n)[(y_1, y_2, ..., \grave{y}_i, y_i, ..., y_n), (x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n])]$$
$$\Delta((y_1, y_2, ..., \grave{y}_i, y_i, ..., y_{n-1}, y_n)(x_2, ..., \grave{x}_i, x_i, ..., x_n)[(y_1, y_2, ..., \grave{y}_i, y_i, ..., y_n), (x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n]) = 0.$$

Now left-multiplying by $\Delta((y_1, y_2, ..., \grave{y}_i, y_i, ..., y_{n-1}, y_n)(x_2, ..., \grave{x}_i, x_i, ..., x_n)[(y_1, y_2, ..., \grave{y}_i, y_i, ..., y_n), (x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n])R$ and right-multiplying by $R[by_n^2, \Delta((y_1, y_2, ..., \grave{y}_i, y_i, ..., y_{n-1}, y_n)(x_2, ..., \grave{x}_i, x_i, ..., x_n)[(y_1, y_2, ..., \grave{y}_i, y_i, ..., y_n), (x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n])]$ with applying Lemma 2 and $aRb \subset Z(R), a, b \in R$, we show that

$$\begin{aligned} &\Delta((y_1, y_2, ..., \grave{y}_i, y_i, ..., y_{n-1}, y_n)(x_2, ..., \grave{x}_i, x_i, ..., x_n)[(y_1, y_2, ..., \grave{y}_i, y_i, ..., y_n), (x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n])R \\ &[by_n^2, \Delta((y_1, y_2, ..., \grave{y}_i, y_i, ..., y_{n-1}, y_n)(x_2, ..., \grave{x}_i, x_i, ..., x_n)[(y_1, y_2, ..., \grave{y}_i, y_i, ..., y_n), (x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)])] = 0. \end{aligned} \quad (24)$$

Light of the semiprimeness of R. We take into account the set $\{P_\alpha\}$ of prime ideals of R such that $\cap P_\alpha = \{0\}$. Compatible with Lemma 1, we have the set $\{P_\alpha\}$ of prime ideals of R is semiprime ideal.

Let $\cap P_\alpha = U$. Consequently, the proof divides into two cases which means either $\Delta((y_1, y_2, ..., \grave{y}_i, y_i, ..., y_{n-1}, y_n)(x_2, ..., \grave{x}_i, x_i, ..., x_n)[(y_1, y_2, ..., \grave{y}_i, y_i, ..., y_n), (x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)]) \in U$ or $[by_n^2, \Delta((y_1, y_2, ..., \grave{y}_i, y_i, ..., y_{n-1}, y_n)(x_2, ..., \grave{x}_i, x_i, ..., x_n)[(y_1, y_2, ..., \grave{y}_i, y_i, ..., y_n), (x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)])] \in U$.

It is possible from the first case, we suppose that

$$\Delta((y_1,y_2,...,\acute{y}_i,y_i,...,y_{n-1},y_n)(x_2,...,\acute{x}_i,x_i,...,x_n)$$
$$[(y_1,y_2,...,\acute{y}_i,y_i,...,y_n),(x_1,x_2,...,\acute{x}_i,x_i,...,x_n]) = 0.$$

Utilization Definition 2 and the fact that $\lambda(R) = -R^2$, this relation becomes

$$\begin{aligned} &-\Delta((y_1,y_2,...,\acute{y}_i,y_i,...,y_{n-1},y_n)(x_3,...,\acute{x}_i,x_i,...,x_n)[(y_1,y_2,...,\acute{y}_i,y_i,...,y_n)(x_1,x_2,...,\acute{x}_i,x_i,...,x_n)])x_2^2 \\ &+y_n d((y_1,y_2,...,\acute{y}_i,y_i,...,y_{n-1})(x_2,...,\acute{x}_i,x_i,...,x_n))[(y_1,y_2,...,\acute{y}_i,y_i,...,y_n),(x_1,x_2,...,\acute{x}_i,x_i,...,x_n)]) = 0. \end{aligned} \tag{25}$$

Writing $-x_2$ instead of x_2, we achieve that

$$2y_n d((y_1,y_2,...,\acute{y}_i,y_i,...,y_{n-1})(x_2,...,\acute{x}_i,x_i,...,x_n))[(y_1,y_2,...,\acute{y}_i,y_i,...,y_n),(x_1,x_2,...,\acute{x}_i,x_i,...,x_n)]) = 0.$$

Due to the fact R is 2-torsion free, we find that

$$y_n d((y_1,y_2,...,\acute{y}_i,y_i,...,y_{n-1})(x_2,x_3,...,\acute{x}_i,x_i,...,x_n))$$
$$[(y_1,y_2,...,\acute{y}_i,y_i,...,y_n),(x_1,x_2,...,\acute{x}_i,x_i,...,x_n]) = 0.$$

Using this result in Relation (25), we arrive to

$$\Delta((y_1,y_2,...,\acute{y}_i,y_i,...,y_{n-1},y_n)(x_3,...,\acute{x}_i,x_i,...,x_n)[(y_1,y_2,...,\acute{y}_i,y_i,...,y_n),(x_1,x_2,...,\acute{x}_i,x_i,...,x_n)])x_2^2 = 0.$$

Left-multiplying by x_2^2R and right-multiplying by $R\Delta((y_1,y_2,...,\acute{y}_i,y_i,...,y_{n-1},y_n)(x_2,...,\acute{x}_i,x_i,...,x_n)[(y_1,y_2,...,\acute{y}_i,y_i,...,y_n),(x_1,x_2,...,\acute{x}_i,x_i,...,x_n])$ with applying Lemma 2 and $aRb \subset Z(R), a,b \in R$, we see that

$$x_2^2R\Delta((y_1,y_2,...,\acute{y}_i,y_i,...,y_{n-1},y_n)(x_2,...,\acute{x}_i,x_i,...,x_n)[(y_1,y_2,...,\acute{y}_i,y_i,...,y_n),(x_1,x_2,...,\acute{x}_i,x_i,...,x_n)])$$
$$= 0.$$

Applying similar arguments with necessary variations as used in previous steps of proof, where R is semiprime ring. This means that the action yields either

$$\Delta((y_1,y_2,...,\acute{y}_i,y_i,...,y_{n-1},y_n)(x_2,...,\acute{x}_i,x_i,...,x_n)[(y_1,y_2,...,\acute{y}_i,y_i,...,y_n),(x_1,x_2,...,\acute{x}_i,x_i,...,x_n)]) \in U.$$

Proceeding inductively we conclude that $\Delta((y_1,y_2,...,\acute{y}_i,y_i,...,y_{n-1},y_n)(x_2,...,\acute{x}_i,x_i,...,x_n)[(y_1,y_2,...,\acute{y}_i,y_i,...,y_n),(x_1,x_2,...,\acute{x}_i,x_i,...,x_n)]) = 0$ for all $\acute{x}_i,x_i,\acute{y}_i,y_i \in R$. This yields a contradiction due to the fact $\Delta \neq 0$.

Or $x_2^2 \in U$. From the latter identity, one can easily obtain that $x_2^2 \in$ *M-set*.

Now we return to discuss the second case of Equation (24) which is $[by_n^2,\Delta((y_1,y_2,...,\acute{y}_i,y_i,...,y_{n-1},y_n)(x_2,...,\acute{x}_i,x_i,...,x_n)[(y_1,y_2,...,\acute{y}_i,y_i,...,y_n),(x_1,x_2,...,\acute{x}_i,x_i,...,x_n)])] \in U$. In particular, we find that

$$[by_n^2,\Delta((y_1,y_2,...,\acute{y}_i,y_i,...,y_{n-1},y_n)(x_2,...,\acute{x}_i,x_i,...,x_n)[(y_1,y_2,...,\acute{y}_i,y_i,...,y_n),(x_1,x_2,...,\acute{x}_i,x_i,...,x_n)])]$$
$$= 0$$

for all $\acute{x}_i,x_i,\acute{y}_i,y_i \in R$.

Furthermore, we conclude that

$$[by_n^2,\Delta((y_1,y_2,...,\acute{y}_i,y_i,...,y_{n-1},y_n)(x_3,...,\acute{x}_i,x_i,...,x_n)$$
$$[(y_1,y_2,...,\acute{y}_i,y_i,...,y_n),(x_1,x_2,...,\acute{x}_i,x_i,...,x_n)])\lambda(x_2)]$$

$$+[by_n^2, y_n d((y_1, y_2, ..., \grave{y}_i, y_i, ..., y_{n-1}, y_n)(x_2, ..., \grave{x}_i, x_i, ..., x_n)$$
$$[(y_1, y_2, ..., \grave{y}_i, y_i, ..., y_n), (x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)])] = 0.$$

In agreement with the fact that λ acts as $\lambda(R) = -R^2$ this relation becomes

$$-[by_n^2, \Delta((y_1, y_2, ..., \grave{y}_i, y_i, ..., y_{n-1}, y_n)(x_3, ..., \grave{x}_i, x_i, ..., x_n)$$
$$[(y_1, y_2, ..., \grave{y}_i, y_i, ..., y_n), (x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)])x_2^2]$$
$$+[by_n^2, y_n d((y_1, y_2, ..., \grave{y}_i, y_i, ..., y_{n-1}, y_n)(x_2, ..., \grave{x}_i, x_i, ..., x_n)$$
$$[(y_1, y_2, ..., \grave{y}_i, y_i, ..., y_n), (x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)])] = 0.$$

We use $x_2 = -x_2$. Combining this result with above relation and applying the torsion restriction of R, we conclude that

$$[by_n^2, y_n d((y_1, y_2, ..., \grave{y}_i, y_i, ..., y_{n-1}, y_n)(x_2, ..., \grave{x}_i, x_i, ..., x_n)$$
$$[(y_1, y_2, ..., \grave{y}_i, y_i, ..., y_n), (x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)])] = 0.$$

Of course, substituting this result for the above relation yields

$$[by_n^2, \Delta((y_1, y_2, ..., \grave{y}_i, y_i, ..., y_{n-1}, y_n)(x_3, ..., \grave{x}_i, x_i, ..., x_n)$$
$$[(y_1, y_2, ..., \grave{y}_i, y_i, ..., y_n), (x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)])x_2^2] = 0$$

Putting $R_1 = (x_1, x_2, x_3, ..., \grave{x}_i, x_i, ..., x_n)$, $R_2 = (y_1, y_2, x_3, ..., \grave{y}_i, y_i, ..., y_n)$ and $y_n = b$, this relation changes to

$$\Delta(R_2R_1[R_2, R_1])[b^3, x_2^2] - [b^3, \Delta(R_2R_1[R_2, R_1])]x_2^2 = 0. \tag{26}$$

In Equation (26), writing b instead of x_2, we are forced to conclude that

$$[b^3, \Delta(R_2R_1[R_2, R_1])]b^2 = 0. \tag{27}$$

Right-multiplying (27) by b, we find that

$$[b^3, \Delta(R_2R_1[R_2, R_1])]b_2^3 = 0.$$

Left-multiplying by b^3r and right-multiplying by $r[b^3, \Delta(R_2R_1[R_2, R_1])]$, for all $r \in R$ with applying Lemma 2 and $aRb \subset Z(R), a, b \in R$, it is possible to obtain that

$$b^3R[b^3, \Delta(R_2R_1[R_2, R_1])] = (0).$$

Based on that R is a semiprime ring with the applying similar manner used in our proof and Lemma 2 and $aRb \subset Z(R), a, b \in R$. It is easy to see that either $b^3 \in U$ or $[b^3, \Delta(R_2R_1[R_2, R_1])] \in U$. Without doubt, the first case give us $b^3 = b^4 = 0$ which satisfy that $b^4 \in$ *M-set*. Basically, we have b is non-zero fixed element of R. The second case produces that $[b^3, \Delta(R_2R_1[R_2, R_1])] = 0$.

Now we emphasis on the result which produces by the second case. Applying it in the relation (26) yields $\Delta(R_2R_1[R_2, R_1])[b^3, x_2^2] = 0$. Based on R is semiprime and employing Lemma 1.

When we continue to carry out the same method as previous, then the last equation proved two options. Either $\Delta(R_2R_1[R_2, R_1]) \in U$ yields a contradiction. While the second case $[b^3, x_2^2] \in U$ implies $[b^3, x_2^2] = 0$ for all $x_2 \in R$.

We continue with the second case. Replacing x_2 by $b + x_2$ with using it, we show that

$$[b^3, bx_2 + x_2b] = 0 \tag{28}$$

In that relation $[b^3, x_2^2] = 0$ putting $x_2 = bx_2 + x_2$, we observe that

$$[b^3, (bx_2)^2] + [b^3, bx_2^2 + xbx] + [b^3, x_2^2] = 0.$$

According to the result $[b^3, x_2^2] = 0$, the first and the third terms finish. In meanwhile, we rewrite the middle term as follows

$$[b^3, (bx_2 + x_2b)x_2] = 0.$$

Simplifying this result and using Relation (28), we see that

$$(bx_2 + x_2b)[b^3, x_2] = 0.$$

Taking rx_2 instead of x_2, we achieve that

$$(brx_2 + rx_2b)r[b^3, x_2] = 0$$

for all $x_2, r \in R$.

Since $x_2, r \in R$ so it is possible to obtain that $y = rx_2$ where y was chosen arbitrary form R. Applying this fact of the above relation and using the same previous technique of the proof. Then, we arrive to $by + yb \in U$. In fact, this result leads to ether $b \circ y = 0$ or $[b^3, x_2] \in U$.

Obviously, from the second case $[b^3, x_2] \in U$, we are forced to find that $b^3 \in Z(R)$. The proof is complete. □

Theorem 10. *For any fixed integer $n \geq 2$, let R be a n-torsion free semiprime ring and D be a non-zero permuting n-generalized semiderivation with a trace μ such that $\mu(R) = a - bR$, $a, b \in R$. If R satisfies $[\mu(x), \delta(y)] \in Z(R)$ for all $x, y \in R$, where δ is a trace of Δ then R has a weak zero divisor.*

Proof. As is easily seen from the main identity, there exists $[\mu(x), \delta(y)] \in Z(R)$ for all $x, y \in R$. The trace μ satisfies the following relation

$$\mu(x + y) = \mu(x) + \mu(y) + \sum_{i=1}^{n-1} \binom{n}{1} D(x, x, ..., x, y..., y)$$

where x appears $n - i$-times and y appears i-times.

In the main identity, replacing x by $x + kz$, for $1 \leq n \leq n - 1$, we find that

$$[\mu(x + kz), \delta(y)] \in Z(R)$$

for all $x, y, z \in R$.

Applying the above form on this relation, we notice that

$$[\mu(x) + \mu(kz) + \sum_{i=1}^{n-1} \binom{n}{1} D(x, x, ..., x, kz, kz..., kz), \delta(y)] \in Z(R)$$

for all $x, y, z \in R$.

In agreement with the main relation this equation reduces to

$$[\sum_{i=1}^{n-1} \binom{n}{1} D(x, x, ..., x, kz, kz..., kz), \delta(y)] \in Z(R)$$

Consequently, the above relation yields,

$$k\varphi_1(x, y, z) + k\varphi_1^2(x, y, z) + ... + k^{n-1}\varphi_1^{n-1}(x, y, z) \in Z(R),$$

where $\varphi_i(x, y, z)$, denote to the sum of the terms in which z appears i times.

In agreement with Lemma 3 and due to the fact R is n-torsion free, we arrive to

$$[D(z, x, x, ..., x), \delta(y)] \in Z(R)$$

for all $x, y, z \in R$.

In particular, for z putting x of this relation, we show that $[D(x, x, x, ..., x), \delta(y)] \in Z(R)$ for all $x, y \in R$.

Arguing in a similar style as we have done of the trace ψ in the previous part of our proof. For the trace δ in the relation $[D(x, x, x, ..., x), \delta(y)] \in Z(R)$ for all $x, y \in R$, we achieve that

$$[D(x, x, x, ..., x), \Delta(x, x, x, ..., x)] \in Z(R)$$

for all $x \in R$. Actually, applying of Theorem 8 gives the required result. □

Corollary 3. *For any fixed integer $n \geq 2$, let R be a n-torsion free semiprime ring and D be a non-zero permuting n-generalized semiderivation with a trace ψ such that $\psi(R) = a - bR$, $a, b \in R$. If R satisfies any one of the following conditions:*

(i) $[\psi(x), \Delta(y)] \in Z(R)$ *for all* $x, y \in R$,
(ii) $[D(x), \delta(y)] \in Z(R)$ *for all* $x, y \in R$,

then R has a weak zero divisor.

5. Permuting n-Semigeneralized Semiderivation of (σ,τ)-Semicommutative Semiprime Rings

In this section, we study the connections between permuting n-semigeneralized semiderivation and semicommutative rings is investigated under some conditions. A ring R is called semicommutative if for any $a, b \in R$, $ab = 0$ implies $aRb = 0$. Also, is called central semicommutative if for any $a, b \in R$, $ab = 0$ implies arb is a central element of R for each $r \in R$ that is, $(aRb \subseteq Z(R))$. Obviously, every semicommutative ring is central semicommutative.

In this note, we find there are many researchers worked and attempted to find some results concerning semicommutative rings. G. Shin [57] showed for a ring R the following statements are equivalent: (i) R is semicommutative. (ii) For any $a, b \in R$, $ab = 0$ implies $aRb = 0$.

In Reference [58] Chan Huh et al. have discussed the relation between semicommutative and reduce ring, where R is a semiprime right Goldie ring. More precisely, they studied the following situations: (i) R is a reduced ring. (ii) R is a semicommutative ring while Tahire Özen et al. [59] proved that, if R is a prime central semicommutative ring, then R does not have any non zero divisors of zero. In References [60,61], the authors investigated on another version of semicommutativity is a weakly semicommutativity and (σ,τ)-generalized derivations with their composition of semiprime rings respectively.

The ring R is called weakly semicommutative if for any $a, b \in R$, $ab = 0$ implies arb is nilpotent for any $r \in R$. Clearly, semicommutative rings are weakly semicommutative. There is no implication between nil-semicommutative rings and weakly semicommutative rings. Furthermore, L. Wang and J. C. Wei [62], introduced a class of rings (we called it a central semicommutative ring) in which if $ab = 0$ implies that arb is central.

For more convenience, we suppose all the results of this section satisfies the relation $aRb \subset Z(R), a, b \in R$. Except Theorem 13.

Now we give the main definition of this section.

Definition 7. *Let R be non empty semicommutative ring with the centre $Z(R)$, the mappings σ and τ are automorphism mapping of R, then R is called (σ,τ)-semicommutative ring if for all $x, y \in R$, $\sigma(x)\tau(y) = 0$ implies $\sigma(x)R\tau(y) = 0$.*

Moreover, a ring R is called central (σ,τ)-semicommutative ring, if for any $x, y \in R$, $\sigma(x)\tau(y) = 0$ implies $\sigma(x)R\tau(y) \subseteq Z(R)$.

Example 3. *Let $R = M_n(\mathbb{F})$ be a ring of $n \times n$ matrices over a field $\mathbb{F}$, $n > 1$ that is:*

$$R = \begin{pmatrix} x_j & x_{j+1} & x_{j+2} & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

for all $x_j \in \mathbb{F}$, $j = 1, 2, ..., n$. Suppose σ and τ are automorphism mappings of R, given by

$$\sigma(R) = \begin{pmatrix} 0 & x_j & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \text{ and } \tau(R) = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ x_j & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Obviously, we achieve the relation $\sigma(x)R\tau(y) = 0$. that is, R is (σ,τ)-semicommutative ring.

Hence, we may show the similar results to a second case central (σ,τ)-semicommutative ring. Due to any arbitrary element of R, we conclude that

$$[\sigma(x)R\tau(y), r] = 0$$

for all $r \in R$. This result indicates to $\sigma(x)R\tau(y) \subseteq Z(R)$.

In this section σ and τ are automorphism mapping of R unless mentioned otherwise.

Theorem 11. *Let R be a 2-torsion free central (σ,τ)-semicommutative semiprime ring, U be an ideal of R and Ω be a non-zero permuting n-semigeneralized semiderivation associated with automorphism functions g and h of R such that $g(R) = Ra - RbR$, $a, b \in R$. Then either $a = 0$ or $d(R) = 0$ or $\Omega_1(R)$ and $\Omega_2(R)$ are commuting.*

Proof. According to our hypothesis, we have R is 2-torsion free central (σ,τ)-semicommutative semiprime and Ω is a permuting n-semigeneralized semiderivation yields that

$$\sigma(\Omega_1(R_1))R\tau(\Omega_2(R_2)) \subseteq Z(R).$$

Obviously, the previous identity can be rewritten as the following

$$\sigma(\Omega_1((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_1, y_2..., \grave{y}_i, y_i, ..., y_n))R\tau(\Omega_2(R_2)) \subseteq Z(R)$$

for all $\grave{x}_i, x_i, \grave{y}_i, y_i \in R$.

Due to the fact t was chosen arbitrary from R, this proved that

$$\begin{aligned} [\sigma(\Omega_1((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2..., \grave{y}_i, y_i, ..., y_n))g(y_1) + h(x_n)D((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1}) \\ (y_1, y_2..., \grave{y}_i, y_i, ..., y_n))R\tau(\Omega_2(R_2)), t] = (0). \end{aligned} \tag{29}$$

By reason of σ is automorphism and the fact that $g(R) = Ra - RbR$, $a, b \in R$, we notice that

$$\begin{aligned} [(\Omega_1((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2..., \grave{y}_i, y_i, ..., y_n))(a - y_1 b) + h(x_n)D((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1}) \\ (y_1, y_2..., \grave{y}_i, y_i, ..., y_n)))R\tau(\Omega_2(R_2)), t] = 0. \end{aligned} \tag{30}$$

Replacing y_1 by $-y_1$ and combining this result with the above relation, we find that

$$2[\Omega_1((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2, y_3, ..., \grave{y}_i, y_i, ..., y_n)aR\tau(\Omega_2(R_2), t] = 0. \tag{31}$$

Applying the fact that R is 2-torsion free property and substituting this result of Equation (30), we conclude that

$$[\Omega_1((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2, y_3, ..., \grave{y}_i, y_i, ..., y_n))aR\tau(\Omega_2(R_2), t] = 0. \tag{32}$$

Moreover, we extend this expression to

$$\begin{aligned}
&\Omega_1((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2..., \grave{y}_i, y_i, ..., y_n))a[\tau(\Omega_2(R_2)), t] \\
&+\Omega_1((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2..., \grave{y}_i, y_i, ..., y_n)[a, t]\tau(\Omega_2(R_2)) \\
&+[\Omega_1((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2..., \grave{y}_i, y_i, ..., y_n), t]a\tau(\Omega_2(R_2)) = 0.
\end{aligned}$$

Further, replacing y_2 by y_1y_2, for σ is a automorphism of R, we find that

$$\sigma(\Omega_1(R_1))a[\tau(\Omega_2(R_2))), t]] + \sigma(\Omega_1(R_1))[a, t]\tau(\Omega_2(R_2))) + [\sigma(\Omega_1(R_1)), t]a\tau(\Omega_2(R_2))) = 0. \tag{33}$$

Employing Lemma 4 and using the fact that R is 2-torsion free, from the main relation $\sigma(\Omega_1(R_1))R\tau(\Omega_2(R_2)) \subseteq Z(R)$, we conclude that $\sigma(\Omega_1(R_1))\tau(\Omega_2(R_2)) \subseteq Z(R)$.
Also, by the same way we receive that $\sigma(\Omega_2(R_2))\tau(\Omega_1(R_1)) \subseteq Z(R)$. This yields

$$[\sigma(\Omega_1(R_1)), \tau(\Omega_2(R_2))] \subseteq Z(R) \tag{34}$$

Furthermore, in Relation (33) replacing t by $\tau(\Omega_2(R_2)$, we conclude that

$$\sigma(\Omega_1(R_1))[a, \tau(\Omega_2(R_2))]\tau(\Omega_2(R_2)) + [\sigma(\Omega_1(R_1)), \tau(\Omega_2(R_2))]b\tau(\Omega_2(R_2)) = 0. \tag{35}$$

Applying Relation (34) of (35), we find that

$$\sigma(\Omega_1(R_1))[a, \tau(\Omega_2(R_2))]\tau(\Omega_2(R_2)) + a\tau(\Omega_2(R_2))[\sigma(\Omega_1(R_1)), \tau(\Omega_2(R_2)] = 0.$$

Now if we continue to carry out the same method as above, we arrive to

$$\sigma(\Omega_1(R_1))a\tau(\Omega_2(R_2)^2) - \sigma(\Omega_1(R_1))\tau(\Omega_2(R_2))b\tau(\Omega_2(R_2) + a\tau(\Omega_2(R_2)[\sigma(\Omega_1(R_1)), \tau(\Omega_2(R_2))] = 0. \tag{36}$$

Here, simplifying the terms $\sigma(\Omega_1(R_1))a\tau(\Omega_2(R_2)^2) - \sigma(\Omega_1(R_1))\tau(\Omega_2(R_2))a\tau(\Omega_2(R_2)$. Replacing y_2 by y_1y_2 of Equation (32) give us $[\Omega_1(R_1)aR\tau(\Omega_2(R_2), t] = 0$. Using this result and

$$\sigma(\Omega_2(R_2))\tau(\Omega_1(R_1)) \in Z(R),$$

we find that

$$\begin{aligned}
&\sigma(\Omega_1(R_1))b\tau(\Omega_2(R_2)^2) - \sigma(\Omega_1(R_1))\tau(\Omega_2(R_2))a\tau(\Omega_2(R_2) \\
&= \sigma(\Omega_1(R_1))a\tau(\Omega_2(R_2))\tau(\Omega_2(R_2)) - a\tau(\Omega_2(R_2)\sigma(\Omega_1(R_1))\tau(\Omega_2(R_2)) \\
&= \tau(\Omega_2(R_2)\sigma(\Omega_1(R_1))a\tau(\Omega_2(R_2)) - a\tau(\Omega_2(R_2)\sigma(\Omega_1(R_1))\tau(\Omega_2(R_2)) \\
&= a\tau(\Omega_2(R_2))\tau(\Omega_2(R_2)\sigma(\Omega_1(R_1)) - a\tau(\Omega_2(R_2))\tau(\Omega_2(R_2)\sigma(\Omega_1(R_1)) \\
&= 0.
\end{aligned}$$

Consequently, this result reduces Equation (36) to

$$a\tau(\Omega_2(R_2)[\sigma(\Omega_1(R_1)),\tau(\Omega_2(R_2))]=0. \tag{37}$$

From Relation (37), where σ and τ are automorphism mappings, we satisfy that

$$a\Omega_2(R_2)[\Omega_1(R_1),\Omega_2(R_2)]=0. \tag{38}$$

Left-multiplying by $[\Omega_1(R_1),\Omega_2(R_2)]t$ and right-multiplying by $ta\Omega_2(R_2)$, with using Lemma 2 and $aRb\subset Z(R), a,b\in R$, this establishes that $a\Omega_2(R_2)R[\Omega_1(R_1),\Omega_2(R_2)]=(0)$.

Light of the semiprimeness of R by using the similar arguments as utilized in the previous theorems, we we observe that either $a\Omega_2(R_2)\in U$ or $[\Omega_1(R_1),\Omega_2(R_2)]\in U$. Now we separate the proof by two cases:

Case I: We have the relation $a\Omega_2(R_2)\in U$ which implies to $a\Omega_2(R_2)=0$. Taking $(x_1,x_2,...,\grave{x}_i,x_i,...,x_n)(y_1,y_2...,\grave{y}_i,y_i,...,y_n)$ instead of R_2, we notice that

$$a(\Omega_2((x_1,x_2,...,\grave{x}_i,x_i,...,x_n)(y_2...,\grave{y}_i,y_i,...,y_n))g(y_1)$$

$$+h(x_n)D((x_1,x_2,...,\grave{x}_i,x_i,...,x_{n-1})(y_1,...,\grave{y}_i,y_i,...,y_n))=0.$$

for all $\grave{x}_i,x_i,\grave{y}_i,y_i\in R$.

Replacing R_2 by $(x_1,x_2,...,\grave{x}_i,x_i,...,x_n)(y_2...,\grave{y}_i,y_i,...,y_n)$ in the relation $a\Omega_2(R_2)=0$, this yields

$$ah(x_n)D((x_1,x_2,...,\grave{x}_i,x_i,...,x_{n-1})(y_1...,\grave{y}_i,y_i,...,y_n)=0. \tag{39}$$

This relation can be rewritten as

$$h(x_n)D((x_1,x_2,...,\grave{x}_i,x_i,...,x_{n-1})(y_2...,\grave{y}_i,y_i,...,y_n)y_1+h(x_n)\psi(x_{n-1})$$

$$\Delta((x_1,x_2,...,\grave{x}_i,x_i,...,x_{n-2})(y_2,y_3,...,\grave{y}_i,y_i,...,y_n)=0.$$

Replacing y_2 by y_1y_2 and employing Equation (39), we arrive to

$$ah(x_n)\psi(x_{n-1})\Delta((x_1,x_2,...,\grave{x}_i,x_i,...,x_{n-2})(y_1,y_2,y_3,...,\grave{y}_i,y_i,...,y_n)=0.$$

By the same argument, it used in previous steps and using Definition 7, we obtain the identity

$$ah(x_n)\psi(x_{n-1})x_{n-2}d((x_1,x_2,...,\grave{x}_i,x_i,...,x_{n-3})(y_2,y_3,...,\grave{y}_i,y_i,...,y_n))=0.$$

Particularly, this relation shows that

$$ah(x_n)\psi(x_{n-1})Rd((x_1,x_2,...,\grave{x}_i,x_i,...,x_{n-3})(y_2,y_3,...,\grave{y}_i,y_i,...,y_n))=0.$$

Based on that R is central $(\sigma\,,\tau)$-semicommutative semiprime. Applying Lemma 1 and the similar processing of previous theorems, we deduce that:

$$\text{either } ah(x_n)\psi(x_{n-1})\in U \text{ or } d((x_1,x_2,...,\grave{x}_i,x_i,...,x_{n-3})(y_2,y_3,...,\grave{y}_i,y_i,...,y_n))\in U.$$

As for the first case, we know that R has not zero divisors, $h(x_n)$ and $\psi(x_{n-1})$ both are non-zero mappings, that indicate to $a=0$.

In the second case, replacing x_{n-3} by x_{n-2},x_{n-1},x_n and y_2 by y_1y_2, this yields $d(R)=0$.

Case II: It is obvious that $[\Omega_1(R_1),\Omega_2(R_2)]=0$. This show that $\Omega_1(R_1)$ and $\Omega_2(R_2)$ are commuting. □

Theorem 12. *Let R be a 2-torsion free semicommutative semiprime ring, U an ideal of R and Ω be a non-zero permuting n-semigeneralized semiderivation associated with automorphism functions g and h of R such that $h(R) = a - Rb, 0 \neq a, b \in R$. Then either $D_1(R_1) = 0$ or $D_2(R_2) = 0$.*

Proof. The hypothesis indicates that R is a 2-torsion free semicommutative semiprime ring and Ω is a non-zero permuting n-semigeneralized semiderivation. Hence, R satisfies the identity

$$\Omega_1(R_1)R\Omega_2(R_2) = \{0\}.$$

For this relation, left-multiplying by $\Omega_2(R_2)$ and right-multiplying by $\Omega_1(R_1)$ with using the fact that R is semicommutative semiprime, we find that

$$\Omega_2(R_2)\Omega_1(R_1) = \{0\}.$$

Writing $(x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_1, y_2, ..., \grave{y}_i, y_i, ..., y_n)$ instead of R_2 in this relation, we achieve that

$$(\Omega_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2..., \grave{y}_i, y_i, ..., y_n)g_2(y_1)$$

$$+h_2(x_n)D_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1..., \grave{y}_i, y_i, ..., y_n))\Omega_1(R_1) = 0$$

for all $\grave{x}_i, x_i, \grave{y}_i, y_i \in R$.

By reason of $h(R) = a - Rb, a, b \in R$. It follows that

$$(\Omega_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2..., \grave{y}_i, y_i, ..., y_n)g_2(y_1)$$

$$+aD_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1..., \grave{y}_i, y_i, ..., y_n)$$

$$-bx_nD_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1..., \grave{y}_i, y_i, ..., y_n))\Omega_1(R_1) = 0.$$

Now in this relation putting $-x_n$ instated of x_n . Combining this result with the above relation, we find that

$$2aD_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1..., \grave{y}_i, y_i, ..., y_n))\Omega_1(R_1) = 0.$$

Since R is 2-torsion free and taking R_1 for $(t_1, t_2, ..., \grave{t}_i, t_i, ..., t_n)(r_1, r_2, ..., \grave{r}_i, r_i, ..., r_n)$, we achieve that

$$aD_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1..., \grave{y}_i, y_i, ..., y_n))(\Omega_1((t_1, t_2, ..., \grave{t}_i, t_i, ..., t_n)(r_2..., \grave{r}_i, r_i, ..., r_n)g_1(r_1)$$

$$+h_1(t_n)D_1((t_1, t_2, ..., \grave{t}_i, t_i, ..., t_{n-1})(r_1..., \grave{r}_i, r_i, ..., r_n)) = 0$$

for all $\grave{x}_i, x_i, \grave{y}_i, y_i, \grave{t}_i, t_i, \grave{r}_i, r_i \in R$.

Based on the fact that $h(R) = a - Rb$, we see that

$$aD_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1..., \grave{y}_i, y_i, ..., y_n))(\Omega_1((t_1, t_2, ..., \grave{t}_i, t_i, ..., t_n)(r_2..., \grave{r}_i, r_i, ..., r_n)g_1(r_1)$$

$$+aD_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1..., \grave{y}_i, y_i, ..., y_n))aD_1((t_1, t_2, ..., \grave{t}_i, t_i, ..., t_{n-1})(r_1..., \grave{r}_i, r_i, ..., r_n))$$

$$-aD_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1..., \grave{y}_i, y_i, ..., y_n))bt_nD_1((t_1, t_2, ..., \grave{t}_i, t_i, ..., t_{n-1})(r_1..., \grave{r}_i, r_i, ..., r_n)) = 0.$$

Taking $-t_n$ instead of t_n of this relation. Combining the result with this relation, we conclude that

$$2aD_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1..., \grave{y}_i, y_i, ..., y_n))aD_1((t_1, t_2, ..., \grave{t}_i, t_i, ..., t_{n-1})(r_1..., \grave{r}_i, r_i, ..., r_n)) = 0.$$

Using the fact R is 2-torsion free. This reason reduces this relation to

$$aD_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1..., \grave{y}_i, y_i, ..., y_n))aD_1((t_1, t_2, ..., \grave{t}_i, t_i, ..., t_{n-1})(r_1..., \grave{r}_i, r_i, ..., r_n)) = 0.$$

Left-multiplying by $aD_1((t_1, t_2, ..., \grave{t}_i, t_i, ..., t_{n-1})(r_1 ..., \grave{r}_i, r_i, ..., r_n))R$ and right-multiplying by $RaD_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1 ..., \grave{y}_i, y_i, ..., y_n))$ with using Lemma 2 and $aRb \subset Z(R)$, we observe that

$$aD_1((t_1, t_2, ..., \grave{t}_i, t_i, ..., t_{n-1})(r_1 ..., \grave{r}_i, r_i, ..., r_n))RaD_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1 ..., \grave{y}_i, y_i, ..., y_n)) = 0.$$

Light of the semiprimeness of R. Suppose we continue to carry out the same style as the proof of Theorem 6, we arrive to two cases:

$$\text{Either } aD_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1 ..., \grave{y}_i, y_i, ..., y_n)) \in U$$

$$\text{or } aD_1((t_1, t_2, ..., \grave{t}_i, t_i, ..., t_{n-1})(r_1 ..., \grave{r}_i, r_i, ..., r_n)) \in U.$$

Based on $a \neq 0$ the first case leads to $D_1((t_1, t_2, ..., \grave{t}_i, t_i, ..., t_{n-1})(r_1 ..., \grave{r}_i, r_i, ..., r_n)) = 0$. While in the second case, we notice that $D_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1 ..., \grave{y}_i, y_i, ..., y_n)) = 0$. The proof of theorem is finished. □

Theorem 13. *Let R be a 2-torsion free (σ,τ)-ring without zero divisors and Ω be a non-zero permuting n-semigeneralized semiderivation associated with a automorphism functions g and h of R such that $g(R) = a - Rb$, $a, b \in R$. If R satisfies $[\sigma(\Omega_1(R_1)) \circ \tau(\Omega_2(R_2)), b] = 0$ then either $\Omega_1(R_1)) \circ \Omega_2(R_2) = 0$ or $[a, b] = 0$.*

Proof. Due to the hypothesis, we have $[\sigma(\Omega_1(R_1)) \circ \tau(\Omega_2(R_2)), b] = 0$.

Replacing R_2 by $(x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_1 ..., \grave{y}_i, y_i, ..., y_n)$, for all $\grave{x}_i, x_i, \grave{y}_i, y_i \in R$.

Immediately following the relation

$$[\sigma(\Omega_1(R_1)) \circ \tau((\Omega_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2 ..., \grave{y}_i, y_i, ..., y_n))g_2(y_1)$$
$$+h_2(x_n)D(((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1 ..., \grave{y}_i, y_i, ..., y_n)))), b] = 0.$$

Since $g_2(R)$ is $g_2(R) = a - Rb$ and τ is automorphism, then this relation becomes

$$[\sigma(\Omega_1(R_1))(\Omega_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2 ..., \grave{y}_i, y_i, ..., y_n))(a - y_1 b)$$
$$+h_2(x_n)D((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1 ..., \grave{y}_i, y_i, ..., y_n))$$
$$+(\Omega_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2 ..., \grave{y}_i, y_i, ..., y_n))(a - y_1 b)$$
$$+h_2(x_n)D((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1 ..., \grave{y}_i, y_i, ..., y_n))\sigma(\Omega_1(R_1)), b] = 0.$$

Moreover, it seems very likely that

$$[\sigma(\Omega_1(R_1)) \circ ((\Omega_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2 ..., \grave{y}_i, y_i, ..., y_n))a$$
$$-\Omega_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2 ..., \grave{y}_i, y_i, ..., y_n))y_1 b$$
$$+h(x_n)D((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1 ..., \grave{y}_i, y_i, ..., y_n))), b] = 0.$$

Writing $-y_1$ instead of y_1 in this relation. Combining the result and this equation gives the relation

$$2[\sigma(\Omega_1(R_1)) \circ ((\Omega_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2 ..., \grave{y}_i, y_i, ..., y_n))a, b] = 0.$$

Furthermore, it is possible this relation modifying

$$\sigma(\Omega_1(R_1)) \circ ((\Omega_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2 ..., \grave{y}_i, y_i, ..., y_n))[a, b]$$

$$+[\sigma(\Omega_1(R_1)) \circ ((\Omega_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2..., \grave{y}_i, y_i, ..., y_n)), b]a = 0.$$

Replacing y_2 by y_1y_2 and using the main relation, we arrive to

$$\Omega_1(R_1) \circ \Omega_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2..., \grave{y}_i, y_i, ..., y_n))[a, b] = 0.$$

Due to the fact that R is without zero divisors, we satisfy the tow cases. Either $\sigma(\Omega_1(R_1)) \circ \tau(\Omega_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2..., \grave{y}_i, y_i, ..., y_n))) = 0$ or $[a, b] = 0$.

Since σ and τ are automorphisms, the first case proved that

$$\Omega_1(R_1)) \circ \Omega_2(R_2) = 0.$$

The proof of theorem is finished. □

6. Permuting Skew n-Antisemigeneralized Semiderivation of Anticommutative Semiprime Rings

In this section, we investigate the relations between the permuting skew n-antisemigeneralized semiderivation with associative rings via anticommutative semiprime ring . Let R denoted an arbitrary associative ring, then a ring R is said anticommutative if $xy = -yx$ for all $x, y \in R$. If a ring R satisfies $x^2 = 0$ for all $x \in R$. Obversely, R is anticommutative ring, but not conversely. Suppose that R is anticommutative. Then it is easy to show that

(i) $((xy - yx)^2 = 0,$
(ii) $(x_2y - yx^2) = 0.$

Note that in an anticommutative ring R, $x^2 = -x^2$ so that $2x^2 = 0$ for all $x \in R$. For each $x, y \in R$, we define the anti-center $\hat{Z}(R)$ of R by $\hat{Z}(R) = \{x \in R \setminus xoy = 0, \forall y \in R\}$. Call R anti-commutative if $\hat{Z}(R) = R$. There are several researchers work in the area of anticommutative rings. In Reference [63], D. MacHale shown that a ring R is anticommutative if and only if for each $x, y \in R$ then there exists an even integer $n(x, y) > 1$ such that $\langle x, y\rangle^{n(x,y)} = \langle x, y\rangle$. For R is a ring not necessarily with 1, then Yen [64] has proved that if, for every $x, y \in R$, either $xy - yx$ is potent or $xy + yx$ is strongly potent then R is either commutative or anticommutative. Call $x \in R$ potent (strongly potent) if $x^n = x$ for some natural (even natural number) $n > 1$. M. S. Putcha, R. S. Wilson and A. Yaqub [65] considered that for each $x, y \in R$, there exists $w = w(x, y) \in \hat{Z}(R)$ such that $(x \circ y)^2 w = x \circ y$. Then R is anti-commutative.

Additionally, A. B. Thaheem [66] revealed that R is an anticommutative semiprime ring. Then it is commutative. Later, Stephen M. Buckley and Desmond MacHale [67] they proved that $ACP(4)$ rings are anticommutative, where we call R a $ACP(S)$ ring, wheresoever $S \subset 2\mathbb{N}$ if, for each $x, y \in R$, t here exists $n(x, y) \in S$ such that $\langle x, y\rangle = \langle x, y\rangle^{n(x,y)}$. Other authors have obtained commutativity and anticommutative rings (viz. [68,69] where further references can be found).

For anticommutative semiprime ring, we post the answer to the question, what is the role of permuting skew n-antisemigeneralized semiderivation in associative rings via anticommutative semiprime.

Theorem 14. *Let R be anticommutative semiprime ring. Let a be a fixed element of R and γ be a permuting skew n-antisemigeneralized semiderivation associated with a automorphism function g of R such that $[a, \sum_{i=1}^{n} \gamma_i(R_i)] = 0$ and $n \geq 1$. Then either $\gamma_1(R) = 0$ or a commute of R.*

Proof. From our hypothesis, we have the main relation $[a, \sum_{i=1}^{n} \gamma_i(R_i)] = 0$, $n \geq 1$.

Particularly, we write the previous relation as

$$[a, \gamma_1(R_1) + \sum_{i=2}^{n} \gamma_i(R_i)] = 0.$$

Replacing R_1 by $(x_1, x_2, x_k' x_k..., x_n)(y_1, y_2, y_k' y_k..., y_n)$, $k = 1, 2, ..., n$ with applying Definition 5, we find that

$$[a, \gamma_1(x_1, x_2, x_k' x_k..., x_n, y_2, y_k' y_k..., y_n) y_1] + [a, g(x_n) D(x_1, x_2, x_k' x_k..., x_{n-1}, y_1, y_2, y_k' y_k..., y_n) = [\sum_{i=2}^{n} \gamma_i(R_i), a]. \quad (40)$$

Now we employ the main relation for $n = 1$ yields

$$[a, \gamma_1(R_1)] = 0. \quad (41)$$

In (41), replacing R_1 by $(x_1, x_2, x_k' x_k..., x_n, y_2, y_k' y_k..., y_n)$ for all $x_k', x_k, y_k', y_k \in R, k = 1, 2, ..., n$ and applying the result of the first term of the left side (40), we observe that

$$\gamma_1(x_1, x_2, x_k' x_k..., x_n, y_2, y_k' y_k..., y_n)[a, y_1] = [\sum_{i=2}^{n} \gamma_i(R_i), a] - [a, g(x_n) D(x_1, x_2, x_k' x_k..., x_{n-1}, y_1, y_2, y_k' y_k..., y_n)]. \quad (42)$$

For this relation, we add the term $\mp y_1 a \gamma_1(x_1, x_2, x_k' x_k..., x_n, y_2, y_k' y_k..., y_n)$ to the left-side with employing the fact that R is anticommutative semiprime ring and Relation (39), we achieve that

$$-(\gamma_1(x_1, x_2, x_k' x_k..., x_n, y_2, y_k' y_k..., y_n) y_1 a + y_1 a \gamma_1(x_1, x_2, x_k' x_k..., x_n, y_2, y_k' y_k..., y_n))$$

$$= [\sum_{i=2}^{n} \gamma_i(R_i), a] - [a, g(x_n) D(x_1, x_2, x_k' x_k..., x_{n-1}, y_2, y_k' y_k..., y_n)]$$

for all $x_k', x_k, y_k', y_k \in R, k = 1, 2, ..., n$.

Applying Equation (40) to the left side of this relation, we find that

$$-(\gamma_1(x_1, x_2, x_k' x_k..., x_n, y_2, y_k' y_k..., y_n) y_1 + y_1 \gamma_1(x_1, x_2, x_k' x_k..., x_n, y_2, y_k' y_k..., y_n)) a$$

$$= [a, \sum_{i=2}^{n} \gamma_i(R_i)] - [a, g(x_n) D(x_1, x_2, x_k' x_k..., x_{n-1}, y_2, y_k' y_k..., y_n)]$$

for all $x_k', x_k, y_k', y_k \in R, k = 1, 2, ..., n$.

Consequently, the fact that R is anticommutative semiprime yields

$$[\sum_{i=2}^{n} \gamma_i(R_i), a] - [a, g(x_n) D(x_1, x_2, x_k' x_k..., x_{n-1}, y_2, y_k' y_k..., y_n)] = 0$$

for all $x_k', x_k, y_k', y_k \in R, k = 1, 2, ..., n$.

Substituting this relation of Equation (42), we find that

$$\gamma_1(x_1, x_2, x_k' x_k..., x_n, y_2, y_k' y_k..., y_n)[a, y_1] = 0.$$

Replacing y_1 by $Rt, t \in R$, we find that

$$\gamma_1(x_1, x_2, x_k' x_k..., x_n, y_2, y_k' y_k..., y_n) R[a, t] = (0).$$

Due to R is a anticommutative semiprime ring, we regard the set $\{P_\alpha\}$ of prime ideals of R such that $\cap P_\alpha = \{0\}$.

According to Lemma 1, we obtain the set $\{P_\alpha\}$ of prime ideals of R is semiprime ideal.

Let $\cap P_\alpha = U$. Hence, we have either $\gamma_1(x_1, x_2, x_k' x_k..., x_n, y_2, y_k' y_k..., y_n) \in U$ for all $x_k', x_k, y_k', y_k \in R, k = 1, 2, ..., n$. Or $[a, t] \in U$ for all $t \in R$.

The case $[a, t] \in U$ for all $t \in R$ impels to $[a, t] = 0$. Consequently, we obtain either $\gamma_1(R) = 0$ or a commute of R. $\square$

Applying a similar approach as above one can prove the following corollary.

Corollary 4. *Let R be an anticommutative semiprime ring, $a \in R$ and γ be a non-zero permuting skew n-antisemigeneralized semiderivation associated with automorphism function g of R such that $[a, \sum_{i=1}^{n} \gamma_i(R_i)] = 0$ and $n \geq 1$. Then $a \in Z(R)$.*

Proof. View of the style of the proof of Theorem 14, we immediately achieve the cases either $\gamma_1(x_1, x_2, x_k^{'} x_k ..., x_n, y_2, y_k^{'} y_k ..., y_n) \in U$, where $\gamma(R) \neq 0$, yields a contradiction. Or $[a, t] \in U$ for all $t \in R$. Hence, we obtain a lies in $Z(R)$. The proof of corollary is completed . □

Theorem 15. *Let R be anticommutative semiprime ring and γ be a permuting skew n-antisemigeneralized semiderivation associated with automorphism function g of R such that $[\sum_{i=1}^{n} D_i(R_i), \sum_{i=1}^{n} \gamma_i(R_i)] = 0$ and $n \geq 1$. Then either $\sum_{i=1}^{n} \gamma_1(R) = 0$ or $D(R) = 0$ or R is commutative.*

Proof. By using similar argument about the proof of the Theorem 14, we arrive to either $\sum_{i=1}^{n} D_i(R_i) = 0$ or $\gamma_i(R_i) = 0$.

Obviously, the first case provides us the relation $[\sum_{i=1}^{n} D_i(R_i), R] = 0$.

Again, applying the same previous technique of the proof of Theorem 14 and using the fact that $R \neq 0$ implies to either $D(R) = 0$ or R is commutative. The proof is completed . The proof of theorem is finished. □

Employing a similar technique with some necessary variations one can prove the following corollary.

Corollary 5. *Let R be an anticommutative semiprime ring and γ be a non-zero permuting skew n-antisemigeneralized semiderivation associated with automorphism function g of R such that $\sum_{i=1}^{n}[D_i(R_i), \gamma_i(R_i)] = 0$ and $n \geq 1$. Then $D(R)$ commute with R.*

Corollary 6. *Any M-set has anticommutative property.*

Proof. Basically, any element belongs to *M-set* satisfies the relation $0 \neq a \in R$, $a^2 = 0$.

Linearization of this equation and using it yields $xy + yx = 0$ for all $x, y \in R$.

Without doubt, *M-set* satisfies the anticommutative property. □

The second branch of the following theorem show the properties anticommutative and commutative coincide of a 2-torsion free semiprime ring R.

Theorem 16. *Let R be a 2-torsion free semiprime ring and Δ be a permuting n-semiderivation with a trace δ such that Δ acts as a homomorphism. Suppose that R admits Δ satisfying the identity $\Delta(R_1) \circ \delta(R_2) \mp (R_1 \circ R_2) \in Z(R)$. If*

(i) δ acts as a surjective mapping then Δ is commuting(resp. centralizing) of R.
(ii) $\delta = 0$ or $\Delta = 0$ then an anticommutative and commutative coincide of R.

Proof. (i) By assumption, we have the main relation $\Delta(R_1) \circ \delta(R_2) \mp (R_1 \circ R_2) \in Z(R)$.

This expression can be rewritten as $[\Delta(R_1) \circ \delta(R_2), r] + [(R_1 \circ R_2), r] = 0$ for all $r \in R$.

Replacing R_2 by R_1, we conclude that $[\Delta(R_1) \circ \delta(R_1), r] + 2[R_1^2, r] = 0$ for all $r \in R$.

If we take R_1 instead of r in this relation, we achieve that $[\Delta(R_1) \circ \delta(R_1), R_1] = 0$.

Compatibility between the two facts Δ is a homomorphism mapping with the trace δ is a surjective mapping, the above relation modifies to

$$[\Delta(R_1)^2, R_1] = 0. \tag{43}$$

Linearization of this equation with using it implies to

$$[\Delta(R_1^2), R_2] + [\Delta(R_2^2), R_1] + [\Delta(R_1 + R_2), R_1 + R_2] = 0.$$

Based on the fact that R is 2-torsion free and replacing R_2 by R_1 and applying relation (43), we arrive to $[\Delta(R_1), R_1] = 0$. Consequently, we observe that Δ is commuting(resp. centralizing) of R. The proof of this branch is complete.

(ii) First of all, the main relation reduces to $(R_1 \circ R_2) \in Z(R)$.

This relation can be rewritten as $[R_1 \circ R_2, r] = 0$ for all $r \in R$.

Putting R_2 for R_1 and $R_1^2 r$ for r and using the fact that R is 2-torsion free. Immediately, it follows that $R_1^2[R_1^2, r] = 0$ for all $r \in R$.

For any arbitrary element $t \in R$. Replacing r by tr of this relation, we observe $R_1^2 t[R_1^2, r] = 0$.

In agreement with our hypothesis R is a semiprime ring, we acknowledge the set $\{P_\alpha\}$ of prime ideals of R such that $\cap P_\alpha = \{0\}$. Compatible with Lemma 1, we observe the set $\{P_\alpha\}$ of prime ideals of R is semiprime ideal. Let $\cap P_\alpha = U$. We hold either $R_1^2 \in U$ that is, $R_1^2 = 0$.

Linearization this relation and using it gives $R_1 \circ R_2 = 0$. Obviously, R is anticommutative . Or $[R_1^2, r] \in U$ for all $r \in R$ implies $[R_1^2, r] = 0$.

Repeating the same previous approach of the first case and applying the fact R is 2-torsion free, we conclude that R is commutative. Hence, the theorem is proved. □

Theorem 17. *Let R be a 2-torsion free anticommutative ring without zero divisors and γ be a permuting skew n-antisemigeneralized semiderivation associated with automorphism function g of R such that $g(R) = a - Rb$, $a, b \in R$. If R satisfies the identity $[[\sigma(\gamma_1(R_1)), \tau(\gamma_2(R_2))], b] = 0$ then either $[a, b] = 0$ or $2D_1(R)aD_2(R) = D_2(R)aD_1(R)$.*

Proof. Without loss of generality, suppose that $[[\sigma(\gamma_1(R_1)), \tau(\gamma_2(R_2))], b] = 0$. Taking $R_2 = (x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_1, y_2 ..., \grave{y}_i, y_i, ..., y_n)$ for all $\grave{x}_i, x_i, \grave{y}_i, y_i \in R$. Based on τ is an automorphism, one can easily find that

$$[[\sigma(\gamma_1(R_1)), \gamma_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2 ..., \grave{y}_i, y_i, ..., y_n))y_1], b]$$

$$+[[\sigma(\gamma_1(R_1)), g_2(x_n)D_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1, y_2 ..., \grave{y}_i, y_i, ..., y_n)), b] = 0.$$

In the main relation $[[\sigma(\gamma_1(R_1)), \tau(\gamma_2(R_2))], b] = 0$, putting $(x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2 ..., \grave{y}_i, y_i, ..., y_n)$ instead of R_1. Using this result of the above relation, we find that

$$\gamma_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2 ..., \grave{y}_i, y_i, ..., y_n))[[\sigma(\gamma_1((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2 ..., \grave{y}_i, y_i, ..., y_n))), y_1], b]$$

$$+[\gamma_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2 ..., \grave{y}_i, y_i, ..., y_n)), b][\sigma(\gamma_1((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2 ..., \grave{y}_i, y_i, ..., y_n))), y_1]$$

$$+[\sigma(\gamma_1((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2 ..., \grave{y}_i, y_i, ..., y_n))), \gamma_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2 ..., \grave{y}_i, y_i, ..., y_n))][y_1, b]$$

$$+[[\sigma(\gamma_1((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2 ..., \grave{y}_i, y_i, ..., y_n))), g_2(x_n)D_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})$$

$$(y_1, y_2 ..., \grave{y}_i, y_i, ..., y_n))], b] = 0.$$

Where $g(R)$ is $g(R) = a - Rb$. Therefore, the following result can be seen as an extension of this relation.

$$\gamma_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2 ..., \grave{y}_i, y_i, ..., y_n))[[\sigma(\gamma_1(R_1)), y_1], b]$$

$$+[\gamma_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2 ..., \grave{y}_i, y_i, ..., y_n)), b][\sigma(\gamma_1(R_1)), y_1]$$

$$+[\sigma(\gamma_1(R_1)), \gamma_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2 ..., \grave{y}_i, y_i, ..., y_n))][y_1, b]$$

$$+[[\sigma(\gamma_1(R_1)), aD_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1, y_2 ..., \grave{y}_i, y_i, ..., y_n)$$

$$-x_n bD_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1, y_2..., \grave{y}_i, y_i, ..., y_n)], b] = 0.$$

Putting $x_n = -x_n$ and combining the result with this relation, we conclude that

$$2[[\sigma(\gamma_1(R_1)), aD_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1, y_2..., \grave{y}_i, y_i, ..., y_n))], b] = 0.$$

Identifiable property of that R is 2-torsion free and simplifying calculation of $\sigma(\gamma_1(R_1))$ which is similar to the previous technique of $\sigma(\gamma_2(R_2))$ modifies the structure of this relation to

$$[[aD_1((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1, y_2, ..., \grave{y}_i, y_i, ..., y_n)), aD_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})$$

$$(y_1, y_2..., \grave{y}_i, y_i, ..., y_n))], b] = 0.$$

Actually, this relation can be rewritten as follows

$$[a^2 D_1((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1, y_2..., \grave{y}_i, y_i, ..., y_n))D_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1, y_2..., \grave{y}_i, y_i, ..., y_n)]$$

$$-a^2 D_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1, y_2..., \grave{y}_i, y_i, ..., y_n))D_1((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1, y_2, ..., \grave{y}_i, y_i, ..., y_n))$$

$$+aD_1((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1, y_2..., \grave{y}_i, y_i, ..., y_n))aD_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1, y_2..., \grave{y}_i, y_i, ..., y_n))$$

$$-aD_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1, y_2, ..., \grave{y}_i, y_i, ..., y_n))aD_1((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1, y_2..., \grave{y}_i, y_i, ..., y_n))$$

$$+a^2 D_1((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1, y_2..., \grave{y}_i, y_i, ..., y_n))D_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1, y_2..., \grave{y}_i, y_i, ..., y_n)), b] = 0.$$

Basically, we have the fact that an anticommutative ring R has the property $2x^2 = 0$ for all $x \in R$. View of our hypothesis, we find that R is 2-torsion free. Hence, the previous fact can rewrite as $x^2 = 0$ for all $x \in R$ and using it of this relation, we achieve that

$$[2a(D_1((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1, y_2..., \grave{y}_i, y_i, ..., y_n))aD_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1, y_2..., \grave{y}_i, y_i, ..., y_n))$$

$$-aD_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1, y_2..., \grave{y}_i, y_i, ..., y_n))aD_1((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})$$

$$(y_1, y_2..., \grave{y}_i, y_i, ..., y_n)), b] = 0.$$

Simplify calculation and employing the fact that $x^2 = 0$ for all $x \in R$ this relation reduces to

$$a[a, b](2D_1((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1, y_2..., \grave{y}_i, y_i, ..., y_n))aD_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1, y_2..., \grave{y}_i, y_i, ..., y_n))$$

$$-D_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1, y_2..., \grave{y}_i, y_i, ..., y_n))aD_1((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1, y_2..., \grave{y}_i, y_i, ..., y_n)) = 0.$$

By reason of R is without zero divisors. Obviously, if we have $a = 0$, then nothing to prove. So far, a must be non-equal to zero. Consequently, we show that:

$$\text{either } [a, b] = 0$$

$$\text{or } 2D_1((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1, y_2..., \grave{y}_i, y_i, ..., y_n)aD_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1, y_2..., \grave{y}_i, y_i, ..., y_n))$$

$$= D_2((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1, y_2..., \grave{y}_i, y_i, ..., y_n))aD_1((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1, y_2..., \grave{y}_i, y_i, ..., y_n)).$$

Replacing x_{n-1} by $x_n x_{n-1}$, the last term can rewire as $2D_1(R)aD_2(R) = D_2(R)aD_1(R))$. This completes the proof. □

Theorem 18. *Let R be a 2-torsion free anticommutative semiprime ring and γ be a permuting skew n-antisemigeneralized semiderivation associated with automorphism function g of R such that $\gamma \in \hat{Z}(R)$. Then $\gamma(R) = 0$.*

Proof. In the beginning, we have $\gamma \in \hat{Z}(R)$. Hence, we observe that

$$\gamma(R) \circ t = 0$$

for all $t \in R$.

Taking $(x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_1, y_1..., \grave{y}_i, y_i, ..., y_n)$ instated of R of this relation, we arrive to

$$(\gamma((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2..., \grave{y}_i, y_i, ..., y_n))y_1$$

$$+g(x_n)D((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1..., \grave{y}_i, y_i, ..., y_n))) \circ t = 0$$

for all $\grave{x}_i, x_i, \grave{y}_i, y_i, t \in R$.

In this relation t by $\gamma((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2..., \grave{y}_i, y_i, ..., y_n))$ yields

$$\gamma((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2..., \grave{y}_i, y_i, ..., y_n))y_1\gamma((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2..., \grave{y}_i, y_i, ..., y_n))$$

$$+g(x_n)D((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1..., \grave{y}_i, y_i, ..., y_n))\gamma((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2..., \grave{y}_i, y_i, ..., y_n))$$

$$+\gamma((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2..., \grave{y}_i, y_i, ..., y_n))^2y_1$$

$$+\gamma((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_1..., \grave{y}_i, y_i, ..., y_n))g(x_n)D((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1..., \grave{y}_i, y_i, ..., y_n)) = 0.$$

However, we emphasise on the main relation that substituting $\gamma(R)$ for t and applying R has 2-torsion free property, we obtain $\gamma(R)^2 = 0$. Using the result of this relation, it should be possible to establish the relation

$$\gamma((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2..., \grave{y}_i, y_i, ..., y_n))y_1\gamma((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2..., \grave{y}_i, y_i, ..., y_n))$$

$$+g(x_n)D((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1..., \grave{y}_i, y_i, ..., y_n))\gamma((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2..., \grave{y}_i, y_i, ..., y_n))$$

$$+\gamma((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2..., \grave{y}_i, y_i, ..., y_n))g(x_n)D((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1..., \grave{y}_i, y_i, ..., y_n)) = 0.$$

We can rewrite this relation as follows

$$\gamma((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2..., \grave{y}_i, y_i, ..., y_n))y_1\gamma((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2..., \grave{y}_i, y_i, ..., y_n))$$

$$+\gamma((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2..., \grave{y}_i, y_i, ..., y_n)) \circ g(x_n)D((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_{n-1})(y_1..., \grave{y}_i, y_i, ..., y_n))$$

$$= 0.$$

In agreement with the main relation is generally suitable for reducing this relation to

$$\gamma((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2..., \grave{y}_i, y_i, ..., y_n))y_1\gamma((x_1, x_2, ..., \grave{x}_i, x_i, ..., x_n)(y_2..., \grave{y}_i, y_i, ..., y_n)) = 0.$$

Light of the semiprimeness of R, there undoubtedly existed on this relation a clearer result which is $\gamma(R) = 0$. We complete the proof. □

Depending on the result of Theorem 18, one can prove the following.

Corollary 7. *Let R be a 2-torsion free anticommutative semiprime ring and γ be a permuting skew n-antisemigeneralized semiderivation associated with a automorphism function g of R such that $\gamma(R) \subseteq \hat{Z}(R)$. Then $D(R) = 0$.*

Immediate consequence of Theorem 13. Suppose that γ is a non-zero permuting skew n-antisemigeneralized semiderivation acting as an automorphism of (σ,τ)-ring. Then, we achieve the corollary.

Corollary 8. *Let R be a 2-torsion free (σ,τ)-ring without zero divisors and γ be a non-zero permuting n-antisemigeneralized semiderivation associated with a automorphism functions g and h of R such that $g(R) = a - Rb$, $a, b \in R$ and γ acts as automorphism of R. If R satisfies $[\sigma(\gamma_1(R_1)) \circ \tau(\gamma_2(R_2)), b] = 0$ then either R is (σ,τ)-anticommutative ring or $[a, b] = 0$.*

Remark 2. *In some results we depend on the condition that the associated function expresses as $g(R) = a - Rb$, $a, b \in R$. Indeed, the condition is not superfluous .*

The following example demonstrates that we cannot exclude the restrictions mandatory on the hypotheses of the results for example $g(R) = a - Rb$, $a, b \in R$.

Example 4. *Let $R = M_n(\mathbb{F})$ be a ring of $n \times n$ matrices over a field $\mathbb{F}$, $n > 1$ that is:*

$$R = \begin{pmatrix} x_j & x_{j+1} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \setminus x_j^2 = x_j,$$

for all $x_j \in \mathbb{F}$, $j = 1, 2, ..., n$. For all $\grave{s}_i s_i \in R$, such that $i = 1, 2, ..., n$. We assume $\Omega(s_1, s_2, ..., \grave{s}_i s_i, ..., s_n)$ defined as

$$\Omega(s_1, s_2, ..., \grave{s}_i s_i, ...s_n) = \begin{pmatrix} 0 & (x_1 x_3 \grave{x}_j x_j \cdots x_{n-1})(x_1 x_3 \grave{x}_j x_j \cdots x_n) & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

It represents the left-side of Definition 3. Also, we chose a and b as fixed elements of R. Hence, we define the function g as

$$g(s_1, s_2, ..., \grave{s}_i s_i, ...s_n) = a - \begin{pmatrix} x_j & x_{j+1} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} b.$$

Now putting

$$a = \begin{pmatrix} \kappa & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

and

$$b = \begin{pmatrix} \varepsilon & 0 & & \cdots & 0 \\ \zeta & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where ζ, κ and ε are constant. Using the values of a and b to find the value of $g(s_1, s_2, ..., \grave{s}_i s_i, ...s_n)$. We arrive to the following

$$g(s_1, s_2, ..., \grave{s}_i s_i, ...s_n) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \varepsilon & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Additionally, we define

$$h(s_1, s_2, ..., \hat{s}_i s_i, ...s_n) = \begin{pmatrix} x_j & 0 & 0 & \cdots & 0 \\ 0 & x_{j+1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

and

$$D(s_1, s_2, ..., \hat{s}_i s_i, ...s_n) = \begin{pmatrix} 0 & (x_1 x_3 \hat{x}_j x_j \cdots x_{n-1})(x_1 x_3 \hat{x}_j x_j \cdots x_n) & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

However, the right side of Definition 3 becomes

$$= \begin{pmatrix} 0 & (x_1 x_3 \hat{x}_j x_j \cdots x_{n-1})(x_1 x_3 \hat{x}_j x_j \cdots x_n) & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} g \begin{pmatrix} x_j & x_{j+1} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} +$$

$$h \begin{pmatrix} x_j & x_{j+1} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 0 & (x_1 x_2 x_j \cdots x_{n-1})(x_1 x_2 x_j \cdots x_n) & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Substituting the values of functions g and h with using the property $x^2 = x$, we conclude that

$$= \begin{pmatrix} 0 & (x_1 x_2 x_j \cdots x_{n-1})(x_1 x_2 x_j \cdots x_n) & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Certainly, we see that $[\Omega_1(R_1), \Omega_2(R_2)] = 0$.

7. Conclusions

In this article, we introduce new generators of a permuting n-derivations which are a permuting n-generalized semiderivation, a permuting n-semigeneralized semiderivation, a permuting n-antisemigeneralized semiderivation and a permuting skew n-antisemigeneralized semiderivation of ring R with their applications. Additionally, we present the definition of a M-set of semiprime ring R. In fact, R has results related to each type of theirs. The presented results have been supported by some examples.

When R admits a permuting n-semiderivations Δ satisfies some identities, we observe that

(i) R has a weak zero-divisor,
(ii) $\Delta^2 \in$ *M-set*,
(iii) $\delta^2(R) \subseteq Z(R)$,
(iv) either $\delta^2(R) \subseteq$ *M-set* or $\Delta(R) \subseteq Z(R)$,
(v) $\Delta(d)$ and $\Delta(R_2)$ commute with R,
(vi) $\Delta(d(R))$ is central of R,
(vii) R is commutative if $\delta(R) = 0$ or $\Delta(R) = 0$,
(viii) either $\Delta(R_1)^2 \subseteq$ *M-set* or $\delta(R_2)$ and $\Delta(R_1^2)$ commute with R.

Let D be a permuting n-generalized semiderivation satisfies some relations of R, we find that

(i) $D(R) \neq 0$ with the property $x_i^2 = x_i$ for all $x \in R$ then either $D(R)$ is central of R or $D(R) \subseteq$ *M-set* or $D(R^2)$ is commuting of R such that $i = 1, 2, ..., n$,
(ii) $D = 0$ then R contains a non-zero central ideal,
(iii) $D(R) \neq 0$ then $D(U^2) \subseteq Z(R)$, U is a non-zero ideal.
(iv) $D(R) = 0$ then $\mu(U^2) \subseteq Z(R)$, μ is a trace of $D(R)$ and U is a non-zero ideal ,
(v) R has a weak zero divisor,
(vi) either $R^2 \subseteq$ *M-set* or $b^4 \in$ *M-set* or $b \circ R = 0$ or $b^3 \in Z(R)$,
(vii) $[\psi(x), \Delta(y)] \in Z(R)$ for all $x, y \in R$,
(viii) $[D(x), \delta(y)] \in Z(R)$ for all $x, y \in R$.

When permuting n-semigeneralized semiderivation Ω satisfies certain conditions of (σ,τ)-semicommutative semiprime, we conclude that

(i) either $a = 0, a \in R$ or $d(R) = 0$ or $\Omega_1(R)$ and $\Omega_2(R)$ are commuting,
(ii) either $D_1(R_1) = 0$ or $D_2(R_2) = 0$,
(iii) either $\Omega_1(R_1)) \circ \Omega_2(R_2) = 0$ or $[a, b] = 0, a, b \in R$.

Let R be an anticommutative semiprime ring admits a permuting skew n-antisemigeneralized semiderivation γ satisfying certain identities, we arrive to

(i) either $\gamma_1(R) = 0$ or a commute of R,$a \in R$,
(ii) $a \in Z(R), a \in R$,
(iii) either $\sum_{i=1}^{n} \gamma_1(R) = 0$ or $D(R) = 0$ or R is commutative,
(iv) any *M-set* has anticommutative property,
(v) δ acts as a surjective mapping then Δ is commuting(resp. centralizing) of R,
(vi) $\delta = 0$ or $\Delta = 0$ then an anticommutative and commutative coincide of R.
(vii) either $[a, b] = 0$ or $2D_1(R)aD_2(R) = D_2(R)aD_1(R)$,
(viii) either R is (σ,τ)-anticommutative ring or $[a, b] = 0, a, b \in R$.

All these results help us to understand rings better and can know about the structure of the rings. In addition to that, it can be helpful for the set of matrices with entries and ring. Further, the calculation of the eigenvalues of matrices, which has multi applications of other sciences, business, engineering and quantum physics.

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