Quantized speeds hidden within the relativistic Dirac energy levels of the H atom

John Suss

There is a hidden construction scheme within the relativistic Dirac energy levels of the H atom. Internal quantized speeds appear in the expression for the levels so that they have the form of mass times quantized speeds. It is possible to represent the levels in terms of normals to a hyperboloid of one-sheet or normals to a hyperboloid of two-sheet in Minkowski space-time. The energy levels reside in the tangent space of either hyperboloids, depending on the choice of representation. The normals do not have defined directions, they really represent entire cone-like regions making quantized hyperbolic angles with the axis of either hyperboloid. The projections of the normals quantize the time-like direction in integral or optionally, half integral units of $1/\alpha$ if the hyperboloid of two-sheet is chosen. It quantizes any space-like direction in integral or optionally, half integral units of $1/\alpha$ if the hyperboloid of one-sheet is chosen.

I. INTRODUCTION

The relativistic Dirac energy levels of the hydrogen and hydrogen-like atoms have been extensively studied. [1, 3] It seems to have been unnoticed in the literature that, with the proper parametrization, the energy levels can be represented in terms of quantized speeds. In this work, it is shown that the energy levels $E_{n\,,\,j+1/2}$ can be expressed as $Mc^2(v_{n\,,\,j+1/2})/c$. So there is a mechanical-like representation of the energy levels. The difference of two energy levels is observable. With the new quantized speeds, whose differences are also quantized speeds, it is seen that whereas light emitted or absorbed by the H atom is ordinary light moving at speed c, but inside of the H atom light has quantized speeds which depend on which energy levels are involved in the emission or absorption. The wavelength of light inside the atom is fixed. It is proportional to the classical radius of the electron proton reduced mass system. The details are explained in the next section.

II. SIMPLE GEOMETRIC CONSTRUCTION BEHIND THE DIRAC ENERGY LEVELS OF THE H ATOM

The quantized energy levels can be written as [1, 3] :

$$E_{nm}/c \;=\; Mc\,\{1+[nm]^{-2}\}^{-1/2}\,, \tag{1}$$

Where the expression [nm] is given by :

$$[nm] = \{n/\alpha - m/\alpha + [m^2/\alpha^2 - 1]^{1/2}\}. \tag{2}$$

In this last expression, α is the fine structure constant, m is j + 1/2, where j is the total angular momentum number, including spin. The principal quantum number n takes the values 1,2,3,…infinity and m = 1,2,3,…,n, is nested within n. The expression for

[nm] contains three terms. The first one involves n, the other two involve m. This suggests that we define hyperbolic angles Θ_m by :

$$\cosh\Theta_m = m/\alpha, \tag{3}$$

$$\sinh\Theta_m = [m^2/\alpha^2 - 1]^{1/2} . \tag{4}$$

This is possible if $\cosh\Theta_m$ is greater or equal to one for all m. Since the inverse of the fine structure constant is equal to 137.037, this condition is automatically satisfied. The first term in [nm] suggests that we define hyperbolic angles Θ_n by :

$$\cosh\Theta_n = n/\alpha, \tag{5}$$

$$\sinh\Theta_n = [\ n^2/\alpha^2 - 1\ ]^{1/2}. \tag{6}$$

Equation (2) now becomes :

$$[nm] = \cosh\Theta_n - \cosh\Theta_m + \sinh\Theta_m . \tag{7}$$

Now define the angles Θ_{nm} by :

$$\sinh\Theta_{nm} = \cosh\Theta_n - \cosh\Theta_m + \sinh\Theta_m , \tag{8}$$

$$\sinh\Theta_{nn} = \sinh\Theta_n . \tag{9}$$

One has :

$$[nm] = \sinh\Theta_{nm} , \tag{10}$$

Therefore :

$$E_{nm}/c = Mc\ [\ 1+(\ \sinh\Theta_{nm}\ )^{-2}\ ]^{-1/2} = Mc\ \tanh\Theta_{nm} . \tag{11}$$

The above expression gives the energy levels in terms of quantized hyperbolic tangents. It is always possible to equate a hyperbolic tangent to an effective speed. In this sense, we can define effective (virtual) internal speeds:

$$v_{nm}/c = \tanh\Theta_{nm} . \tag{12}$$

The energy levels become:

$$E_{nm}/c = Mc\ v_{nm}/c = Mc\ \tanh\Theta_{nm} . \tag{13}$$

The binding energy is $- Mc\ [\ 1 - v_{nm}/c\ ]$. The free state Mc occurs at $v_{nm} = c$. The principal quantum energy levels are obtained of course by setting n = m.

With the quantized speeds v_{nm}, the Dirac energy levels begin to take on the appearance of a kind of mechanical system. We can pursue this idea further by defining:

$$Mc[nm] = Mc\ \sinh\Theta_{nm} = Mc\ v_{nm}/c\ \{1 - v_{nm}^2 / c^2\}^{-1/2} = P_{nm} , \tag{14}$$

and

$$Mc \cosh\Theta_{nm} = Mc \{1 - v_{nm}^2 / c^2\}^{-1/2} = \varepsilon_{nm}/c . \qquad (15)$$

We see that the $Mc \cosh\Theta_{nm}$ has the form of an internal fourth component of relativistic momentum and that $Mc \sinh\Theta_{nm}$ has the form of an internal relativistic 3-momentum. The ratio of the two gives:

$$P_{nm}\, c / \varepsilon_{nm} = v_{nm} /c . \qquad (16)$$

The energy levels are:

$$E_{nm} /c = Mc^2 P_{nm} / \varepsilon_{nm} . \qquad (17)$$

We also have:

$$\varepsilon^2_{nm}/c^2 = P^2_{nm} + M^2 c^2 . \qquad (18)$$

When $n = m$, we get:

$$\varepsilon_n /c = Mc\, n/\alpha , \qquad (19)$$

$$P_n = Mc [n^2/\alpha^2 - 1]^{1/2} . \qquad (20)$$

We see that it is the "fourth component" of the internal relativistic energy "4 - vector" which is quantized in integral numbers.

It is instructive to write the energy levels in a variety of mechanical-like forms in order to be able to compare them more clearly with ordinary relativistic mechanics.

We have:

$$P_{nm} = [(\varepsilon_n - \varepsilon_m)/c + P_m] , \qquad (21)$$

$$\varepsilon_{nm} /c = \{[(\varepsilon_n - \varepsilon_m)/c + P_m]^2 + M^2 c^2\}^{1/2} . \qquad (22)$$

When $n = m$ we have:

$$\varepsilon_n /c = \{P^2_n + M^2 c^2\}^{1/2} . \qquad (23)$$

Compare with Eq. (18). The energy levels E_{nm} /c are the ratio of Eqs. (21, 22) times Mc^2.

Equations (17, 21, 22) show that the energy levels are constructed entirely out of ε_n and ε_m which are quantized in integral numbers. If we think of ε_n and ε_m as masses, because of the equivalence between mass and energy, we see that ε_n and ε_m are quantized multiples of 137.037 times the mass of the electron. The energy levels themselves, viewed as masses, are much tinier masses. *We have an example of very small masses (the energy levels of the hydrogen atom) constructed out of much larger masses (integral multiples of 137.037 times the mass of the electron).*

If n = m we get:

$$E_n /c = Mc\, P_n /\{P^2_n + M^2 c^2\}^{1/2} . \qquad (24)$$

It is possible to rewrite the expressions for P_{nm}, ε_{nm} /c, E_{nm} /c in yet another way. Since:

$$\cosh\Theta_m - \sinh\Theta_m = e^{-\Theta m} , \qquad (25)$$

one has :

$$\sinh\Theta_{nn} - \sinh\Theta_{nm} = [e^{-\Theta m} - e^{-\Theta n}] , \qquad (26)$$

so that :

$$Mc [\sinh\Theta_{nn} - \sinh\Theta_{nm}] = Mc [e^{-\Theta m} - e^{-\Theta n}] . \qquad (27)$$

We can therefore rewrite Eqs. (21, 22) in terms of Mc $e^{-\Theta m}$ and Mc $e^{-\Theta n}$. The energy levels given by Eq. (17) can therefore also be expressed in terms of these *"exponential masses".*

Up to now, only the energy levels which are unobservable have been discussed. The difference between two energy levels are observable. We can rewrite the difference between any two energy levels in a variety of interesting ways.

We have:

$$E_{n'm'} /c - E_{nm} /c = Mc (v_{n'm'} /c - v_{nm} /c)$$

$$= Mc (\tanh\Theta_{n'm'} - \tanh\Theta_{nm})$$

$$= Mc \sinh (\Theta_{n'm'} - \Theta_{nm}) / \cosh\Theta_{n'm'} \cosh\Theta_{nm} . \qquad (28)$$

The difference between two energy levels involves the difference of the angles $(\Theta_{n'm'} - \Theta_{nm})$ as well as the angles $\Theta_{n'm'}$, Θ_{nm} .

Is it possible to find an effective quantized speed for the difference between two energy levels?

Define:

$$\tanh\Theta_{n'm' ; nm} = [\tanh\Theta_{n'm'} - \tanh\Theta_{nm}] . \qquad (29)$$

We first must make sure that $\tanh\Theta_{n'm' ; nm}$ always lies between 0 and 1. The difference of energy levels can be equal to 0. It would mean that there is no emission or absorption of light. The greatest difference of energy levels is the difference between the free state Mc and the ground state E_1 /c. The absolute value of Mc $\tanh\Theta_{n'm' ; nm}$ varies from 0 to Mc $- E_1$ /c. We can therefore write:

$$E_{n'm'} /c - E_{nm} /c = Mc \tanh\Theta_{n'm' ; nm} = Mcv_{n'm' ; nm}/c . \qquad (30)$$

This defines a new effective quantized speed which can be used for the difference between energy levels. This speed is measurable.

We can also represent the difference between any two energy levels in terms of effective quantized virtual accelerations. Putting $Mc = e^2/cL_o$, where L_o is the classical radius of the electron, one obtains:

$$E_{n'm'}/c - E_{nm}/c = e^2/c\,[v_{n'm'} - v_{nm}]/cL_o\,, \qquad (31)$$

with $cT_o = L_o$ and $e^2/c = \hbar\alpha$, one has the acceleration-like term :

$$E_{n'm'}/c - E_{nm}/c = \hbar\alpha\,[v_{n'm'} - v_{nm}]/c^2T_o\,. \qquad (32)$$

In the case of the principal energy levels, we can make connections between the ε_n /c and harmonic oscillators. Using Eq. (19), we have:

$$\varepsilon_n/c = n\hbar\alpha/L_o\,. \qquad (33)$$

In this form, the "fourth component of the quantized four-vector" ε_n /c takes the form of a harmonic oscillator. The same is of course true of ε_m /c.

III The energy levels and undetermined "4 - vectors ". Representations with hyperboloids of one-sheet or two-sheet.

We observe that the combination $[P_{nm}, \mathcal{E}_{nm}/c]$ and $[Mc, E_{nm}/c]$ can be considered as time-like "4-vectors" except for the fact that there are no defined time-like or space-like directions. We can nevertheless define an undetermined unit time-like "4-vector" $\hat{U}$ and an undetermined unit "4-vector" $\hat{A}$ such that $\hat{A}.\hat{U} = 0$; $\hat{A}.\hat{A} = 1$; $\hat{U}.\hat{U} = -1$; We can think of the "vector" $\hat{A}$ as representing the totality of all unit "4-vectors" perpendicular to a given $\hat{U}$ and then repeating the process for all possible $\hat{U}$ directions.

We can then define : $\mathbf{P}_{nm} = P_{nm}\hat{A} + \mathcal{E}_{nm}/c\,\hat{U}$ (1.a) $\mathbf{P}_{nm}^* = \mathcal{E}_{nm}/c\,\hat{A} + P_{nm}\hat{U}$ (1.b)

We also have $\mathbf{E}_{nm}/c = E_{nm}/c\,\hat{A} + Mc\,\hat{U}$ (2.a) $\mathbf{E}_{nm}^*/c = Mc\,\hat{A} + E_{nm}/c\,\hat{U}$ (2.b)

Therefore,

$\mathbf{P}_{nm} = Mc\,[\sinh\Theta_{nm}\hat{A} + \cosh\Theta_{nm}\hat{U}]$ (3.a) $\mathbf{P}_{nm}^* = Mc\,[\cosh\Theta_{nm}\hat{A} + \sinh\Theta_{nm}\hat{U}]$ (3.b)

$\mathbf{E}_{nm}/c = Mc\,[\tanh\Theta_{nm}\hat{A} + \hat{U}]$ (4.a) $\mathbf{E}_{nm}^*/c = Mc\,[\hat{A} + \tanh\Theta_{nm}\hat{U}]$ (4.b)

Equations (3.a, b) suggest that one should define two perpendicular unit "4-vectors" $\hat{A}_{nm}$ and $\hat{U}_{nm}$.
$\hat{A}_{nm} = [\cosh\Theta_{nm}\hat{A} + \sinh\Theta_{nm}\hat{U}]$ (5.a) $\hat{U}_{nm} = [\sinh\Theta_{nm}\hat{A} + \cosh\Theta_{nm}\hat{U}]$ (5.b)

We see that the undetermined perpendicular unit "4-vectors" $\hat{A}_{nm}$, $\hat{U}_{nm}$ are connected to $\hat{A}$, $\hat{U}$ through what looks formally like Lorentz transformations through the angles Θ_{nm}. Equations (3 a, b) and (4.a, b) can now be written as :

$\mathbf{P}_{nm} = Mc\,\hat{U}_{nm}$ (6a) $\mathbf{P}_{nm}{}^* = Mc\,\hat{A}_{nm}$ (6b)

$\mathbf{E}_{nm}/c = (Mc / \cosh\Theta_{nm})\,\hat{U}_{nm}$ (6c) $\mathbf{E}_{nm}{}^*/c = (Mc/\cosh\Theta_{nm})\,\hat{A}_{nm}$ (6d)

Note that $Mc / \cosh\Theta_{nm} = Mc\,\mathrm{sech}\Theta_{nm}$ resembles a quantized 4-vector potential.

Every mass discussed so far: Mc, $Mc \sinh\Theta_{nm}$, $Mc \cosh\Theta_{nm}$, $Mc \tanh\Theta_{nm}$, $Mc\,\mathrm{sech}\Theta_{nm}$,

$Mc\, e^{-\Theta nm}$, can be multiplied by $\hat{A}$, $\hat{U}$, $\hat{A}_{nm}$, $\hat{U}_{nm}$ to make time-like or space-like "4-vectors".

We can also multiply by $(\hat{U} + \hat{A})$ or $(\hat{U}_{nm} + \hat{A}_{nm})$ to make null "4-vectors".

Thus we can have as null "4-vectors" :

$\mathcal{E}_{nm}/c\ (\hat{U} + \hat{A})$, $P_{nm}(\hat{U} + \hat{A})$, $Mc\,(\hat{U} + \hat{A})$, $E_{nm}/c\,(\hat{U} + \hat{A})$, etc. and the same with $(\hat{U}_{nm} + \hat{A}_{nm})$.

Note that $E_{nm}/c\,(\hat{U} + \hat{A})$ gives us the energy levels as a null "4-vector". This enables us to write the difference of any two energy levels as null "4-vectors". We can understand the meaning of the above equations by using a hyperboloid of two-sheet or a hyperboloid of one-sheet imbedded in Minkowski space-time each with axis in the $\hat{U}$ direction.

a) Representation with a hyperboloid of two-sheet.

If we use the hyperboloid of two-sheet, its axis is in the future $\hat{U}$ direction. The $\hat{U}_{nm}$ will be unit normals to the hyperboloid making angles Θ_{nm} with the $\hat{U}$ direction, all drawn from a center O.
The projection of $\hat{U}_{nm}$ onto the $\hat{U}$ axis will be $\cosh\Theta_{nm}$, i.e. $(\hat{U}_{nm} . \hat{U})\,\hat{U} = \cosh\Theta_{nm}\,\hat{U}$ (7) If $\hat{U}_{nm}$ and $\hat{U}$ were fully specified 4-vectors, then $\sinh\Theta_{nm}\,\hat{A}$ would be the component of $\hat{U}_{nm}$ perpendicular to $\hat{U}$, but the direction $\hat{A}$ is not fully defined, only the angle Θ_{nm} is. Therefore, take all possible directions $\hat{A}$ making an angle Θ_{nm} around $\hat{U}$. This describes cones centered at O cut by spheres of radii $\sinh\Theta_{nm}$ which lies on the hyperboloid of two-sheet. The quantized momentum $P_{nm}\,\hat{A} = Mc \sinh\Theta_{nm}\,\hat{A}$ is generated by the intersection of the cones and the hyperboloid. The axis of the hyperboloid can be any axis $\hat{U}$ from center O . Even without specified directions, the angles Θ_{nm} and the cones' intersections with the hyperboloid remain. In the tangent space of the hyperboloid, the cones' intersections are spheres of radii $\tanh\Theta_{nm}$ in all possible directions $\hat{A}$. When multiplied by Mc this gives us the energy levels. $(E_{nm}/c)\,\hat{A} = Mc \tanh\Theta_{nm}\,\hat{A}$ (8) . *The free state, in the tangent space of the hyperboloid centered at O, is : $Mc\,(\hat{U} + \hat{A})$. The mass Mc is therefore represented as a light cone which intersect the tangent space in a sphere of radius Mc.*

If $n = m$ then $\cosh\Theta_{nm} = \cosh\Theta_n = n/\alpha$. $n = 1, 2, 3\ldots$

The virtual time along the axis $\hat{U}$ is therefore measured in integral multiple of $1/\alpha$. So, superficially, one could say that time has been quantized. This phrase is potentially misleading, however, and should be used with care. $\mathcal{E}_{nm}/c$ therefore becomes $\mathcal{E}_n/c = Mc\,(n/\alpha)$. If we think of $\mathcal{E}_n/c$ as an auxiliary mass from which the energy levels are obtained, we can see that $\mathcal{E}_1$ is 137 Mc^2 so it is much larger than Mc^2. The other $\mathcal{E}_n$ are even larger. We see that, *even in the case of the energy levels of the H atom, smaller masses are obtained from larger masses. In the case of the H atom, it is simply* *because* $\sinh\Theta_{nm}$ *is projected onto the tangent space of the hyperboloid of two-sheet through a central projection and becomes* $\tanh\Theta_{nm}$ *.*

b) Representation with a hyperboloid of one-sheet.

We can use a hyperboloid of one-sheet to represent the energy levels. The hyperboloid is centered at O. Its axis is in the direction $\hat{U}$ and its unit normals are given by $\hat{A}$ and the $\hat{A}_{nm}$. The angles Θ_{nm} are now the angles between the horizontal axis $\hat{A}$ and the normals $\hat{A}_{nm}$. $\mathcal{E}_{nm}/c$ can now be assigned a direction $\hat{A}$ to become the "4-vector" $\mathcal{E}_{nm}/c\ \hat{A}$. P_{nm} then has direction $\hat{U}$ and becomes the "4-vector" $P_{nm}\ \hat{U}$. Since $\hat{A}$ represents all possible directions perpendicular to $\hat{U}$, the normals $\hat{A}_{nm}$ form cones with axis $\hat{U}$ which make angles Θ_{nm} with $\hat{A}$. The cones cut the hyperboloid of one-sheet in spheres of radii $\cosh\Theta_{nm}$.

The energy levels are harder to visualize. They lie in the tangent space of the hyperboloid of one-sheet. The tangent space is a cylinder whose axis is in the $\hat{U}$ direction. Its basis is a sphere of radius 1 if unit vectors are used or Mc if the masses are used. The energy levels are given by : $(E_{nm}/c)\ \hat{U} = Mc \tanh\Theta_{nm}\ \hat{U}$ (9) . If $n = m$, $\cosh\Theta_{nm} = \cosh\Theta_n = n/\alpha$. The quantization n/α is no longer in the time-like direction $\hat{U}$ but in the direction $\hat{A}$ i.e all space-like directions perpendicular to $\hat{U}$. *This means that what is quantized, using the hyperboloid of one-sheet representation of the energy levels, are all the possible space-like directions.*

IV The energy levels in integer representation. Energy levels differences

It is possible to write the difference of two energy levels in a variety of suggestive forms. Since :

$(E_{n'm'} - E_{nm})/c = Mc\,[\tanh\Theta_{n'm'} - \tanh\Theta_{nm}] = Mc \sinh(\Theta_{n'm'} - \Theta_{nm})/\cosh\Theta_{n'm'}\cosh\Theta_{nm}$

We can express functions of the differences of hyperbolic angles in terms of the energy levels.

$\sinh(\Theta_{n'm'} - \Theta_{nm}) = [\tanh\Theta_{n'm'} - \tanh\Theta_{nm}]\cosh\Theta_{n'm'}\cosh\Theta_{nm}$ (6a)

$\cosh(\Theta_{n'm'} - \Theta_{nm}) = \{1 + [\tanh\Theta_{n'm'} - \tanh\Theta_{nm}]^2\cosh^2\Theta_{n'm'}\cosh^2\Theta_{nm}\}^{1/2}$ (6b)

$\tanh(\Theta_{n'm'} - \Theta_{nm})$ is the ratio of the two previous expressions.

Since the physical quantities are the differences between energy levels, it would be convenient to have them geometrized with the help of a hyperboloid of one-sheet or two-sheet. For this we need to find an angle relative to the axis of the hyperboloids.

Let $\tanh\Theta_{n'm';nm} = [\tanh\Theta_{n'm'} - \tanh\Theta_{nm}]$ (7a).

We first must make sure that $\tanh\Theta_{n'm';nm}$ always lies between 0 and 1. This is possible because it is equivalent to placing the differences between the two levels at the origin in the tangent space of the hyperboloid of two-sheet centered at O or the hyperboloid of one-sheet centered at O . The difference of energy levels can be equal to 0 . It would mean that there is no emission or absorption of light. Since the difference between any two energy levels can vary between a minimum of 0 to a maximum of $Mc - E_1/c$, $\tanh\Theta_{n'm';nm}$ can vary from 0 to a maximum of $1 - \tanh\Theta_1$ which is less than 1 , so that $\tanh\Theta_{n'm';nm}$ is properly defined in the tangent space centered at O .

$\tanh\Theta_{n'm';nm} = [\tanh\Theta_{n'm'} - \tanh\Theta_{nm}] = \sinh(\Theta_{n'm'} - \Theta_{nm})/\cosh\Theta_{n'm'}\cosh\Theta_{nm}$ (7b)

This equation gives us the $\Theta_{n'm' ; nm}$ in terms of $(\Theta_{n'm'} - \Theta_{nm})$.

We now can calculate the other trigonometric functions.

$\cosh\Theta_{n'm' ; nm} = 1/ \{ 1 - [\tanh\Theta_{n'm'} - \tanh\Theta_{nm}]^2 \}^{1/2}$ (7c)

$\sinh\Theta_{n'm' ; nm} = [\tanh\Theta_{n'm'} - \tanh\Theta_{nm}] / \{ 1 - [\tanh\Theta_{n'm'} - \tanh\Theta_{nm}]^2 \}^{1/2}$ (7d).

V Enery levels. Half integer representation. Angular momentum-like states inside the levels.

We can rewrite all the relations of the previous sections using a half integer quantization of the time-like direction using the hyperboloid of two-sheet, or the space-like directions using the hyperboloid of one-sheet by the following device :

Let $\cosh\Theta_n = n/\alpha = j_n + ½$; $n = 1, 2, 3\ldots$; $j_n = 1/2 , 3/2 , 5/2\ldots$;
Let $\cosh\Theta_m = m/\alpha = j_m + ½$; $m = 1, 2, \ldots, n$; $j_m = 1/2 , 3/2 , \ldots, j_n$;
Let $\cosh\Theta j_n = j_n /\alpha$; $\cosh\Theta j_m = j_m /\alpha$; $\cosh\Theta_{1/2} = 1/ 2\alpha$, $\sinh^2\Theta_{1/2} = (1/4\alpha^2 - 1)$;
$Mc = \hbar\alpha/L_o$; $n^2/\alpha^2 = [j_n (j_n + 1)/\alpha^2 + 1/ 4 \alpha^2] = [j_n (j_n + 1) - 1/ 2 (1 /2 - 1)]/\alpha^2$ (1a)
$n^2/\alpha^2 = [j_n (j_n + 1) - 1/ 2 (1 /2 + 1) + 1] /\alpha^2$ (1b)
with the same equations for m^2/α^2 with n replaced by m .

Therefore : $\mathcal{E}_n / c = \hbar \{ [j_n (j_n + 1) - 1/ 2 (1 /2 - 1)] \}^{1/2} /L_o$ (1c)

with a similar equation for $\mathcal{E}_m / c$.

We notice that $\mathcal{E}_n / c$ and $\mathcal{E}_m / c$ have a form similar to that of angular momentum eigenvalues of operator J– operating on angular momentum states $| j_n, j_{1/2})$ and $| j_m, j_{1/2})$, divided by L_o .[2]
However, a subtle point needs to be mentioned because the relations (1a,b) are valid in general even when n = 1, 2, 3,… infinity and m = 1, 2, 3, … infinity , separately.
In that case, we are dealing with two different principal quantum numbers, not sublevels of n, and $\mathcal{E}_n / c$, $\mathcal{E}_m / c$ refer to two different principal quantum numbers. If m is bounded by n then m refers to sublevels of n and they are involved in constructing P_{nm} and $\mathcal{E}_{nm} / c$. We can check whether there is further resemblance of form by looking at the squares of the $\mathcal{E}$'s. We have:

$(\mathcal{E}_n^2 - \mathcal{E}_m^2) / c^2 = \hbar^2 \{[j_n (j_n + 1)] - [j_m (j_m + 1)]\} /L_o^2.$ (2a)

This resembles the eigenvalues of the operators J- J+ operating on the state $| j_n , j_m)$.[2] The resemblance becomes greater if m is bounded by n .
Since $(m - 1)^2 / \alpha^2 = j_m (j_m - 1) / \alpha^2 + 1/4 \alpha^2 = [j_m (j_m - 1) - 1/2 (1/2 - 1)] /\alpha^2$ (2b)
We have: $(\mathcal{E}^2_n - \mathcal{E}^2_{m-1}) /c^2 = \hbar^2[j_n (j_n + 1) - j_m (j_m - 1)]/L_o^2$ (2c)
This has a form similar to those of the eigenvalues of the angular momentum of operators J+ J– operating on angular momentum states $| j_n , j_m)$ divided by L_o^2 . [2]
We also have : $[j_n (j_n + 1) - j_m (j_m - 1)] = [(j_n^2 - j_m^2) + (j_n + j_m)]$ (3a)

$(j_n + 1/2) + (j_n^2 - 1/4) = [j_n (j_n + 1) - 1/2 (1/2 - 1)]$ (3b)

The P_n , P_m are found using $\sinh\Theta_n$, $\sinh\Theta_m$ we need to find expressions for them.
Using (1a, b) and $\cosh\Theta_{1/2} = 1/2\alpha$, $\sinh^2\Theta_{1/2} = (1/4\alpha^2 - 1)$; we have :

$\mathrm{Sinh}^2\Theta_n = [n^2/\alpha^2 - 1] = [(j_n + ½)^2/\alpha^2 - 1] = [j_n(j_n + 1)/\alpha^2 + (1/4\alpha^2 - 1)]$ (4a) $\mathrm{Sinh}^2\Theta_n = j_n(j_n + 1)/\alpha^2 + \sinh^2\Theta_{1/2}$ (4b) $\mathrm{Sinh}^2\Theta_n = j_n(j_n + 1)/\alpha^2 - 1/2\,(1/2 + 1)/\alpha^2 + (1/\alpha^2 - 1)$ (4c)
$\mathrm{Sinh}^2\Theta_n = j_n(j_n + 1)/\alpha^2 - 1/2\,(1/2 + 1)/\alpha^2 + \sinh^2\Theta_1$ (4d)
$\mathrm{Sinh}^2\Theta_n = [j_n(j_n + 1)/\alpha^2 - 1/2\,(1/2 - 1)/\alpha^2 - 1]$ (4e) so we have :
$P^2_n = \hbar^2[j_n(j_n + 1) - 1/2\,(1/2 - 1)]/L_o^2 - M^2c^2$ (5a)

$P^2_n = \{\hbar^2[j_n(j_n + 1) - 1/2\,(1/2 + 1)]/L_o^2 + \mathcal{E}_1{}^2/c^2\}$ (5b)

$P^2_n = \hbar^2[j_n(j_n + 1) - 1/2\,(1/2 - 1)]/L_o^2 - \hbar^2\alpha^2/L_o{}^2$ (5c) with similar eqs. for P_m .

P^2_n have the extra term $M^2c^2 = \mathcal{E}_1{}^2/c^2 = \hbar^2\alpha^2/L_o{}^2$ added to $\mathcal{E}_n{}^2/c^2$ so they are less angular momentum-like than the $\mathcal{E}_n{}^2/c^2$. The energy levels E_{nm}/c are formed with $\mathcal{E}_n$, $\mathcal{E}_m$, P_m .

$P_{nm} = [(\mathcal{E}_n - \mathcal{E}_m)/c + P_m]$; $\mathcal{E}^2{}_{nm}/c^2 = \{[(\mathcal{E}_n - \mathcal{E}_m)/c + P_m]^2 + M^2c^2\}$;

$E_{nm}/c = Mc\,(P_{nm}\,c/\mathcal{E}_{nm})$;

We can now write the P_{nm} in half integer form :
$P_{nm} = (\hbar/L_o)(j_n - j_m) + (\hbar/L_o)\{[j_m(j_m + 1) - 1/2\,(1/2 - 1)] - \alpha^2\}^{1/2}$ (6a)

This permits us to obtain $\mathcal{E}_{nm}$ in half integral form. We can obtain E_{nm}/c in half integral form by taking the ratio of P_{nm} to $\mathcal{E}_{nm}$. It is also possible to obtain simple relations between angular momentum-like quantities and the expressions $\sinh(\Theta_{n'} - \Theta_n)$ and $\sinh(\Theta_{n'} + \Theta_n)$ discussed previously where n' = 1, 2, 3…infinity and n = 1, 2, 3…infinity. We have for example : $\sinh(\Theta_{n'} - \Theta_n)\sinh(\Theta_{n'} + \Theta_n) = [j_{n'}(j_{n'} + 1) - j_n(j_n + 1)]/\alpha^2$ (7)
where $n' = j_{n'} + 1/2$ and $n = j_n + 1/2$; $j_{n'} = 1, 2, 3\ldots$ infinity ; $j_n = 1, 2, 3\ldots$ infinity.

VI Dirac energy levels and quantized virtual hyperbolic motions. Rindler-like coordinates.

Dirac energy levels and quantized virtual hyperbolic motions.

A) Quantized speeds.

The expression for the energy levels: $E_{nm}/c = Mc[n,m]/\{1+[n,m]^2\}^{1/2}$ (1)

Suggest putting them in the following form: $E_{nm}/c = Mc[gt_{final}/c]_{nm}/\{1+[gt_{final}/c]^2{}_{nm}\}^{1/2}$ (2)

With $[gt_{final}/c]_{nm} = [n,m] = \sinh\Theta_{nm}$ (3)

In this expression, g and t_f appear. One can choose either t_f freely then g is quantized or g is chosen freely and t_f is quantized or both g and t_f are quantized. The idea is to pretend that the energy levels are obtained from a virtual hyperbolic motion from $t = t_{initial}$ to $t = t_{final}$ in such a way that we get $\sinh\Theta_{nm}$ at t_{final} . Hyperbolic motion is expressed by :

$v(t)/c/\{1-[v(t)/c]^2\}^{1/2} - v(t_o)/c/\{1-[v(t_o)/c]^2\}^{1/2} = g(t-t_o)/c$ (4)

putting $t_o = 0$ and $v(t_o)/c = 0$ we get:

$v(t)/c/\{1-[v(t)/c]^2\}^{1/2} = gt/c = \sinh\Theta(t)$ (5)

Therefore : $v_{nm}(t_f)/c/\{1-[v_{nm}(t_f)/c]^2\}^{1/2} = (gt_{final}/c)_{nm} = \sinh\Theta_{nm} = [n,m]$ (6)

We also have: $(v(t_f)/c)_{nm} = (gt_f/c)_{nm}/\{1+(gt_f/c)^2_{nm}\}^{1/2} = \tanh\Theta_{nm} = [n,m]/\{1+[n,m]^2\}^{1/2}$ (7)

Eqs. (6, 7) can be interpreted as meaning that v_{nm}/c arises from a virtual hyperbolic motion from a time $t = 0$ to t_{final} which can be freely chosen as being the same for all nm or that v_{nm}/c arises from a constant acceleration g which is the same for all nm . In the first case, we can write the t_{final} as T_f . T_f is completely arbitrary. In that case, the constant acceleration is quantized and depends on nm . These quantized accelerations will be denoted by g_{nm} . Then :

$g_{nm} = c \sinh\Theta_{nm}/T_f$ (8a) $T_f = t_{final}$ arbitrary.

Eq. (7) becomes : $[v(T_f)/c]_{nm} = (g_{nm} T_f/c)/\{1+(g_{nm} T_f/c)^2\}^{1/2} = \tanh\Theta_{nm}$ (8b)

The angle $\Theta_{nm} = g_{nm}\tau_{nm}/c = c \sinh\Theta_{nm}\tau_{nm}/c T_f$ (8c) τ_{nm} is the proper time.

$\tau_{nm}/T_f = \Theta_{nm}/\sinh\Theta_{nm}$ (8d)

In the second case, g is arbitrary and the same for all nm . Therefore:

$gT_{nm} = c \sinh\Theta_{nm}$; $T_{nm} = c \sinh\Theta_{nm}/g$ (8e) g arbitrary . $T_{nm} = (t_{final})_{nm}$

and $[v_{nm}(T_{nm})/c] = (gT_{nm}/c)/\{1+(gT_{nm}/c)^2\}^{1/2}$ (8f)

Note that in the case where T_f is completely arbitrary, it can be chosen to be Planck's time L_P/c , where L_P is the Planck length. It can be chosen as the age of the universe T_u = 13.6 billion years or the classical radius L_o of the reduced proton-electron mass M etc. The last one is particularly useful.

The angle $\Theta_{nm} = g\tau_{nm}/c$ (8g) $\tau_{nm} = c\Theta_{nm}/g$ (8h)

B) Quantized accelerations.

The three- acceleration $dv(t)/dt = a(t)$ is given by:

$a(t) = g/\{1+[gt/c]^2\}^{3/2}$ (9a) therefore : $a(t)t/c = (gt/c)/\{1+[gt/c]^2\}^{3/2}$ (9b) $a(0) = g$ (9c)

we want:

$[a(t_f)t_f/c]_{nm} = [gt_f/c]_{nm}/\{1+[gt_f/c]^2_{nm}\}^{3/2} = \tanh\Theta_{nm}/[\cosh\Theta_{nm}]^2 = \sinh\Theta_{nm}/[\cosh\Theta_{nm}]^3$ (9d)

We have, as before, the restriction: $[gt_f/c]_{nm} = \sinh\Theta_{nm}$ (9e)

If we choose $t_{final} = T_f$, arbitrary, then:

$a_{nm}(T_f) = g_{nm}/\{1+[g_{nm}T_f/c]^2\}^{3/2}$ (9f) $g_{nm} = c\sinh\Theta_{nm}/T_f$; T_f arbitrary .

From which we get: $a_{nm}(T_f)/c^2 = (v_{nm}/c)(1-v^2_{nm}/c^2)/cT_f$ (9g)

and from eq. (9a) we get:

$a_{nm}(T_f)/(1-v^2_{nm}/c^2)^{3/2} = g_{nm}$ (9h)

$a_{nm}(T_f) = g_{nm}(1-v^2_{nm}/c^2)^{3/2} = c\sinh\Theta_{nm}/T_f\cosh^3\Theta_{nm}$ (9i)

If we also choose $T_f = L_o/c$, we get from eqs. (9d, e, f) , a_{nm} entirely in terms of the parameters of the Dirac energy levels. In the second situation, if we choose g arbitrary, then:

$a_{nm}(T_{nm}) = g / \{ 1+ [g\ T_{nm} / c]^2 \}^{3/2}$ (10a) g arbitrary ; $T_{nm} = c \sinh\Theta_{nm} / g = (T_f)_{nm}$ (10b)

$a_{nm}(T_{nm}) / (1 - v^2_{nm}/c^2)^{3/2} = g$ (10c) $a_{nm}(T_{nm}) = g (1 - v^2_{nm}/c^2)^{3/2} = g / \cosh^3\Theta_{nm}$ (10d)

Since g is arbitrary, we can choose it to be : $g = c^2 / L_o$ (10.e) ; then

$cT_{nm} = L_o \sinh\Theta_{nm}$ (10f) $a_{nm}(T_{nm}) = c^2 / L_o \cosh^3\Theta_{nm} = c^2 (1 - v^2_{nm}/c^2)^{3/2} / L_o$ (10g)

Compare eq. (10g) with eq. (9i).

An electromagnetic field is a field of mass x acceleration per unit charge. *There is a suggestion that the expression $M a_{nm} / e$ represent quantized electromagnetic fields.* We would have:

$(Mc^2/ e)\ v_{nm}/c\ (1 - v^2_{nm}/c^2) / cT_f = (\hbar \alpha c / e L_o)\ v_{nm}/c\ (1 - v^2_{nm}/c^2) / cT_f$ using (9i)

or $(Mc^2/ e)\ g / (1 - v^2_{nm}/c^2)^{3/2} = (\hbar \alpha c / e L_o)\ g / (1 - v^2_{nm}/c^2)^{3/2}$ using (10d)

Such fields do not seem to have been mentioned in the literature.

C) Quantized positions.

The virtual positions associated with the virtual hyperbolic trajectories are easily calculated and are closely connected to Rindler-like coordinate systems as will be shown. (4)

$dx(t)/cdt = v(t)/c = gt/c/\{1+(gt/c)^2\}^{1/2}$ (11a) $dx = gct/c^2/\{1+(gct/c^2)^2\}^{1/2} cdt$ (11b)

Integrate dx from 0 to X and d(ct) from 0 to T ; We get:

$X(T) = c^2/g\{1+[gcT/c^2]\}^{1/2} - c^2/g$ (11c) ; with $X(0) = 0$;

$[1+gX(T)/c^2] = \{1+(gT/c)^2\}^{1/2}$ (11d) ; $[1+gX(T)/c^2]^2 = 1+(gT/c)^2$ (11e)

$[1+gX(T)/c^2]^2 - [gT/c]^2 = 1$ (12a)

$[1+gX(T)/c^2] = \cosh\Theta(T)$ (12b) $gT/c = \sinh\Theta(T)$ (12c)

We need to evaluate eqs. (11c to 12c) at $T_{final} = T_f$. And we want:

$(gT_f/c)_{nm} = \sinh\Theta_{nm} = [n,m]$ (13a) $\{ 1+[gX(T_f)/c^2]_{nm}\} = \cosh\Theta_{nm} = \{1+[n,m]^2\}^{1/2}$ (13b)

The quantization conditions are :

$[gX(T_f)/c^2]_{nm} = \{1+[n,m]^2\}^{1/2} - 1$ (14a) $[gT_f/c]_{nm} = [n,m] = \sinh\Theta_{nm}$ (14b)

with either g or T_f freely chosen.

a) Choose T_f arbitrary, then: $g_{nm} = c \sinh\Theta_{nm} / T_f$ (15a). From eq. (14a) , we get:

$g_{nm} X_{nm}(T_f) / c^2 = \cosh\Theta_{nm} - 1$ (15b) $X_{nm}(T_f) = c^2 [\cosh\Theta_{nm} - 1] / (c \sinh\Theta_{nm} / T_f)$ (15c)

$X_{nm}(T_f) = [c T_f / \tanh\Theta_{nm} - c T_f / \sinh\Theta_{nm}]$ (15d)

This very important relation involves the inverse of $\tanh\Theta_{nm} / cT_f$ and $\sinh\Theta_{nm} / cT_f$ which are proportional to E_{nm} and ε_{nm} respectively. If T_f is chosen as L_o/c we have an immediate and interesting connection with the Dirac energy levels.

$X_{nm}\,(L_o/c\,) = [\,L_o / \tanh\Theta_{nm} - L_o / \sinh\Theta_{nm}\,]$ (16a)
Since $Mc \sinh\Theta_{nm} = \hbar\,\alpha \sinh\Theta_{nm} / L_o$ and $Mc \tanh\Theta_{nm} = \hbar\,\alpha \tanh\Theta_{nm} / L_o$,

Finally : $X_{nm}\,(L_o/c\,) = [\hbar\,\alpha\, c / E_{nm} - \hbar\,\alpha\, c / \mathcal{E}_{nm}\,] = \hbar\,\alpha\, c\,[\,1/ E_{nm} - 1/ \mathcal{E}_{nm}\,]$ (16b)

Eqs. (16.a, b) can easily be represented using a hyperboloid of two-sheet. They represent a kind of Rindler-like coordinate system whose positions from an origin O are :

$c^2 /g^*_{nm} = L_o / \sinh\Theta_{nm}$ (17a) $c^2 /g^{**}_{nm} = L_o / \tanh\Theta_{nm} = L_o \cosh\Theta_{nm} / \sinh\Theta_{nm}$ (17b)

Thus, a Rindler-like coordinate system is intimately connected with the inverse of the energy levels E_{nm} /c and the inverse of the auxiliary momenta $\mathcal{E}_{nm} /c$.

There are two other recognizable terms in the series (17a, b) i.e:

$c^2 /g^{***}_{nm} = L_o \cosh^2\Theta_{nm} / \sinh\Theta_{nm} = L_o \cosh\Theta_{nm} / \tanh\Theta_{nm}$ (18a)

$c^2 /g^{****}_{nm} = L_o \cosh^3 \Theta_{nm} / \sinh\Theta_{nm} = L_o \cosh^2 \Theta_{nm} / \tanh\Theta_{nm}$ (18b)

We notice that from eq. (9.i) $a_{nm}\,(L_o /c) /c^2 = [\,\tanh\Theta_{nm} / L_o \cosh^2 \Theta_{nm}\,]$ (18c)

Therefore: $c^2 /g^{****}_{nm} = c^2 / a_{nm}\,(L_o /c)$ (18d)

We now have the Rindler-like coordinate sequence:

$X_{nm}\,(L_o/c\,) = [\,L_o / \tanh\Theta_{nm} - L_o / \sinh\Theta_{nm}\,] = [\,c^2 /g^{**}_{nm} - c^2 /g^*_{nm}\,]$ (19a)

$X^*_{nm}\,(L_o/c\,) = [\,L_o \cosh\Theta_{nm} / \tanh\Theta_{nm} - L_o / \tanh\Theta_{nm}\,] = [\,c^2 /g^{***}_{nm} - c^2 /g^{**}_{nm}\,]$ (19b)

$X^{**}_{nm}\,(L_o/c\,) = [\,L_o \cosh^2\Theta_{nm} / \tanh\Theta_{nm} - L_o \cosh\Theta_{nm} / \tanh\Theta_{nm}\,] = [\,c^2 /g^{****}_{nm} - c^2 /g^{***}_{nm}\,]$ (19c)

b) If g is chosen as arbitrary, then $T_{nm} = c \sinh\Theta_{nm} / g$.

$g\, X_{nm}\,(\,T_{nm}\,) / c^2 = \cosh\Theta_{nm} - 1$ (20a) $X_{nm}\,(\,T_{nm}\,) = c^2 \cosh\Theta_{nm} / g - c^2 / g$ (20b)

Again we have a Rindler-like coordinate system with :

c^2/g , $c^2/g^+ = (c^2/g) \cosh\Theta_{nm}$, $c^2/ g^{++} = (c^2/g) \cosh^2\Theta_{nm}$, $c^2/ g^{+++} = (c^2/g) \cosh^3 \Theta_{nm}$ (20c) etc.

and $X_{nm}\,(\,T_{nm}\,) = c^2 \cosh\Theta_{nm} / g - c^2 / g = c^2/g^+ - c^2/g$ (21a)

$X^+_{nm}\,(\,T_{nm}\,) = c^2 \cosh^2 \Theta_{nm} / g - c^2 \cosh\Theta_{nm} / g = c^2/g^{++} - c^2/g^+$ (21b)

$X^{++}_{nm}\,(\,T_{nm}\,) = c^2 \cosh^3 \Theta_{nm} / g - c^2 \cosh^2 \Theta_{nm} / g = c^2/g^{+++} - c^2/g^{++}$ (21c) etc.

The previous X_{nm} sequence are points on the horizontal axis of the Rindler-like coordinate system. The points represent the initial points of a virtual motion which ends on an axis making an angle Θ_{nm} with the horizontal axis. This is true for each nm. The initial horizontal axis and a final axis open up through an angle Θ_{nm} issued from a center O. The vertical time axis sequence is :

$T_{nm} = c \sinh\Theta_{nm} / g$ (22a) $T^+_{nm} = c \sinh\Theta_{nm} \cosh\Theta_{nm} / g = c \sinh\Theta_{nm} /g^+$ (22b)

$T^{++}_{nm} = c \sinh\Theta_{nm} \cosh^2 \Theta_{nm} / g = c \sinh\Theta_{nm} /g^{++}$ (22c) etc.

T_{nm}, T^{+}_{nm}, T^{++}_{nm} represent the time of duration of the virtual motions at each position of the Rindler-like coordinate system.

If g is chosen as c^2/L_o then we get $c\,T_{nm} = L_o \sinh\Theta_{nm}$ (22d) and

$X_{nm}(T_{nm}) = L_o \cosh\Theta_{nm} - L_o$ (22e)

Repeat the previous process to get the Rindler-like coordinates entirely in terms of the Dirac energy levels parameters. The same is also true of course of all the expressions in previous section a).

VII Geometric representation of the Klein-Gordon energy levels.

The Klein-Gordon energy levels in a Coulomb potential are given by [(1)] :

$E_{nl}/c = Mc\{1 + [n,l]^{-2}\}^{-1/2}$; $l = 0,1,2,3,\ldots,n-1$; $n = 1,2,3,\ldots,$infinity ;

$[n,l] = \{ n/\alpha - (l + 1/2)/\alpha + [(l + 1/2)^2/\alpha^2 - 1]^{1/2} \}$;

If we can write $[n, l] = \sinh\Theta_{nl}$, we get: $E_{nl}/c = Mc\tanh\Theta_{nl}$;

We want to be able to compare these energy levels with those of the Dirac energy levels in order to see if we can find a construction method for the $\sinh\Theta_{nl}$.

Let $n = j_n + 1/2$ (1a) $m = j_m + 1/2$ (1b) $l = m - 1$ (1c)

then $l + 1/2 = m - 1 + 1/2 = j_m + 1/2 - 1 + 1/2 = j_m$; $(l + 1/2) = j_m$ (1d)

where $n = 1, 2, \ldots,$ infinity , $m = 1, 2, \ldots, n$ as before.

We have: $[n, l] = \{ 1/2\,\alpha + (j_n - j_m)/\alpha + [j_m^2/\alpha^2 - 1]^{1/2} \}$ (2a)

Let $[j_n, j_m] = (j_n - j_m)/\alpha + [j_m^2/\alpha^2 - 1]^{1/2}$ (2b)

We notice that eq. (2b) has the same form as [n,m] of eq. (II 1b)
except that $j_n = 1/2, 3/2, \ldots$and $j_m = 1/2, 3/2, \ldots, j_n$ replace $n = 1, 2, \ldots$ and $m = 1,2,\ldots,n$

We have : $[n, l] = [j_n, j_m] + (1/2)/\alpha$ (3)

We can therefore set $\sinh\Theta_{jn,jm} = \cosh\Theta_{jn} - \cosh\Theta_{jm} + \sinh\Theta_{jm} = [j_n, j_m]$ (4a)

With $\cosh\Theta_{jn} = j_n/\alpha$ (4b) $\cosh\Theta_{jm} = j_m/\alpha$ (4c)

The quantization of timelike direction is therefore half integer. Same with the equivalent spacelike representation. The geometric construction of $\sinh\Theta_{jn,jm}$ is exactly the same as that of $\sinh\Theta_{n,m}$ so $\sinh\Theta_{jn,jm}$ can be constructed. We can obtain the Klein-Gordon energy levels with the assignment $[n, l] = \sinh\Theta_{nl}$. We get the important relation :

$[n, l] = \sinh\Theta_{nl} = \sinh\Theta_{jn,jm} + (1/2)/\alpha$ (5a)

$\cosh\Theta_{nl} = \{1 + [\sinh\Theta_{jn,jm} + (1/2)/\alpha]^2\}^{1/2}$ (5b) The energy levels are :

$E_{n,l}/c = Mc\tanh\Theta_{n,l} = Mc[\sinh\Theta_{jn,jm} + (1/2)/\alpha] / \{1 + [\sinh\Theta_{jn,jm} + (1/2)/\alpha]^2\}^{1/2}$ (5c)

The energy levels can easily be constructed and represented geometrically in terms of a hyperboloid of one-sheet or a hyperboloid of two-sheet. Note also the presence of the extra term $(1/2)/\alpha = \cosh\Theta_{1/2}$ in eqs. (5a, b, c) .

Setting :

$\mathcal{E}_{jn} / c = Mc\, j_n / \alpha = \hbar\, j_n / L_o$ (6a) $\mathcal{E}_{jm} / c = Mc\, j_m / \alpha = \hbar\, j_m / L_o$ (6b)

We have $P_{jm} = Mc \{ (j_m / \alpha)^2 - 1 \}^{1/2} = \{ (\hbar\, j_m / L_o)^2 - M^2 c^2 \}^{1/2}$ (6c)

and $P_{jn,\, jm} = \mathcal{E}_{jn} / c - \mathcal{E}_{jm} / c + P_{jm}$ (7a)

$P_{n,\, l} = Mc [n, l] = Mc \sinh\Theta_{nl} = Mc [\sinh\Theta_{jn\,,\, jm} + (1/2)/\alpha]$ (7b)

$P_{n,\, l} = P_{jn,\, jm} + Mc / 2\alpha = P_{jn,\, jm} + \mathcal{E}_{1/2}$ (7c) $\mathcal{E}_{1/2} = Mc / 2\alpha$ (7d)

$(\mathcal{E}_{n,\, l} / c)^2 = [(P_{n,\, l})^2 + M^2 c^2]$ (8a) $(\mathcal{E}_{n,\, l} / c)^2 = \{ [P_{jn,\, jm} + \mathcal{E}_{1/2}]^2 + M^2c^2 \}$ (8b)

The energy levels $E_{n,\, l} / c$ are then given by : $E_{n,\, l} / c = Mc^2\, P_{n,\, l} / \mathcal{E}_{n,\, l}$ (9a)

$E_{n,\, l} / c = Mc [P_{jn,\, jm} + \mathcal{E}_{1/2}] / \{ [P_{jn,\, jm} + \mathcal{E}_{1/2}]^2 + M^2c^2 \}^{1/2}$ (9b)

The energy levels are therefore constructed from masses $Mc \sinh\Theta_{nl}$ and $Mc \cosh\Theta_{nl}$ which are themselves constructed from masses $Mc \sinh\Theta_{jn\,,\, jm}$, $Mc / 2\alpha$, which are themselves constructed from masses $Mc\, j_n /\alpha$, $Mc\, j_m /\alpha$ and Mc. Note that since $Mc\, j_n /\alpha$ and $Mc\, j_m /\alpha$ can be written as $\hbar\, j_n / L_o$ and $\hbar\, j_m / L_o$ and $Mc / 2\alpha = \hbar /2 L_o$. The energy levels can be constructed out of these harmonic oscillator-like masses and Mc.

VIII Dirac energy levels of hydrogen-like atoms.

The energy levels are [1, 3] :

$E_{nm} /c = Mc \{ 1 + [nm]^{-2} \}^{-1/2}$ (1a) $[nm] = \{ n/ Z\alpha - m/ Z\alpha + [m^2/ Z^2\alpha^2 - 1]^{1/2} \}$ (1b)

$n = 1, 2, 3,\ldots$ infinity ; $m = 1, 2, \ldots,n$; $Z = 1, 2, 3, \ldots$;

We have: $\cosh\Theta_n = n / Z\alpha$ (2a) $\cosh\Theta_m = m / Z\alpha$ (2b)

$\sinh\Theta_n = [n^2/ Z^2\alpha^2 - 1]^{1/2}$ (2c) $\sinh\Theta_m = [m^2/ Z^2\alpha^2 - 1]^{1/2}$ (2d)

$\sinh\Theta_{nm} = [n/ Z\alpha - m/ Z\alpha + [m^2/ Z^2\alpha^2 - 1]^{1/2}] = [n,m]$ (3)

The time-like quantization using the hyperboloid of two-sheet becomes:

$\cosh\Theta_n = 1/Z\alpha , 2/Z\alpha , 3/Z\alpha \ldots$ but Z can also take the values 1, 2, 3, …

If $Z = 1$ we have the hydrogen atom which has already been discussed.
Since $\cosh\Theta_n = n /Z\alpha$ must always be > or = 1 , we have to verify under what conditions $\cosh\Theta_n$ and $\cosh\Theta_m$ are > or = 1 . If $Z = 1$, $\cosh\Theta_n$ is always > or = 1 . This is also true for $Z = 1, 2, 3,$ up to 137 . Since $\cosh\Theta_n = 137n/Z + 0.037n/Z$ (4)
We can expect that $\cosh\Theta_n$ will become < 1 when $Z > 137.037n$, i.e.
Z > or = to 138n and similarly $\cosh\Theta_m < 1$ when Z > or = to 138m.
The lower states fade away as Z increases until eventually all states are extinguished. This of course does not take into account other reasons for extinctions discussed in the literature. [1]
The half integer representation of the H energy levels is modified for hydrogen-like atoms. We have for example :

$m^2 / Z^2 \alpha^2 = [j_m (j_m + 1) / Z^2 \alpha^2 + 1 / 4Z^2 \alpha^2] = [j_m (j_m + 1) - 1/2 (1/2 - 1)] / Z^2 \alpha^2$ (5a)

$Mc = \hbar \alpha / L_o$; M is the electron mass. $ZMc = \hbar Z \alpha / L_o$ (5b)

$Z^2 M^2 c^2 \cosh^2\Theta_m = \hbar^2 [j_m (j_m + 1) - 1/2 (1/2 - 1)] / L^2_o$ (6a)

$Z^2 M^2 c^2 \sinh^2\Theta_m = \hbar^2 \{ [j_m (j_m + 1) - 1/2 (1/2 - 1)] - Z^2 \alpha^2 \} / L^2_o$ (6b)

So, we get the angular momentum-like entities if we use $Z Mc$ instead of Mc .

IX Conclusion

The existence of quantized speeds $v_{n,m}$ hidden within the Dirac energy levels of the H or H-like atoms opens up a variety of intriguing possibilities. Since the auxiliary masses (or energies) $\varepsilon_n = Mc^2 n/Z\alpha$, are much higher than the mass of an electron, the present work might help uncover new regularities among masses of elementary particles. In addition, if the hydrogen or hydrogen-like atoms have hidden relativistic speeds, why not other atoms? The Helium atom might be a possible candidate to generalize the present work. It might perhaps also be possible to use the present scheme to include some molecules, or even some atomic nuclei. An intriguing possibility would be applications to the theory of fluids or condensed matter systems. The energy momentum tensor for a perfect fluid for example might be expressible in terms of quantized internal speeds in some cases.

X References

[1] W. Greiner, *Relativistic Quantum Mechanics, Wave Equations* 2nd edition (Springer- Verlag, New York, 1994). See page 182 for the Dirac energy level formula. For the Klein-Gordon energy levels, see page 39, eq. (21a). See page 182 and 183 for the extinction of the Dirac hydrogen-like atoms energy levels.

[2] W. Greiner and B. Müller , *Quantum Mechanics, Symmetries* 2nd edition (Springer-Verlag, New York, 1994). For eq. (1c), section V, see page 57, eq. (2.18b). For eqs. (2a, 2c), section V, see page 55, eq. (2.11).

[3] A. Messiah, *Quantum Mechanics*, (John Wiley & Sons, New York, 1962), Vol. II. See page 931, equations (XX.179, XX.179') for the Dirac energy level formula.

[4] W. Rindler, *Introduction To Special Relativity* 2nd edition (Oxford University Press, New York, 1991).