Uniqueness theorem of the curvature tensor

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Abstract. This paper develops the uniqueness theorem of the curvature tensor, which states that the Riemann-Christoffel tensor (and its linear combinations) is the only tensor that depends on the connection and is linear with respect to the second derivatives of the metric tensor. From this result, Cartan's theorem is obtained, according to which Einstein's tensor is the only second-order tensor that depends on the metric tensor, on its first derivatives, is linear with respect to the second derivatives of the metric tensor and its covariant divergence is null, admitting that the coefficients of these second derivatives are tensors derived from the metric tensor.

1. Gravitational Field Equations

In November 1915, Hilbert and Einstein almost simultaneously obtained the gravitation equation in the presence of matter [1]. Hilbert reached the final equation through a variational principle, with the same procedure as today [2], using density Lagrangian

$$\boldsymbol{L} = \sqrt{g}R$$

where g is the absolute value of the metric tensor determinant, and R is the scalar curvature [3].

Einstein "suggests" as an equation for the external case, or in the absence of matter, the annulment of the Ricci tensor R_{ik}, and subsequently generalizes the equation for the presence of matter [4].

As an argument to suggest the equation $R_{ik} = 0$, Einstein noted that "there is a minimum of arbitrariness in the choice of these equations. Because apart from R_{ik} there is no second-order tensor that is formed by the g_{ik} and its derivatives, not containing derivatives higher than the second and being linear in these derivatives".

In his work on Relativity, Lichnerowic [5] pointed out another procedure to obtain the gravitational field equations. In this derivation, the Poisson equation $\nabla^2\phi = -4\pi G\rho$ of Newtonian gravitation is generalized; in the first term of this equation, there are the second derivatives of the potential ϕ, and in the second, the density of matter, which is the gravitational source. By analogy, Lichnerowicz looks for a tensor equation of the type

$$S_{ik} = -\chi T_{ik}$$

S_{ik} is a tensor of geometric content, and T_{ik} is the tensor energy-momentum. The previous equation must have two conditions; the first is that S_{ik} is a tensor that depends exclusively on the potentials (that is, the metric tensor) and its first-order derivatives, being linear with respect to the derivatives of second-order, in similarity with the Poisson equation. Moreover, that the gradient of S_{ik} is null, in order to satisfy the energy-momentum conservation equation $D_i T^i_k = 0$.

In 1922 Èlie Cartan [6] demonstrated that the only tensor that satisfies the above conditions is given by

$$S_{ik} = h\left[R_{ik} - \frac{1}{2}\left(R + k\right)g_{ik}\right]$$

h and k are constants to be determined. If S_{ik} is equal to the energy-momentum tensor, the relativistic gravitation equation is obtained.

The previous Cartan theorem has a more general version, which we call uniqueness theorem of the curvature tensor: the only tensor that depends on the metric tensor, its first derivatives and is linear with respect to its second derivatives is the tensor

$$R_{ijpq}+\varphi g_{ij}g_{pq}+\gamma g_{ip}g_{jq}+\eta g_{iq}g_{jp}$$

R_{ijpq} is the curvature tensor of Riemann-Christoffel, and ϕ, γ, η are arbitrary numerical constants. Of this, directly derives Cartan's theorem cited above.

Our intention in this research is the demonstration of the uniqueness theorem, which, although other authors have treated it, deserves a broader treatment [7].

2. Definitions

A manifold of Riemann is characterized by having a symmetric metric tensor, a symmetrical connection and the covariant derivative of the metric tensor identically null [8]. From these properties follows that the connection of a manifold of Riemann are the symbols of Christoffel

$$\Gamma^r_{pq}=\frac{1}{2}g^{rs}\left(\partial_p g_{sq}+\partial_q g_{sp}-\partial_s g_{pq}\right). \tag{1}$$

In an invertible coordinate transformation $x'^p=x'^p\left(x^k\right)$ the connection is transformed according to the law

$$\Gamma'^r_{pq}=B^m_pB^n_qA^r_s\Gamma^s_{mn}+B^s_{pq}A^r_s \tag{2}$$

with the definitions

$$A^r_s=\frac{\partial x'^r}{\partial x^s};\quad B^m_p=\frac{\partial x^m}{\partial x'^p};\quad B^s_{pq}=\frac{\partial^2 x^s}{\partial x'^p\partial x'^q};\quad A^s_{ri}=\frac{\partial^2 x'^s}{\partial x^r\partial x^i}.$$

By the nullity of the covariant derivative of the metric tensor $D_i g_{pq}=0$ follows that its partial derivative in function of the connection is

$$D_i g_{pq}=0\quad\Rightarrow\quad\partial_i g_{pq}=g_{sp}\Gamma^s_{iq}+g_{sq}\Gamma^s_{ip}. \tag{3}$$

The fourth-order Riemann-Christoffel curvature tensor R_{psir} is defined by

$$R_{psir}=g_{pk}R^k_{sir}=g_{pk}\left(\partial_i\Gamma^k_{sr}-\partial_r\Gamma^k_{si}+\Gamma^n_{sr}\Gamma^k_{ni}-\Gamma^n_{si}\Gamma^k_{nr}\right) \tag{4}$$

that has the symmetries

$$R_{psir}=R_{irps};\quad R_{psir}=-R_{psri};\quad R_{psir}=R_{spri}.$$

If the connection of a manifold is symmetric, there is a coordinate system for each point for which the connection is null at that point, but its derivatives do not have to be null. This coordinate system is called locally inertial. The transformation to locally inertial coordinates does not modify the components of a tensor.

The locally inertial coordinate system is not unique. We can change from one to another through a coordinate transformation that has the condition $B^s_{pq}=0$ at the point considered.

3. Tensor linearly dependent on the second derivatives of the metric tensor

In the following calculations, we refer to a locally inertial system, that is, in which Christoffel's symbols are null at a given point, this coordinate system always exists in a manifold of Riemann for having symmetric connection.

We look for a tensor T_{ijpq} that depends on the connection and is linear with respect to the second derivatives of the metric tensor g_{ik}. In a locally inertial system, the components of the connection are null; therefore, T_{ijpq} only depends on the second derivatives of g_{ik}. The maximum number of second derivatives on which T_{ijpq} can depend is six, taking into account the symmetry of the metric tensor and that the order of the derivation can be altered, then

$$T_{ijpq}=\alpha\partial_{ij}g_{pq}+\beta\partial_{ip}g_{jq}+\chi\partial_{qi}g_{jp}+\delta\partial_{qp}g_{ij}+\varepsilon\partial_{pj}g_{qi}+\phi\partial_{qj}g_{ip} \tag{5}$$

$\alpha, \beta, \chi, \delta, \varepsilon, \phi$ are numerical constants. Deriving (3), the second derivatives of the metric tensor in a locally inertial system are obtained

$$\begin{aligned}
\partial_{ij}g_{pq}&=g_{sp}\partial_i\Gamma^s_{jq}+g_{sq}\partial_i\Gamma^s_{jp}\\
\partial_{ip}g_{jq}&=g_{sj}\partial_i\Gamma^s_{pq}+g_{sq}\partial_i\Gamma^s_{pj}\\
\partial_{qi}g_{jp}&=g_{sj}\partial_q\Gamma^s_{ip}+g_{sp}\partial_q\Gamma^s_{ij}\\
\partial_{qp}g_{ij}&=g_{si}\partial_q\Gamma^s_{jp}+g_{sj}\partial_q\Gamma^s_{ip}\\
\partial_{pj}g_{qi}&=g_{sq}\partial_p\Gamma^s_{ij}+g_{si}\partial_p\Gamma^s_{jq}
\end{aligned} \tag{6}$$

$$\partial_{qj} g_{ip} = g_{si} \partial_q \Gamma^s_{jp} + g_{sp} \partial_q \Gamma^s_{ji} \tag{6}$$

expressions that when replacing them in (5) result

$$\begin{aligned} T_{ijpq} &= g_{sp}\left(\alpha \partial_i \Gamma^s_{jq} + \chi \partial_q \Gamma^s_{ij} + \phi \partial_q \Gamma^s_{ij}\right) + g_{sq}\left(\alpha \partial_i \Gamma^s_{jp} + \beta \partial_i \Gamma^s_{jp} + \varepsilon \partial_p \Gamma^s_{ij}\right) + \\ &+ g_{sj}\left(\beta \partial_i \Gamma^s_{pq} + \chi \partial_q \Gamma^s_{ip} + \delta \partial_q \Gamma^s_{ip}\right) + g_{si}\left(\delta \partial_q \Gamma^s_{jp} + \varepsilon \partial_p \Gamma^s_{jq} + \phi \partial_q \Gamma^s_{jp}\right). \end{aligned} \tag{7}$$

We now consider a coordinate transformation that leads to a new locally inertial system. We want what T_{ijpq} to be a tensor with respect to transformations of generic coordinates; therefore, it must also be a tensor with respect to the transformation of a locally inertial system to another system of the same characteristic.

From the law of transformation of the connection (2) follows that the law of inverse transformation is

$$B^w_r A^q_v A^p_t \Gamma'^r_{pq} = \Gamma^w_{tv} + A^q_v A^p_t B^w_{pq}$$

making the simplification

$$-A^q_v A^p_t B^w_{pq} = -\frac{\partial x'^q}{\partial x^v}\frac{\partial x'^p}{\partial x^t}\frac{\partial^2 x^w}{\partial x'^p \partial x'^q} + \frac{\partial}{\partial x'^s}\left(\frac{\partial x'^r}{\partial x^t}\frac{\partial x^w}{\partial x'^r}\right)\frac{\partial x'^s}{\partial x^v},$$

the derivative of the expression in parentheses is null. Developing the previous expression

$$-A^q_v A^p_t B^w_{pq} = A^r_{vt} B^w_r \tag{8}$$

therefore the inverse of the law of connection transformation is

$$\Gamma^w_{tv} = B^w_r A^q_v A^p_t \Gamma'^r_{pq} + A^r_{vt} B^w_r,$$

solving A^r_{vt} and deriving

$$A^m_{vtn} = A^m_w \partial_n \Gamma^w_{tv} - A^q_v A^p_t A^s_n \partial'_s \Gamma'^m_{pq}$$

where has been taken into account that the two coordinate systems are locally inertial and therefore null the Christoffel symbols at the point considered. From the previous expression, we deduce

$$\partial'_c \Gamma'^m_{ba} = B^v_a B^t_b B^n_c A^m_w \partial_n \Gamma^w_{tv} - B^v_a B^t_b B^n_c A^m_{vtn}. \tag{9}$$

In a coordinate transformation, the tensor T'_{ijpq} is

$$\begin{aligned} T'_{ijpq} &= g'_{sp}\left(\alpha \partial'_i \Gamma'^s_{jq} + \chi \partial'_q \Gamma'^s_{ij} + \phi \partial'_q \Gamma'^s_{ij}\right) + g'_{sq}\left(\alpha \partial'_i \Gamma^s_{jp} + \beta \partial'_i \Gamma'^s_{jp} + \varepsilon \partial'_p \Gamma'^s_{ij}\right) + \\ &+ g'_{sj}\left(\beta \partial'_i \Gamma'^s_{pq} + \chi \partial'_q \Gamma'^s_{ip} + \delta \partial'_q \Gamma'^s_{ip}\right) + g'_{si}\left(\delta \partial'_q \Gamma'^s_{jp} + \varepsilon \partial'_p \Gamma'^s_{jq} + \phi \partial'_q \Gamma'^s_{jp}\right), \end{aligned}$$

using (9) and since g'_{ik} is a covariant second-order tensor, we get

$$\begin{aligned} T'_{ijpq} &= B^v_i B^t_j B^e_p B^n_q T_{vten} - (\alpha+\chi+\phi) B^v_q B^t_j B^n_i B^d_s B^e_p A^s_{vtn} g_{de} - (\alpha+\beta+\varepsilon) B^v_p B^t_j B^n_i B^d_s B^e_q A^s_{vtn} g_{de} - \\ &- (\beta+\chi+\delta) B^v_q B^t_p B^n_i B^d_s B^e_j A^s_{vtn} g_{de} - (\delta+\varepsilon+\phi) B^v_j B^t_p B^n_q B^d_s B^e_i A^s_{vtn} g_{de}, \end{aligned} \tag{10}$$

defining

$$Z_{qjip} = B^v_q B^t_j B^n_i B^d_s B^e_p A^s_{vtn} g_{de}$$

for T_{ijpq} to be a tensor it is necessary that

$$(\alpha+\chi+\phi) Z_{qjip} + (\alpha+\beta+\varepsilon) Z_{pjiq} + (\beta+\chi+\delta) Z_{qpij} + (\delta+\varepsilon+\phi) Z_{jpqi} = 0 \tag{11}$$

as the coordinate transformation is arbitrary, with the only condition that at the point considered $B^s_{pq} = 0$, then a transformation can always be found with which the system of homogeneous linear equations (11) has a rank of four *, just like the number of unknowns, and therefore the system will be determined compatible, and the only solution is the trivial, that is

$$\alpha+\chi+\phi = 0; \quad \alpha+\beta+\varepsilon = 0; \quad \beta+\chi+\delta = 0; \quad \delta+\varepsilon+\phi = 0. \tag{12}$$

Under these conditions, T_{ijpq} it is a fourth-order covariant tensor. When solving (12), there are two indeterminate unknowns, if we choose α and χ, then

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* An example of a coordinate transformation that converts a locally inertial system into another with the same characteristic is

$$x'^i = \alpha^i_j \left(x^j - x^j_0\right) + \alpha^i_{jkr}\left(x^j - x^j_0\right)\left(x^k - x^k_0\right)\left(x^r - x^r_0\right)$$

transformation that satisfies the condition $B^s_{pq} = 0$ at point $x^i_0 = 0$ as required, which follows from (8)

$$\beta=-\alpha-\chi;\quad \delta=\alpha;\quad \varepsilon=\chi;\quad \phi=-\alpha-\chi. \tag{13}$$

By applying conditions (12) to the tensor defined in (7), we obtain from equation (10)

$$T'_{ijpq}=B^v_i B^t_j B^e_p B^n_q T_{vten}$$

which demonstrates that T_{ijpq} has tensor character at least for transformations between locally inertial coordinate systems and at the point where they are defined.

Applying (13) in (7) and as the curvature tensor (4) in a locally inertial system is

$$R_{psir}=g_{pk}\left(\partial_i\Gamma^k_{sr}-\partial_r\Gamma^k_{si}\right) \tag{14}$$

then

$$T_{ijpq}=\alpha R_{pjiq}+\alpha R_{qjip}+\beta R_{jpiq}+\varepsilon R_{ijpq}=\alpha R_{pjiq}+\alpha R_{qjip}-\left(\alpha+\chi\right)R_{jpiq}+\chi R_{ijpq}$$

which demonstrates that the tensor T_{ijpq}, that is, one that depends on the connection and is linear with respect to the second derivatives of the metric tensor, has as a more general expression, in a locally inertial system, a linear combination of the curvature tensor.

4. The curvature tensor as a function of the second derivatives of the metric tensor in a locally inertial system

Making the derivative of the Christoffel symbols (1) and taking into account that we do the calculations in a locally inertial system where Γ^j_{ik} is null and by (3) the first derivatives of the metric tensor are also null

$$\partial_t\Gamma^r_{pq}=\frac{1}{2}g^{rs}\left(\partial_{tp}g_{sq}+\partial_{tq}g_{sp}-\partial_{ts}g_{pq}\right) \tag{15}$$

which allows us to put (14) depending on the second derivatives of the metric tensor

$$R_{psir}=\frac{1}{2}\left(\partial_{si}g_{rp}-\partial_{pi}g_{sr}-\partial_{sr}g_{ip}+\partial_{pr}g_{si}\right) \tag{16}$$

expression that is valid exclusively at the point where the locally inertial system is defined.

5. The tensor T_{ijpq} based on four second derivatives of the metric tensor

We have verified that six is the maximum number of second derivatives of the metric tensor with which the tensor T_{ijpq} can be formed, which is a tensor that depends linearly on the second derivatives of g_{ik}. We wonder, what is the smallest number of second derivatives of g_{ik} that can generate the tensor T_{ijpq}?

The determinant of the coefficients of the system of homogeneous linear equations (12) is

$$\begin{vmatrix}1&0&1&0&0&1\\1&1&0&0&1&0\\0&1&1&1&0&0\\0&0&0&1&1&1\end{vmatrix}$$

has rank four; that is to say, the number of unknowns, which is six, is higher than the rank of the determinant of the coefficients; therefore, the system is indeterminate compatible and, therefore, with infinite solutions.

If instead of six unknowns the system of equations (12) had five unknowns, we would again have an undetermined compatible system, in effect, suppose that the tensor T_{ijpq} depended on five second derivatives of g_{ik} [eg, in (5) $\phi=0$], so

$$T_{ijpq}=\alpha\left[\partial_{ij}g_{pq}-\partial_{qi}g_{jp}+\partial_{qp}g_{ij}-\partial_{pj}g_{qi}\right]=2\alpha R_{pijq}$$

if instead of ϕ we nullify another of the coefficients of (12) we would find that the tensor T_{ijpq} is still proportional to the curvature tensor.

Are possible combinations of four unknowns that generate determinants of the coefficients

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considering that $A^r_{vt}=0$ at point $x^i_0=0$. The coefficients α^i_j and α^i_{jkr} are arbitrary numerical values, and x^i_0 are the coordinates of the point where it is defined the locally inertial system. Given the arbitrariness of the numerical coefficients, we can always find some coefficients with which to ensure that the determinant of the coefficients of the system of equations (11) is of rank four and therefore has the trivial solution.

of rank three, and therefore being less than the number of unknowns would have different solutions from the trivial. By direct inspection, we found that the combinations of unknowns that make the system (12) have non-trivial solutions are

$$\begin{array}{ll} (1)\quad \alpha=0;\quad \delta=0 \quad\Rightarrow & \chi+\phi=0;\quad \beta+\varepsilon=0;\quad \beta+\chi=0;\quad \varepsilon+\phi=0 \\ & \beta=-\chi;\quad \varepsilon=\chi;\quad \phi=-\chi \\ (2)\quad \beta=0;\quad \phi=0 \quad\Rightarrow & \alpha+\chi=0;\quad \alpha+\varepsilon=0;\quad \chi+\delta=0;\quad \delta+\varepsilon=0 \\ & \alpha=-\chi;\quad \delta=-\chi;\quad \varepsilon=\chi \\ (3)\quad \chi=0;\quad \varepsilon=0 \quad\Rightarrow & \alpha+\phi=0;\quad \alpha+\beta=0;\quad \beta+\delta=0;\quad \delta+\phi=0 \\ & \beta=-\alpha;\quad \delta=\alpha;\quad \phi=-\alpha \end{array} \tag{17}$$

also, by immediate inspection, we verify that there are no systems with 3, 2, or 1 unknowns that have non-trivial solutions; therefore, the minimum number of second derivatives on which the T_{ijpq} tensor can depend is four.

6. The tensor T_{ijpq} as a function of four second derivatives of the metric tensor

From (5), (16) and (17) we can derive the tensor T_{ijpq} that in a locally inertial system depends exclusively on four second derivatives of the metric tensor

$$\begin{aligned} T^{(1)}_{ijpq} &= \chi\left(-\partial_{ip}g_{jq}+\partial_{iq}g_{jp}+\partial_{jp}g_{iq}-\partial_{jq}g_{ip}\right)=2\chi R_{pqij} \\ T^{(2)}_{ijpq} &= \chi\left(\partial_{ij}g_{pq}-\partial_{iq}g_{jp}+\partial_{pq}g_{ij}-\partial_{jp}g_{iq}\right)=2\chi R_{piqj} \\ T^{(3)}_{ijpq} &= \alpha\left(\partial_{ij}g_{pq}-\partial_{ip}g_{jq}+\partial_{pq}g_{ij}-\partial_{jq}g_{ip}\right)=2\alpha R_{pjiq} \end{aligned}$$

all three possibilities are proportional to the curvature tensor.

We shown that at the point where the locally inertial system is defined, the only tensor that depending on the connection and is linear with respect to the second derivatives of the metric tensor is the Riemann-Christoffel curvature tensor or a linear combination of this tensor. Now we have to show that this property extends to every point of the manifold and for all coordinate systems.

7. Uniqueness theorem of the curvature tensor of Riemann-Christoffel

At any point in a locally inertial system, the expression of the Riemann-Christoffel tensor is given by (4); therefore, it is a tensor that depends linearly on four of the second derivatives of the metric tensor, and this same property will have in a coordinate system general. Now we have to see if this tensor is the only with that property.

Suppose a tensor H_{ijpq} that in any coordinate system has the property of linearly depending on four of the second derivatives of the metric tensor. If we make a change to a locally inertial coordinate system, we will find that at the point where this system is defined it will be fulfilled

$$H_{ijpq}=R_{ijpq}$$

since, as we have seen before, R_{ijpq} is the only tensor with the property considered *. The above is a tensor relationship that will be maintained for any transformation of coordinates and any point in space because, for each of them, it is possible to define a locally inertial system. Therefore, we verify the validity of the uniqueness theorem of the curvature tensor that states: that the only tensor that depends on the connection and is linear with respect to the second derivatives of the metric tensor is the Riemann-Christoffel curvature tensor or its linear combinations.

8. Generalization of the uniqueness theorem

We modify the statement of the uniqueness theorem to allow the tensor T_{ijpq} to depend on the metric tensor, leaving, therefore, the statement: the only tensor that depends on the metric tensor, its first derivatives ** and is linear with respect to the second derivatives of the metric tensor is the tensor

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* Strictly speaking, it should be $H_{ijpq}=\kappa R_{ijpq}$, where κ is a numerical coefficient, but for practical purposes, we consider proportional tensors as identical.

** The dependence of the connexion is not the same as the dependence of the first derivatives of the metric tensor; in the latter case, there is not only dependence on the connexion but also the metric tensor.

$$R_{ijpq} + \varphi g_{ij} g_{pq} + \gamma g_{ip} g_{jq} + \eta g_{iq} g_{jp} \tag{18}$$

or their linear combinations, being φ, γ, η arbitrary numerical coefficients. Note that the only covariant fourth-order tensor we can build with the metric tensor is its product by itself.

9. The extended version of the uniqueness theorem

In the previous reasoning, when we talk about linearity, we have assumed that the proportionality coefficients are numerical constants. The theorem can be generalized by admitting that the proportionality coefficients are geometric tensors of any order $X_{uvt...}$ *. There is a substantial limitation of the geometric tensors with which $X_{uvt...}$ ** can be formed, among which is the metric tensor (in its covariant or contravariant form), the Kronecker symbols δ_i^k and the Levi-Civita tensor $\Delta^{xyz...}$ § whose order is the same as the dimension of the variety §§.

For simplicity, we assume that the tensor X_{uv} is of second order. We are looking for a T_{ijpquv} tensor that depends linearly on the second derivatives and that in a locally inertial system has the form

$$T_{ijpquv} = \prod_{i,j,p,q,u,v} X_{uv} T_{ijpq} =$$
$$X_{uv}\left(A_1 \partial_{ij} g_{pq} + A_2 \partial_{ip} g_{jq} + A_3 \partial_{qi} g_{jp} + A_4 \partial_{qp} g_{ij} + A_5 \partial_{pj} g_{qi} + A_6 \partial_{qj} g_{ip}\right) +$$
$$+ X_{ui}\left(B_1 \partial_{vj} g_{pq} + B_2 \partial_{vp} g_{jq} + B_3 \partial_{qv} g_{jp} + B_4 \partial_{qp} g_{vj} + A_5 \partial_{pj} g_{qv} + B_6 \partial_{qj} g_{vp}\right) + ...$$

the symbol Π represents the sum of all possible permutations of the indexes *i, j, p, q, u* and *v.* By (6) we express the previous equation based on the first derivatives of the connection

$$T_{ijpquv} = X_{uv}\begin{bmatrix} g_{sp}\left(A_1 \partial_i \Gamma^s_{jq} + A_3 \partial_q \Gamma^s_{ij} + A_6 \partial_q \Gamma^s_{ij}\right) + g_{sq}\left(A_1 \partial_i \Gamma^s_{jp} + A_2 \partial_i \Gamma^s_{jp} + A_5 \partial_p \Gamma^s_{ij}\right) + \\ + g_{sj}\left(A_2 \partial_i \Gamma^s_{pq} + A_3 \partial_q \Gamma^s_{ip} + A_4 \partial_q \Gamma^s_{ip}\right) + g_{si}\left(A_4 \partial_q \Gamma^s_{jp} + A_5 \partial_p \Gamma^s_{jq} + A_6 \partial_q \Gamma^s_{jp}\right) \end{bmatrix} +$$
$$+ X_{ui}\begin{bmatrix} g_{sp}\left(B_1 \partial_v \Gamma^s_{jq} + B_3 \partial_q \Gamma^s_{vj} + B_6 \partial_q \Gamma^s_{vj}\right) + g_{sq}\left(B_1 \partial_v \Gamma^s_{jp} + B_2 \partial_v \Gamma^s_{jp} + B_5 \partial_p \Gamma^s_{vj}\right) + \\ + g_{sj}\left(B_2 \partial_v \Gamma^s_{pq} + B_3 \partial_q \Gamma^s_{vp} + B_4 \partial_q \Gamma^s_{vp}\right) + g_{sv}\left(B_4 \partial_q \Gamma^s_{jp} + B_5 \partial_p \Gamma^s_{jq} + B_6 \partial_q \Gamma^s_{jp}\right) \end{bmatrix} + ...$$

when doing a coordinate transformation to another locally inertial system, we obtain using (9) and taking into account that X_{uv} is a covariant second-order tensor

$$T'_{ijpquv} = B_i^a B_j^b B_p^c B_q^d B_u^r B_v^t T_{abcd} - \left(A_1 + A_3 + A_6\right) B_q^a B_j^b B_i^c B_s^d B_p^e B_u^r B_v^t A^s_{abc} X_{rt} g_{de} -$$
$$-\left(A_1 + A_2 + A_5\right) B_p^a B_j^b B_i^c B_s^d B_q^e A^s_{abc} B_u^r B_v^t X_{rt} g_{de} - \left(A_2 + A_3 + A_4\right) B_q^a B_p^b B_i^c B_s^d B_j^e B_u^r B_v^t A^s_{abc} X_{rt} g_{de} -$$
$$-\left(A_4 + A_5 + A_6\right) B_j^a B_p^b B_q^c B_s^d B_i^e B_u^r B_v^t A^s_{abc} X_{rt} g_{de} + ...$$

by identical reasoning that those followed in section 3, we find that the conditions for T_{ijpquv} to be a tensor are

$$A_1 + A_3 + A_6 = 0; \quad A_1 + A_2 + A_5 = 0; \quad A_2 + A_3 + A_4 = 0; \quad A_4 + A_5 + A_6 = 0$$
$$B_1 + B_3 + B_6 = 0; \quad B_1 + B_2 + B_5 = 0; \quad B_2 + B_3 + B_4 = 0; \quad B_4 + B_5 + B_6 = 0$$

and equal relationships for the remaining coefficients *C, D*, ... Then we find that

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* The demonstration can be done with contravariant or mixed tensors, reaching the same result.

** With symmetric connection, like Christoffel's symbols, we cannot form a tensor. Any combination of this connection would not have a tensor character since it is always possible to choose at any point a locally inertial system, where that combination is canceled an if it is null in a coordinate system, it remains null in any other coordinate system.

§ Levi-Civita's tensor is defined as

$$\Delta^{xyz...} = J \varepsilon^{xyz...}$$

where $\varepsilon^{xyz...}$ are the completely antisymmetric symbols, and J is a parameter that in a coordinate transformation is changed according to

$$J' = |A| J$$

where $|A|$ is the determinant of the transformation matrix.

§§ There are also the torsion tensor, the torsion vector and the non-metricity tensor $D_i g_{pq}$, which are nulls in a manifol odf Riemann that has a symmetrical connection.

$$T_{ijpquv} = \prod_{i,j,p,q,u,v} X_{uv} T_{ijpq} \; (19)$$

it is a tensor, and since T_{ijpq} is a linear combination of curvature tensors, then from (19), is found that the tensor T_{ijpquv} is a linear combination of different tensors of the form $X_{uv} R_{ijpq}$. Since the tensor X can have any order, we find that, in general, tensor type $T_{ijpquvt...} = T_{ijpq} X_{uvt.....}$ are linear combinations of the second derivatives of the metric tensor, admitting that the proportionality coefficients are tensors. With reasoning similar to that of section (7), we show that these tensors are unique.

10. The tensor $X_{uvt....}$

We have already seen that we can form tensors of any order that are linear combinations of the second derivatives of the metric tensor, admitting that the proportionality coefficients are tensors. The tensors obtained by contraction from $T_{ijpquvt...} = T_{ijpq} X_{uvt.....}$ have the same property of being linear with respect to the second derivatives the metric tensor.

To obtain contracted tensors, the tensor $X_{uvt...}$ can only be formed by the metric tensor, since the Kronecker symbols do not generate a new tensor and the contraction of the Levi-Civita tensor with the curvature tensor is zero. Indeed, in a locally inertial system and if the manifold has four dimensions

$$\Delta^{psir} R_{psir} = \frac{1}{2} \Delta^{psir} \left(\partial_{si} g_{rp} - \partial_{pi} g_{sr} - \partial_{sr} g_{ip} + \partial_{pr} g_{si} \right),$$

any permutation of two indices of the Levi-Civita tensor is an odd permutation, and therefore the sign changes, but each of the terms of the curvature tensor has two pairs of symmetric indices and therefore when multiplied by the Levi-Civita tensor gives a null result *. For example, for the first term of the curvature tensor exchanging the indexes s and i

$$\Delta^{psir} \partial_{si} g_{rp} = -\Delta^{pisr} \partial_{si} g_{rp} = -\Delta^{psir} \partial_{is} g_{rp} = -\Delta^{psir} \partial_{si} g_{rp} \quad \Rightarrow \quad \Delta^{psir} \partial_{si} g_{rp} = 0$$

and the same with the remaining terms and therefore

$$\Delta^{psir} R_{psir} = 0.$$

We conclude, therefore, that the metric tensor can only form the tensor $X_{uvt...}$.

11. Uniqueness theorem for second-order tensors

The Ricci tensor is the contraction of the curvature tensor

$$R_{si} = g^{pr} R_{psir} = R^{r}_{sir} = g^{pr} g_{pk} \left(\partial_i \Gamma^k_{sr} - \partial_r \Gamma^k_{si} + \Gamma^n_{sr} \Gamma^k_{ni} - \Gamma^n_{si} \Gamma^k_{nr} \right) =$$
$$= \partial_i \Gamma^k_{sk} - \partial_k \Gamma^k_{si} + \Gamma^n_{sk} \Gamma^k_{ni} - \Gamma^n_{si} \Gamma^k_{nk},$$

in a locally inertial system

$$R_{si} = \frac{1}{2} g^{pr} \left(\partial_{si} g_{rp} - \partial_{pi} g_{sr} - \partial_{sr} g_{ip} + \partial_{pr} g_{si} \right)$$

which proves that it is a symmetric tensor in a manifold of Riemann.

The other possible contraction of the Riemann-Christoffel tensor is the homothetic curvature

$$V_{ir} = g^{ps} R_{psir} = \partial_i \Gamma^s_{sr} - \partial_r \Gamma^s_{si}$$

but it is null for a manifold with symmetric connection. The remaining contractions of the curvature tensor turn out to be the Ricci tensor, either positive or negative.

A second contraction of the curvature tensor gives the scalar curvature R

$$R = g^{si} R_{si} = g^{si} \left(\partial_i \Gamma^k_{sk} - \partial_k \Gamma^k_{si} + \Gamma^n_{sk} \Gamma^k_{ni} - \Gamma^n_{si} \Gamma^k_{nk} \right),$$

from here we get the second order tensor $g_{ik} R$.

Therefore the only second-order tensor that can be formed depending on the metric tensor, its first derivatives and is linear with respect to the second derivatives is

* The reasoning of the text is extensible to any number of dimensions. The contracted product of the metric tensor with the Levi-Civita tensor is also null because it is the product of a symmetric tensor by another antisymmetric.

$$T_{ik} = \alpha R_{ik} + \beta g_{ik} R + \gamma g_{ik}$$

or linear combinations of the previous tensors, where α, β and γ are arbitrary numerical values.

It's called Einstein's tensor

$$E_{ik} = R_{ik} - \frac{1}{2} g_{ik} R - \lambda g_{ik} \tag{20}$$

that has the property that its covariant gradient is null [9]

$$D_r \left(R_k^r - \frac{1}{2} \delta_k^r R - \lambda \delta_k^r \right) = 0$$

therefore we conclude that the only tensor that, depending on the metric tensor, on its first derivatives, is linear with respect to the second derivatives and has a covariant gradient null is the tensor (20), which we have previously called Cartan's theorem.

12. References

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