ON (AN)ABELIAN GEOMETRY AND DEFORMATION THEORY WITH APPLICATIONS TO QUANTUM PHYSICS

LUCIAN M. IONESCU

Abstract. The Betti-de Rham period isomorphism ("Abelian Geometry") is related to algebraic fundamental group (Anabelian Geometry), in analogy with the classical context of Hurewicz Theorem. To investigate this idea, the article considers an *"Abstract Galois Theory"*, as a separated abstract structure from, yet compatible with, the Theory of Schemes, which has its historical origin in Commutative Algebra and motivation in the early stages of Algebraic Topology.

The approach to **Motives via Deformation Theory** was suggested by Kontsevich in [Kon] as early as 1999, and suggests *Formal Manifolds* [KS] §4, with local models formal pointed manifolds, are the source of motives, and perhaps a substitute for a "universal Weil cohomology".

The proposed research aims to gain additional understanding of periods via a concrete project, the discrete algebraic de Rham cohomology, a follow-up of PI's previous work.

The connection with *arithmetic gauge theory* [3] should provide additional intuition, by looking at covering maps as "flat connection spaces", and considering branching covers of the Riemann sphere as the more general case.

The research on Feynman/Veneziano Amplitudes and Gauss/Jacobi sums accumulated as part of the prior IHES visit, allows to deepen the parallel between the continuum and discrete frameworks: an analog of Virasoro algebra in finite characteristic.

A larger project is briefly considered, consisting in deriving Motives from the Theory of Deformations, as suggested by Kontsevich [Kon]. Following Soibelman and Kontsevich [KS], the idea of defining *Formal Manifolds* as groupoids of *pointed formal manifolds* (after Maurer-cartan "exponentiation"), with associated torsors as "gluing data" (transition functions) is presented. This framework seems to be compatible with the ideas from Theory of Periods, sheaf theory / etale maps and Grothedieck's development of Galois Theory (Anabelian Geometry).

The article is a preliminary evaluation of a research plan of the author. Further concrete problems are included, since they are related to the general ideas mentioned above, and especially relevant to understanding the applications to scattering amplitudes in quantum physics.

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Contents

Preamble

There are several areas which need be considered in concerted effort:

1) Theory of Periods, starting from the algebraic de Rham cohomology, which we will refer to as: *Abelian Geometry*;

2) Theory of em algebraic fundamental group, the starting point of Grothendieck's *Anabelian Geometry*;

3) *Theory of Motives* developed based on *Deformation Theory*, as initially suggested by Kontsevich in 1999 [Kon]**, §4 p.24** (in the context of deformation quantization technique of Chen iterated integrals), **in the context of simplicial Maurer-Cartan complex** [Markl][1];

4) The *Gauge Theory* interpretations of the above (Galois Categories with fiber functor as analog of principal bundle theory, etale maps as "flat connections" with representations of the fundamental group etc.), which for example Minhyong Kim calls *Arithmetic Gauge Theory* [3].

1) - 4) will clarify the role of the associahedron in Quantum Physics.

But first, a few concrete current projects will be presented, leading to a better understand of periods and scattering amplitudes.

[1]Therein, the application to Kontsevich's approach to deformation quantization is presented as an application in *Ch.7 Homotopy invariance and quantization*; together with the considerations from [Kon] §4, they should provide the foundations for a general presentation, based directly on *Deformation Theory*, conform [KS].

1. Research Projects Regarding Algebraic de Rham Cohomology

The Theory of Periods at the level of *algebraic de Rham cohomology* of an algebraic variety, inherits much more from Algebraic Number Theory, via a generalization of Galois Theory in the spirit of Grothendieck, who also introduced the *algebraic fundamental group* [1].

1.1. A Study of the Period Isomorphism ($char = 0$).

Periods can be viewed as the coefficients of the Period Isomorphism of an algebraic variety defines in a homological markup Betti basis. This additional structure is traditionally associated to a Hodge filtration.

Problem I: **Filtrations vs. Bialgebra**.

The PI will investigate if the filtration and associated graded structure play the role of "reducing the structure group", by comparison with Gram-Schmidt O.N. procedure and Jordan-Holder Theorems.

Problem II: **Frobenius forms and bialgebra structures**. The integration pairing is supplemented by the Betti homological bases, yielding the periods and the Period de Rham Isomorphism. The PI will investigate if this additional structure can be modeled by a co-pairing, which plays the role of such a basis, and compatible with integration. The Frobenius algebra and Frobenius form will be taken as prototypical examples. The analogy with 2D-TQFTs associated to Frobenius algebras will be used to help the intuition view periods in the context of an "arithmetic gauge theory". The work of [3] will be considered, for additional ideas and tools to be applied in this project.

1.2. Discrete de Rham Periods ($char \neq 0$).

It is useful to understand the relation between the Theory of Periods in characteristic zero and finite characteristics [12].

Prior to the above research on periods, the present author noticed that POSet formulation theory of G.-C. Rota yielding a "discrete Fundamental Theorem of Calculus" (Mobius function and zeta function as kernels for finite differences and sumation), coincide with topologists' formulation of cohomology of finite Abelian groups [6].

By some "meta-uniqueness", this should be the algebraic de Rham cohomology for Abelian groups, and would constitute a usefull analogy for what periods are in the discrete case. It probably would relate to Gauss sums / periods, and Jacobi sums.

G.-C. Rota also noticed the "sets" analog of Galois Theory, and introduced the *lattice of periods* of a group action (1970s) [4]. This is related to the topics discussed, as a "toy model" (A kind of "Galois Theory over the field with one element").

To keep the proposal short, the main claims to be made precise are listed below as problems.

Problem III: The discrete de Rham cohomology of [6] and algebraic de Rham cohomology (Kahler differentials etc.) applied to abelian groups coincide.

This work will set the foundations of what a "finite string" is.

1.3. **Gauss-Jacobi Sums and Virasoro Algebra in Finite Characteristic.** Gauss and jacobi sums are the analogs of Gamma function and Euler beta integral, which was used by Veneziano to model the amplitude with the same name in String Theory.

Jacobi sum is also a Hochschild 2-cocycle, inviting to consider the corresponding central extension, and relate it with projective representations, similar to the framework of Quamtum Mechanics (Schrodinger representation etc.).

The String Theory starts from considering derivations on the unit circle, i.e. Witt algebra, and then extend it as a central extension: Virasoro algebra.

It seems that the *same* concepts can be applied to the *finite circle* Z/nZ (cyclotomic extensions, when, and if connecting with Galois Theory).

Problem IV: A) Define the Virasoro algebra in finite characteristic: central extension via Jacobi 2-cocycle of the derivations on the finite circle:

$$J = dg_{Hochschild}, \quad J(c, c') = g(c)g(c')/g(cc').$$

B) Compare Veneziano amplitudes [9, 10], based on the Gamma function, with the Jacobi sums based on Gauss sums (Fourier coefficient of multiplicative characters). Investigate the connections with resolvents / Gauss periods associated to cyclotomic subfields, and implications to Galois Theory: from Abelian case to Anabelian Geometry.

1.4. **Is there a "Finite/Algebraic String Theory"?** To explore how to formulate a Finite String Theory, one would relate the above considerations regarding discrete de Rham and periods, discrete Virasoro and amplitudes (Abelian 2D case), with algebraic Riemann surfaces (Belyi Theorem), and their Platonic tesselations: Hurewitz surfaces (Anabelian 3D case) [15, 16, 17, 18, 19].

It would probably "relate well" with the current research in Rational CFTs and p-adic String Theory.

1.5. **The Special case of Elliptic Curves.** The special case of elliptic curves, in char zero and finite, represents a non-trivial, yet well studied setup for investigating the above problems.

One specific goal is to understand the Legendre relation between periods in the context of a bialgebra structure, as an alternative approach to Hodge filtration.

Also instructive is to see how the group law on elliptic curves relates to the torsor structure: consider the fundamental group picture of Abel-Jacobi Theorem (path integrals, lattice of periods) and the formal derivation of associativity of geometric addition (divisor line intersected with cubic) which *fails*, due to some additional algebraic structure in this "analitic-geometric side" (associativity is obvious on the "algebraic side" C/Λ via Weierstrass coordinates). In other words we would like to better understand why here the "associator" is trivial; this investigation may benefit from the ideas in Drinfeld's article on quasi-Hopf algebras and Grothendieck-Teichmuller group (connection with Galois groups, viewed as algebraic fundamental groups).

In the finite characteristic Weil Conjectures and Weil zeros, and Weil's Lefschets fixed point approach to Riemann Hypothesis will be revisited in relation with the study of Gauss/Jacobi sums as sketched above in §1 (Research Projects).

2. Further Developments Regarding Galois Theory and Motives

Galois Theory can benefit from a formulation in terms of Galois Categories [2], §4.2, p.49, but *separated from sheaf theory* which favors covering/etale maps (epi), via the fiber functor, to extensions (mono), as for example in the case of field extensions when one has to rephrase them in terms of sheaf theory.

2.1. **Abstract Galois Theory and Arithmetic Gauge Theory.** This *Abstract Galois Theory* is a theory of representations of algebraic fundamental group, via the representation of the fiber functor of a Galois category.

This theory can be developed separately from the theory of schemes needed to "convert" the theory of field extensions (Commutative Algebra) into a theory of covering maps (etale). It can be developed by considering a "moduli space" functor dual to the fiber functor; the adjunction between Sets and k-vector Spaces (groups and k-algebras) allows for a clearer picture (e.g. what "field with one element" is; galois Theory a la Rota or Artin; etc.).

The theory of Fibrations (principal bundles, etale maps or abstract epis), interfaces well with *Arithmetic Gauge Theory* [3] (local systems / flat connections provide reps of the fundamental group). It would benefit from the duality epi / mono, whenever such a "good duality" exists.

2.2. **Applications to Periods and Motives.** A reasonable research direction is to view *Periods* as "extremals" in the sense of Lagrangian systems, the way Minyung Kim uses Arithmetic Gauge Theory to study rational solutions of Diophantine Equations.

More precisely, instead of integration (associating a number to a geometric / action pair), one may consider a representable functor, like the fiber functor on a Galois category, as in Grothendieck's approach to the algebraic fundamental group. Then one expects that this "Anabelian Geometry" (Abstract Galois Theory) will yield in the case of algebraic varieties periods as spectral values for extremal cycles under an "arithmetic Lagrangian".

2.3. **A Deformation Theory approach to Motives and Periods.** In [Kon] Kontsevich remarks the ubiquity of iterated integrals, and looks for the source of motives in Deformation Quantization; behind it is, of course, *Deformation Theory*.

In the arithmetic framework (Adelic Number Theory), it was pointed out by the author [13] that p-adic numbers *are* deformations of the "finite fields" F_p, when looking at them from the Galois Theory point of view (Z/nZ and its symmetries;

corresponding to cyclotomic extensions and recovering Abelian Class Field Theory from "Arithmetic Galois Theory" [14]).

A preliminary investigation of developing the theory of *pointed formal manifolds* in [KS] will try to develop the suggestions from [Kon] and the use of torsors in the theory of periods, from [7].

The main idea (so far) is that pointed formal manifolds $g^\bullet$ are a Lie-type of object, and solving maurer-Cartan equation plays the role of expeonentiation, yielding the group-like object. Taking quotients $h^\bullet \to g^\bullet$, in the sense of fibers and associated moduli space (compare with Abstract Galois Theory / covering spaces / etale maps), will yield the "homogeneous spaces", a candidate for the *formal manifolds*, perhaps.

Torsors, as equivalent structures, could play the role of transition functions, leading to a different "design" of the theory at this stage.

References

[1] Frans Oort, "The algebraic fundamental group", in *Geometric Galois Actions*, p.67-84.

[2] J. P. Murre, "Lectures on An Introduction to Grothendiecks Theory of the Fundamental Group", Tata Institute of Fundamental Research, Bombay 1967, http://www.math.tifr.res.in/ publ/ln/tifr40.pdf

[3] Minhyong Kim, "Arithmetic Gauge Theory: a brief introduction", arxiv.org/abs/1712.07602

[Kon] M. Kontsevich, "Operads and Motives in Deformation Quantization", https://arxiv.org/abs/math/9904055

[KS] M. Kontsevich and Y. Soibelman, "Deformation Theory. I", https://www.math.ksu.edu/ soibel/Book-vol1.ps

[Markl] Martin Markl, "Deformation Theory of Algebras and Their Diagrams", Regional Conference Series in Mathematics, CBMS No.116,

[4] G.-C. Rota and D. A. Smith, "Enumaration under group action", Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e series, tome 4, no.4 (1977), p.637-646.

[5] Wikipedia, Anabelian geometry, https://en.wikipedia.org/wiki/Anabelian_geometry

[6] L. M. Ionescu, A discrete analog of de Rham cohomology on finite abelian groups as manifolds, JP Journal of Algebra, Number Theory and Applications Volume 39, Issue 6, Pages 891 - 906 (December 2017).

[7] M. Kontsevich and D. Zagier, Periods, IHES 2001, https://www.maths.ed.ac.uk/ v1ranick/papers/kontzagi.pdf

[8] Christian Bogner and Francis Brown, "Feynman integrals and iterated integrals on moduli spaces of curves of genus zero", https://arxiv.org/abs/1408.1862

[9] L. M. Ionescu, "Using the Local to Global Principal in the study of Periods, Feynman integrals and Jacobi sums", Nov. 2016, IHES GP: http://my.ilstu.edu/ lmiones/ISUP/ISU-GP.html

[10] L. M. Ionescu, "Periods, Feynman Integrals and Jacobi Sums", ver. 1, 2018 (after ISU talk 2017), 2018, http://my.ilstu.edu/~lmiones/

[11] L. M. Ionescu, "Periods: Variation sur un theme de Kontsevich", IHES presentation, feb. 2018, slides: http://my.ilstu.edu/~lmiones/

[12] L. M. Ionescu, "Periods: from global to local", https://arxiv.org/abs/1806.08726

[13] L. M. Ionescu, "A study of p-adic Frobenius lifts and p-adic periods, from a deformation theory viewpoint", Advances in Pure Mathematics, Vol.8 No.4, p.408-418, April 2018; arxiv.org/abs/1801.07570

[14] L. M. Ionescu, "Arithmetic Galois Theory", work in progress, 2019.
[15] G. V. Belyi, "On Galois extensions of a maximal cyclotomic field", Math. USSR Izvestija, Vol. 14 (1980), No.2.
[16] G. Jones and D. Singerman, "Belyi functions, hypermaps and Galois groups", Bulletin of the London Mathematical Society 28(6):561-590, 1996; www.researchgate.net/publication/250730420_Belyi_Functions_Hypermaps_and_Galois_Groups
[17] Sergei K. Lando and Alexander K. Zvonkin, "Graphs on Surfaces and Their Applications" (Appendix by Don B. Zagier), Springer, 2004.
[18] A. Grothendieck, "Sketch of a programme", http://www.landsburg.com/grothendieck/EsquisseEng.pdf
[19] , Leila Schneps, "Grothendieck's Long March through Galois Theory", https://webusers.imj-prg.fr/~leila.schneps/SchnepsLM.pdf
[20] J. Zhao, "Multiple Zeta Functions, multiple polylogarithms and their special values,", Series on Number Theory and Its Applications Vol.12, World Scientific, 2016.
[21] B. Friedrich, Periods and algebraic de Rham cohomology, Thesis, 1979.
[22] C. N. Yang, "Quantum numbers, Chern classes, and a bodhisattva", Physics Today 65, 1, 33 (2012), https://physicstoday.scitation.org/doi/10.1063/PT.3.1398
[23] Gregory L. Naber, "Topology, Geometry and Gauge fields: Foundations", Texts in Applied Mathematics, No.25, 2011; "Topology, Geometry and gauge Fields: Interactions", Applied Mathematical Sciences, 2011.
[24] V. I. Volovich, "Number theory as the ultimate physical theory", 1987.
[25] L. M. Ionescu, "The Platonic Exceptional Universe" (2018) and "The Qubit Model" (2019), http://my.ilstu.edu/ lmiones/

Department of Mathematics, Illinois State University, IL 61790-4520
E-mail address: lmiones@ilstu.edu