An theorem about square root

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Abstract

Use Properties of exponents and Characteristic equation to derive an formula of square root.
Keyword: Square root

1.Introduction

if $\sqrt{a\sqrt{a\sqrt{a}}}\ldots$(there are *n square roots piled up),*
how to get the value easier.

2.Inference

$\sqrt{a} = a^{\frac{1}{2}}$

$\Downarrow$

$\sqrt{a\sqrt{a}} = (a \times a^{\frac{1}{2}})^{\frac{1}{2}} = a^{\frac{3}{4}}$

$\Downarrow$

$\sqrt{a\sqrt{a\sqrt{a}}} = [a(a \times a^{\frac{1}{2}})^{\frac{1}{2}}]^{\frac{1}{2}} = a^{\frac{7}{8}}$

Let us pay attention to ***a****'s exponents, we can find the rules:*

the numerator's value=the denominator's value-1

the denominator's value=2^n*(****n*** *is the number of root)*

And invent a theorem according to the rules:

The value of $\sqrt{a\sqrt{a\sqrt{a}}}\ldots$ *will be*

$$a^{\frac{2^n-1}{2^n}}$$

3.Proof

Let pick up the Exponents and make $\frac{2^n-1}{2^n} = b_n$.

if

$$n = 1 \qquad b_1 = \frac{1}{2}$$

$$n = 2 \qquad b_2 = \frac{1}{2}(1 + \frac{1}{2}) = \frac{1}{2}(1 + b_1)$$

$$n = 3 \qquad b_3 = \frac{1}{2}[1 + \frac{1}{2}(1 + \frac{1}{2})] = \frac{1}{2}(1 + b_2)$$

$$\vdots$$

$$n = k(k \in \mathbb{Z}) \qquad b_k = \frac{1}{2}(1 + b_{k-1})$$

$$\downarrow$$

$$b_n = \frac{1}{2}(1 + b_{n-1}) \ldots (1)$$

Let use $2 \times \frac{3}{4} = 1 + \frac{1}{2}$ *find the answer.*

$2b_n = 1 + b_{n-1}$

$-)2 \times \frac{3}{4} = 1 + \frac{1}{2}$

$2(b_n - \frac{3}{4}) = b_{n-1} - \frac{1}{2}$

$b_n - \frac{3}{4} = \frac{b_{n-1}}{2} - \frac{1}{4}$

$b_n = \frac{1}{2}(1 + b_{n-1})$

$= (1)$

$= \frac{2^n-1}{2^n}$

Q.E.D

Remark

Understand that the square root and Squaring are inverse.

if

n≦0

ex.

$$n = -1 \qquad a^{\frac{2^{-1}-1}{2^{-1}}} = a^{\frac{\frac{-1}{2}}{\frac{1}{2}}} = a^{-1}$$

The ex. contradit the property, so we need to add the range limits:

$$n > 0$$

4.Conclusions

The value of $\sqrt{a\sqrt{a\sqrt{a}}}\ldots$ *(there are n square roots piled up) will be*

$$a^{\frac{2^n-1}{2^n}} \quad (n \in \mathbb{Z}, n > 0)$$