A REMARK ON THE ERDÓS-STRAUS CONJECTURE

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Abstract. In this paper we discuss the Erdós-Straus conjecture. Using a very simple method we show that for each $L \in \mathbb{N}$ with $L > n-1$ there exist some $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ with $x_i \neq x_j$ for all $1 \leq i < j \leq n$ such that

$$\frac{n}{L} \ll \sum_{j=1}^{n} \frac{1}{x_j} \ll \frac{n}{L}$$

In particular, for each $L \geq 3$ there exist some $(x_1, x_2, x_3) \in \mathbb{N}^3$ with $x_1 \neq x_2$, $x_2 \neq x_3$ and $x_3 \neq x_1$ such that

$$c_1 \frac{3}{L} \leq \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} \leq c_2 \frac{3}{L}$$

for some $c_1, c_2 > 1$.

1. **Introduction**

The Erdós-Straus conjecture is the assertion that for each $n \in \mathbb{N}$ for $n \geq 3$ there exist some $x_1, x_2, x_3 \in \mathbb{N}$ such that

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = \frac{4}{n}.$$

More formally the conjecture states

Conjecture 1.1. For each $n \geq 3$, does there exist some $x_1, x_2, x_3 \in \mathbb{N}$ such that

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = \frac{4}{n}?$$

Despite its apparent simplicity, the problem still remain unresolved. However there has been some noteworthy partial results. For instance it is shown in [1] that the number of solutions to the Erdós-Straus Conjecture is bounded poly-logarithmically on average. The problem is also studied extensively in [2] and [3]. In this paper, using a somewhat different and a much simpler method, we show that

Theorem 1.1. *For each $L \in \mathbb{N}$ with $L > n-1$ there exist some $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ with $x_i \neq x_j$ for all $1 \leq i < j \leq n$ such that*

$$\frac{n}{L} \ll \sum_{j=1}^{n} \frac{1}{x_j} \ll \frac{n}{L}$$

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In particular, for each $L \geq 3$ there exist some $(x_1, x_2, x_3) \in \mathbb{N}^3$ with $x_1 \neq x_2$, $x_2 \neq x_3$ and $x_3 \neq x_1$ such that

$$\frac{3}{L} \ll \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} \ll \frac{3}{L}.$$

2. Main result

In this section we introduce the notion of compression of points $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ for $x_j \neq 0$ for all $j = 1, 2, \ldots, n$. This notion is of independent interest and can be developed further but we deem it necessary in our studies. We launch the following language in that regard.

Definition 2.1. By the compression of scale $m \geq 1$ on $\mathbb{R}^n$ we mean the map $\mathbb{V} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that

$$\mathbb{V}_m[(x_1, x_2, \ldots, x_n)] = \left(\frac{m}{x_1}, \frac{m}{x_2}, \ldots, \frac{m}{x_n}\right)$$

for $n \geq 2$ and with $x_i \neq 0$ for all $i = 1, \ldots, n$.

Remark 2.2. The notion of compression is in some way the process of re scaling points in $\mathbb{R}^n$ for $n \geq 2$. Thus it is important to notice that a compression pushes points very close to the origin away from the origin by certain scale and similarly draws points away from the origin close to the origin.

Proposition 2.1. *A compression of scale $m \geq 1$ with $\mathbb{V}_m : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a bijective map.*

Proof. Suppose $\mathbb{V}_m[(x_1, x_2, \ldots, x_n)] = \mathbb{V}_m[(y_1, y_2, \ldots, y_n)]$, then it follows that

$$\left(\frac{m}{x_1}, \frac{m}{x_2}, \ldots, \frac{m}{x_n}\right) = \left(\frac{m}{y_1}, \frac{m}{y_2}, \ldots, \frac{m}{y_n}\right).$$

It follows that $x_i = y_i$ for each $i = 1, 2, \ldots, n$. Surjectivity follows by definition of the map. Thus the map is bijective. $\square$

Remark 2.3. In order to get a handle on the general case of the problem, we introduce the notion of the mass of compression on points $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ for $n \geq 2$.

Definition 2.4. By the mass of a compression of scale $m \geq 1$ we mean the map $\mathcal{M} : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) = \sum_{i=1}^{n} \frac{m}{x_i}.$$

Remark 2.5. We find the following elementary estimate useful.

Lemma 2.6. *The estimate remain valid*

$$\sum_{n \leq x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right)$$

where $\gamma = 0.5772 \cdots$.

Remark 2.7. Next we prove upper and lower bounding the mass of the compression of scale $m \geq 1$. We use these estimates as a black box in obtaining our result.

Proposition 2.2. *Let $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ for $n \geq 2$. Then the estimates hold*

$$m \log\left(1 - \frac{n-1}{\sup(x_j)}\right)^{-1} \ll \mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) \ll m \log\left(1 + \frac{n-1}{\mathrm{Inf}(x_j)}\right)$$

for $n \geq 2$ with $x_j \geq 1$.

Proof. Let $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ for $n \geq 2$ with $x_j \geq 1$. Then it follows that

$$\begin{aligned}\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) &= m \sum_{j=1}^{n} \frac{1}{x_j} \\ &\leq m \sum_{k=0}^{n-1} \frac{1}{\mathrm{Inf}(x_j) + k}\end{aligned}$$

and the upper estimate follows by applying Lemma 2.6. The lower estimate also follows by noting the lower bound and applying Lemma 2.6

$$\begin{aligned}\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) &= m \sum_{j=1}^{n} \frac{1}{x_j} \\ &\geq m \sum_{k=0}^{n-1} \frac{1}{\sup(x_j) - k}.\end{aligned}$$

□

Theorem 2.8. *There exist some $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ with $x_i \neq x_j$ for all $1 \leq i < j \leq n$ for each $n \geq 2$ with $x_j \geq 1$ such that*

$$m \frac{n}{L_1} \ll \mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) \ll m \frac{n}{L_2}$$

for some $L_1, L_2 \in \mathbb{N}$.

Proof. First choose $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ such that $\sup(x_j) > \mathrm{Inf}(x_j) > n - 1$ for $j = 1, \ldots n$. Then from Proposition 2.2, we have the upper bound

$$\begin{aligned}\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) &\ll m \log\left(1 + \frac{n-1}{\mathrm{Inf}(x_j)}\right) \\ &= m \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left(\frac{n-1}{\mathrm{Inf}(x_j)}\right)^k \\ &\ll m \frac{n}{\mathrm{Inf}(x_j)}.\end{aligned}$$

The lower bound also follows by noting that

$$\begin{aligned}\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) &\gg m \log\left(1 - \frac{n-1}{\sup(x_j)}\right)^{-1} \\ &= m \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{n-1}{\sup(x_j)}\right)^k \\ &\gg m \frac{n}{\sup(x_j)}\end{aligned}$$

and the inequality follows by taking $\sup(x_j) = L_1$ and $\mathrm{Inf}(x_j) = L_2$. □

Remark 2.9. It is quite implicit from the estimates in Theorem 2.8 that the implicit constants arising from the inequality are each of scale bigger than 1.

Theorem 2.8 is redolent of the Erdòs-Straus conjecture. Indeed It can be considered as a weaker version of the conjecture. It is quite implicit from Theorem 2.8 that there are infinitely many points in $\mathbb{N}^n$ that satisfy the inequality with finitely many such exceptions. Therefore in the opposite direction we can assert that there are infinitely many $L_1, L_2 \in \mathbb{N}$ that satisfies the inequality. We state a consequence of the result in Theorem 2.8 to shed light on this assertion.

Corollary 2.1. For each $L \in \mathbb{N}$ with $L > n-1$ there exist some $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ with $x_i \neq x_j$ for all $1 \leq i < j \leq n$ such that

$$\frac{n}{L} \ll \sum_{j=1}^{n} \frac{1}{x_j} \ll \frac{n}{L}$$

In particular, for each $L \geq 3$ there exist some $(x_1, x_2, x_3) \in \mathbb{N}^3$ with $x_1 \neq x_2$, $x_2 \neq x_3$ and $x_3 \neq x_1$ such that

$$\frac{3}{L} \ll \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} \ll \frac{3}{L}.$$

Proof. Let $L \in \mathbb{N}$ with $L > n-1$ and take $K = \sup(x_j)$ and $L = \mathrm{Inf}(x_j)$ for any such point $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$. Then it follows that

$$\frac{n}{L} \ll \frac{n}{K} \ll \sum_{j=1}^{n} \frac{1}{x_j} \ll \frac{n}{L}.$$

The special case follows by taking $n = 3$. $\square$

It is important to recognize that the condition $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ with $x_i \neq x_j$ for $1 \leq i < j \leq n$ in the statement of the result is not only a quantifier but it a necessity; otherwise, the estimate for the mass of compression will be flawed completely. To wit, suppose that we take $x_1 = x_2 = \ldots = x_n$, then it will follow that $\mathrm{Inf}(x_j) = \sup(x_j)$, in which case the mass of compression of scale m satisfies

$$m \sum_{k=0}^{n-1} \frac{1}{\mathrm{Inf}(x_j) - k} \leq \mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) \leq m \sum_{k=0}^{n-1} \frac{1}{\mathrm{Inf}(x_j) + k}$$

and it is easy to notice that this inequality is absurd. By extension one could also try to equalize the sub-sequence on the bases of assigning the Supremum and the Infimum and obtain an estimate but that would also contradict the mass of compression inequality after a slight reassignment of the sub-sequence. Thus it is important for the estimates to make any good sense to ensure that any tuple $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ must satisfy $x_i \neq x_j$ for all $1 \leq i < j \leq n$. Thus it is required in our result that any tuple we use has to have distinct entry.

3. Further discussions

The result can be interpreted as saying that for each $L \geq 3$ there exist some $(x_1, x_2, x_3) \in \mathbb{N}^3$ with $x_1 \neq x_2$, $x_2 \neq x_3$ and $x_3 \neq x_1$ such that

$$c_1 \frac{3}{L} \leq \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} \leq c_2 \frac{3}{L}$$

for some constants $c_1, c_2 > 1$. The Erdós-Straus conjecture will follow if we can take $c_1 = c_2 = \frac{4}{3}$. Investigating the scale of these constants is the motivation for our next quest, which we do not pursue in this paper. Indeed the method we have employed not only does it put a constraint on the unit sum of possible solutions to the Erdós-Straus conjecture but also provides a lower threshold below which the size of the constituent triple should not fall. On much general setting, the result has the following twist

Theorem 3.1. *For each $L \in \mathbb{N}$ with $L > n-1$, there exist some $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ with $x_i \neq x_j$ for all $1 \leq i < j \leq n$ and some constant $c_1, c_2 > 1$ such that*

$$c_1 \frac{n}{L} \leq \sum_{j=1}^{n} \frac{1}{x_j} \leq c_2 \frac{n}{L}.$$

[1].

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