THE LITTLE ℓ FUNCTION

1 author:

Theophilus Agama
African Institute for Mathematical Sciences Ghana

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THEOPHILUS AGAMA

ABSTRACT. In this short note we introduce a function which iteratively behaves in a similar fashion compared to the factorial function. However the growth rate of this function is not as dramatic and sudden as the factorial function. We also propose an approximation for this function for any given input, which holds for sufficiently large values of n.

1. Introduction

The factorial function is of fundamental importance in many areas of pure and applied mathematics. The function is one among the many transcedental functions that has been studied over centuries. For a detailed historical account concerning the evolution of the factorial function, See [2]. It is defined as $n! = n(n-1)(n-2)\cdots 2\cdot 1$. For sufficiently large values of n, we have the famous Stirling formula (See[1]), given by

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n,$$

which gives an approximation for the factorial function. In this paper, we introduce the little ℓ function, which behaves in some sense as the factorial function.

2. The little ℓ function

Definition 2.1. Let n be a natural number, then we set

$$\ell(n) := \begin{cases} \frac{n}{2}\ell(\frac{n}{2})\cdots 1 \quad \text{if} \quad n \quad \text{is} \quad \text{even} \\ \frac{n-1}{2}\ell(\frac{n-1}{2})\cdots 1 \quad \text{if} \quad n>1 \quad \text{is} \quad \text{odd} \\ 1 \quad \text{if} \quad n=1. \end{cases}$$

The little ℓ function, though completely different from the factorial function, can be seen to be performing a role similar to the factorial function. The factorial function on an integer n iterates downwards on n till it's value gets to 1, so does the little ℓ function. That is, $n! = n(n-1)!$ where as $\ell(n) = \frac{n}{2}\ell(\frac{n}{2})$ or $\ell(n) = \frac{n-1}{2}\ell(\frac{n-1}{2})$, according as n is odd or even.

Example 2.2.

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(a) $\ell(24) = 12 \cdot 6 \cdot 3 \cdot 1 = 216$, $\ell(100) = 50 \cdot 25 \cdot 12 \cdot 6 \cdot 3 \cdot 1 = 270000$.

(b) $\ell(55) = 27 \cdot 13 \cdot 6 \cdot 3 \cdot 1 = 6318$, $\ell(127) = 63 \cdot 31 \cdot 15 \cdot 7 \cdot 3 \cdot 1 = 615195$.

Remark 2.3. The little ℓ function is somewhat akin to the factorial function. However the growth rate of the factorial function supersedes that of the ℓ function. Below is a table that gives the distribution of the little ℓ function for the first thirty values of the integers.

3. Distribution of the little ℓ function

Values of n	The little ℓ function on n	Values of the little $\ell(n)$ function
1	$\ell(1)$	1
2	$\ell(2)$	1
3	$\ell(3)$	1
4	$\ell(4)$	2
5	$\ell(5)$	2
6	$\ell(6)$	3
7	$\ell(7)$	3
8	$\ell(8)$	8
9	$\ell(9)$	8
10	$\ell(10)$	10
11	$\ell(11)$	10
12	$\ell(12)$	18
13	$\ell(13)$	18
14	$\ell(14)$	21
15	$\ell(15)$	21
16	$\ell(16)$	64
17	$\ell(17)$	64
18	$\ell(18)$	72
19	$\ell(19)$	72
20	$\ell(20)$	100
21	$\ell(21)$	100
22	$\ell(22)$	110
23	$\ell(23)$	110
24	$\ell(24)$	216
25	$\ell(25)$	216
26	$\ell(26)$	234
27	$\ell(27)$	234
28	$\ell(28)$	294
29	$\ell(29)$	294
30	$\ell(30)$	315

We can observe from the above table that $\ell(n) = \ell(n+1)$ for any even number n.

Conjecture 3.1. For any natural number n, we have that

$$\ell(n) \sim C_0 \left(\frac{n}{2}\right)^{\log n} \left(\frac{1}{2}\right)^{\frac{(\log n)^2 - \log n}{2}}$$

for some constant $C_0 > 0$.

3.1. **Heuristics in support of Conjecture 3.1.** Conjecture 3.1 is not completely surprising. We believe this is a good approximation of the ℓ function. Heuristically, choose any integer n at random and apply the little ℓ function on n. We remark that n can either be odd or even. Without loss of generality let us assume n is even, then the little ℓ on n is given by $\ell(n) = \frac{n}{2}\ell(\frac{n}{2})$. Again $\frac{n}{2}$ can either be odd or even. Without loss of generality, let us suppose $\frac{n}{2}$ is odd, then $\ell(n) = \frac{n}{2}(\frac{n/2-1}{2})\ell(\frac{n/2-1}{2}) \approx (\frac{n}{2})(\frac{n}{2})\frac{1}{2}\ell(\frac{n/2-1}{2})$. Again $\frac{n/2-1}{2}$ can either be odd or even. Without loss of generality, let us assume $\frac{n/2-1}{2}$ is even, then we find that $\ell(n) \approx (\frac{n}{2})^2\frac{1}{2}(\frac{n/2-1}{4})\ell(\frac{n/2-1}{4}) \approx (\frac{n}{2})^3(\frac{1}{2})^3\ell(\frac{n/2-1}{4})$. Let $P(\ell(n))$ denotes the number of factors of the $\ell(n)$ function, or the frequency at which the $\ell(n)$ converges to 1, then it can be seen that $P(\ell(n)) = (1+o(1))\log n$. Then by continous iteration the little $\ell(n)$ function should be approximately

$$\ell(n) \sim C_0 \left(\frac{n}{2}\right)^{\log n} \left(\frac{1}{2}\right)^{\frac{(\log n)^2 - \log n}{2}}.$$

4. Comparism to flipping a coin

Infering from the above heuristics, the little $\ell(n)$ function can intuitively be thought of as flipping a coin at finite number of times, where in our case the frequency of this experiment is given to be roughly $P(\ell(n)) = (1+o(1))\log n$ and where $\ell(n) = \frac{n}{2}\ell(\frac{n}{2})$ or $\ell(n) = \frac{n-1}{2}\ell(\frac{n-1}{2})$, each with probability $\frac{1}{2}$. Roughly speaking, the estimate

$$\ell(n) \approx n^{\log n} \left(\frac{1}{2}\right)^{\frac{(\log n)^2 - \log n}{2}} \left(\frac{1}{2}\right)^{\log n}.$$

obtained from the above heuristics, can be seen as the probability of obtaining either a head or tail in $(1+o(1))\log n$ number of trials, for sufficiently large values of n. [1].

References

1. Mortici, Cristinel *A class of integral approximations for the factorial function*, Computers & Mathematics with Applications, vol. 59.6, Elsevier, 2010, pp.2053–2058.
2. Dutka, Jacques *The early history of the factorial function*, Archive for history of exact sciences, vol. 43.3, Springer, 1991, pp.225–249.

Department of Mathematics, African Institute for Mathematical science, Ghana

E-mail address: theophilus@aims.edu.gh/emperordagama@yahoo.com

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