q-analogs of sinc sums and integrals

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q-analogs of sum equals integral relations $\sum_{n\in\mathbb{Z}} f(n) = \int_{-\infty}^{\infty} f(x)dx$ for sinc functions and binomial coefficients are studied. Such analogs are already known in the context of q-hypergeometric series. This paper deals with multibasic 'fractional' generalizations that are not q-hypergeometric functions.

Surprizing properties of sinc sums and integals were first discovered by C. Stormer in 1895 [1,2]. The more general properties of band limited functions were known to engineers from signal processing and to physicists. For example, K.S. Krishnan viewed them as a rich source for finding identities [3]. R.P. Boas has studied the error term when approximating a sum of a band limited function with corresponding integral [5]. More recently these properties were studied and popularized in a series of papers [6–8].

sinc function is a special case of binomial coefficients

$$\binom{2}{1+x} = \frac{\Gamma(3)}{\Gamma(1+x)\Gamma(1-x)} = \frac{2\sin\pi x}{\pi x} = 2\,\mathrm{sinc}(\pi x).$$

Therefore only sums with binomial coefficients will be studied in the following. It is known that binomial coefficients are band limited (e.g., see [10])

$$\binom{a}{u} = \frac{1}{2\pi}\int_{-\pi}^{\pi}(1+e^{it})^a e^{-iut}\,dt,$$

i.e. their Fourier spectrum is limited to the band $|t| < \pi$. According to general theorems [5, 6] whenever Fourier spectrum of a function $f(x)$ is limited to the band $|t| < 2\pi$ one expects that

$$\sum_{n=-\infty}^{\infty} f(n) = \int_{-\infty}^{\infty} f(x)dx. \tag{1}$$

Bandwidth of a product of bandlimited functions is the sum of their bandwidths [8]. In case of binomial coefficients this together with the theorem mentioned above implies that

$$\sum_{n=-\infty}^{\infty}\binom{a}{\alpha n}^l = \int_{-\infty}^{\infty}\binom{a}{\alpha x}^l dx, \qquad 0<\alpha\le\frac{2}{l}. \tag{2}$$

For a general band limited function the above formula would have been valid only when $\alpha < \frac{2}{l}$. The validity of (2) when $\alpha = \frac{2}{l}$ is explained by the fact that spectral density of binomial coefficient vanishes at boundary values $t = \pm\pi$.

q-analog of the Gamma function is defined as

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}$$

and the q-binomial coefficients

$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \frac{\Gamma_q(a+1)}{\Gamma_q(b+1)\Gamma_q(a-b+1)},$$

with the standard notations for the q-shifted factorials

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \qquad (a_1,\ldots,a_r;q)_n = \prod_{k=1}^{r}(a_k;q)_n, \qquad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

In the limit $q \to 1^-$ one has $\Gamma_q(a) \to \Gamma(a)$, i.e. standard values of the Gamma function and binomial coefficients are recovered.[11]

q-analog of the property of bandlimitedness has been studied in the literature [12]. This paper has a much more narrow scope and only deals with sums of binomial coefficients. We will find that (2) with $0 < \alpha \le 1/l$ has a very natural q-analog. However no such simple direct q-analog of (2) with $1/l < \alpha \le 2/l$ is known. Nevertheless there is a formula that in the limit $q \to 1^-$ can be brought to the form (2) after a series of simple steps.

In Theorem 2 we will use a method of functional equations [13] (see also [11], sec. 5.2) combined with an idea to to G. Gasper [14] to find a Laurent series for a certain integral of an infinite product. First we need the following theorem taken from the book [15].

Theorem 1. *Let*

$$F(z) = \int_\gamma f(\zeta,z)d\zeta, \tag{3}$$

where the following conditions are satisfied

(1) *γ is an infinite picewise continous curve*

(2) *the function $f(\zeta,z)$ is continous in (ζ,z) at $\zeta \in \gamma,\ z \in D$, where D is a domain in the complex z plane,*

(3) *for each fixed $\zeta \in \gamma$ the function $f(\zeta,z)$ viewed as a function of z is regular in D,*

(4) *integral (3) converges uniformly in $z \in D'$, where D' is an arbitrary closed subdomain of D.*

Then $F(z)$ is regular in D.

Lemma 1. *Let p and q two real numbers that satisfy $0 < p < q < 1$, then*

$$F(z) = \int_{-\infty}^{\infty} \frac{\left(bq^\zeta, aq^{-\zeta}; p\right)_\infty}{\left(-zq^\zeta, -q^{1-\zeta}/z; q\right)_\infty}\, d\zeta$$

is regular in the half plane $\operatorname{Re} z > 0$.

Proof. Put in the theorem above $f(\zeta,z) = \frac{(bq^\zeta,aq^{-\zeta};p)_\infty}{(-zq^\zeta,-q^{1-\zeta}/z;q)_\infty}$, $\gamma = (-\infty,+\infty)$, and D an arbitrary domain in the half plane Re $z > 0$. Then (1),(2) and (3) are obviously satisfied. To prove (4) let $p = e^{-\omega}$, $q = p^\alpha$, $\omega > 0$, $0 < \alpha < 1$ and consider the asymptotics of $f(\zeta,z)$ when $\zeta \to +\infty$. In this limit one has $(bq^\zeta;p)_\infty \to 1$, $(-zq^\zeta;q)_\infty \to 1$. According to an asymptotic formula ([11], p. 118)

$$\operatorname{Re}[\ln(p^s;p)_\infty] = \frac{\omega}{2}(\operatorname{Re} s)^2 + \frac{\omega}{2}(\operatorname{Re} s) + O(1), \qquad \operatorname{Re} s \to -\infty,$$

we have

$$|(aq^{-\zeta};p)_\infty| = |(p^{-\alpha\zeta-\omega^{-1}\ln a};p)_\infty| = O\left(|a|^{\alpha\zeta}q^{-(\alpha\zeta^2-\zeta)/2}\right),$$

$$|(-q^{1-\zeta}/z;q)_\infty| = |(q^{1-\zeta+\alpha^{-1}\omega^{-1}\ln z};q)_\infty| = O\left(|q/z|^\zeta q^{-(\zeta^2-\zeta)/2}\right).$$

So

$$f(\zeta,z) = O\left(|za^\alpha/q|^\zeta q^{(1-\alpha)\zeta^2/2}\right), \quad \zeta \to +\infty.$$

Similarly

$$f(\zeta, z) = O\left(|b^{\alpha}/z|^{-\zeta} q^{(1-\alpha)\zeta^2/2}\right), \quad \zeta \to -\infty.$$

It is now easy to see that the integral (*) converges. Hence according to Weierstrass M-Test integral $F(z)$ converges uniformly in z when $\operatorname{Re} z \geq \delta > 0$. As a result the function

$$f(a,b,z) = \frac{(-z,-q/z;q)_\infty}{\ln\frac{1}{q}} \int\limits_0^\infty \frac{(bt/z, pz/at;p)_\infty}{(-t,-q/t;q)_\infty} \frac{dt}{t}$$

is regular when $\operatorname{Re} z > 0$ □

Lemma 2. *The function*

$$f(a,b,z) = \frac{(-z,-q/z;q)_\infty}{\ln\frac{1}{q}} \int_0^\infty \frac{(bt, a/t;p)_\infty}{(-zt,-q/(zt);q)_\infty} \frac{dt}{t}$$

satisfies the functional equations

$$f(a,b,z) = f(a,bp,z) - bf(a,bp,qz), \tag{4}$$

$$f(a,b,z) = f(ap,b,z) - af(ap,b,z/q). \tag{5}$$

Proof. After a series of simple manipulations of the infinite products we find

$$\begin{aligned}
f(a,b,qz) &= \frac{(-qz,-1/z;q)_\infty}{\ln\frac{1}{q}} \int_0^\infty \frac{(bt,a/t;p)_\infty}{(-qzt,-1/(zt);q)_\infty} \frac{dt}{t} \\
&= \frac{(-z,-q/z;q)_\infty}{z\ln\frac{1}{q}} \int_0^\infty \frac{z\,(bt,a/t;p)_\infty}{(-zt,-q/(zt);q)_\infty}\, dt \\
&= \frac{p(-z,-q/z;q)_\infty}{b\ln\frac{1}{q}} \int_0^\infty \frac{bt}{p} \frac{(bt,a/t;p)_\infty}{(-zt,-q/(zt);q)_\infty} \frac{dt}{t} \\
&= \frac{p}{b}(f(a,b,z) - f(a,b/p,z)).
\end{aligned}$$

This is equivalent to (4). Similarly or using the first functional equation and the formula $f(a,b,z) = f(b,a,q/z)$ we find

$$\begin{aligned}
f(a,b,z) = f(b,a,q/z) &= f(b,ap,q/z) - af(b,ap,q^2/z) \\
&= f(ap,b,z) - af(ap,b,z/q),
\end{aligned}$$

as required. □

Theorem 2. *Let p and q two complex numbers such that $|p| < |q| < 1$, then*

$$\sum_{n=-\infty}^{\infty} (bq^n, aq^{-n};p)_\infty z^n q^{n(n-1)/2} = \frac{(-z,-q/z;q)_\infty}{\ln\frac{1}{q}} \int_0^\infty \frac{(bt/z, az/t;p)_\infty}{(-t,-q/t;q)_\infty} \frac{dt}{t}.$$

Proof. First consider the case $0 < p < q < 1$. The function $f(a,b,z)$ from Lemma 2 can be written in the form

$$f(a,b,z) = (-z,-q/z;q)_\infty \int_{-\infty}^\infty \frac{\left(bq^{\zeta}/z, azq^{-\zeta};p\right)_\infty}{\left(-q^{\zeta},-q^{1-\zeta};q\right)_\infty}\, d\zeta.$$

According to Lemma 1 $f(a,b,z)$ is a regular function of z in the region Re$z>0$. As a result $f(a,b,z)$ has the Laurent series expansion

$$f(a,b,z)=\sum_{n=-\infty}^{\infty} c_n(a,b)z^n, \qquad \mathrm{Re}\, z>0.$$

Functional equation (4) gives the following recursion relation for coefficients $c_n(a,b)$

$$c_n(a,b)=(1-bq^n)c_n(a,bp).$$

This recursion means that

$$c_n(a,b)=(bq^n;p)_\infty c_n(a,0).$$

The functional equation (5) gives

$$c_n(a,b)=(1-aq^{-n})c_n(a/p,b),$$

from which one obtains

$$c_n(a,b)=(aq^{-n};p)_\infty c_n(0,b).$$

By combining these equations one gets

$$c_n(a,b)=(bq^n;p)_\infty c_n(a,0)=(bq^n,aq^{-n};p)_\infty c_n(0,0).$$

It is known that ([11], ex. 6.16)

$$\int_0^\infty \frac{1}{(-t,-q/t;q)_\infty}\frac{dt}{t}=(q;q)_\infty \ln\frac{1}{q}.$$

According to Jacobi triple product formula

$$(q,-z,-q/z;q)_\infty=\sum_{n=-\infty}^{\infty} z^n q^{n(n-1)/2}$$

this implies that $c_n(0,0)=z^n q^{n(n-1)/2}$, so finally

$$c_n(a,b)=(bq^n,aq^{-n};p)_\infty z^n q^{n(n-1)/2}.$$

Now one needs to continue the result established for Re $z>0, 0<p<q<1$ analytically to complex values of parameters z,p,q to complete the proof. □

Series containing infinite products $(bq^n,aq^{-n};p)_\infty$ have been studied in [12]. It appears that the series in Theorem 2 have been first considered in the paper [17] which also contains a different representation for this sum in terms of an integral over a unit circle.

Corollary 1. *The formula in Theorem 2 can be written in symmetric form*

$$\sum_{n=-\infty}^{\infty}\frac{(bq^n,aq^{-n};p)_\infty}{(-zq^n,-q^{1-n}/z;q)_\infty}=\int_{-\infty}^{\infty}\frac{(bq^x,aq^{-x};p)_\infty}{(-zq^x,-q^{1-x}/z;q)_\infty}\,dx,$$

or in terms of q-binomial coefficients

$$\sum_{n=-\infty}^{\infty}\begin{bmatrix} a\\ b+\alpha n\end{bmatrix}_p \frac{1}{(-zq^n,-q^{1-n}/z;q)_\infty}=\int_{-\infty}^{\infty}\begin{bmatrix} a\\ b+\alpha x\end{bmatrix}_p \frac{1}{(-zq^x,-q^{1-x}/z;q)_\infty}\,dx, \tag{6}$$

where $q=p^\alpha$, $0<\alpha<1$.

This gives an example of function for which sum equals integral. The case $|p| = |q| < 1$, $|b/a| < |z| < 1$ was known to Ramanujan. In this case, the series is Ramanujan's ${}_1\psi_1$ sum and the integral is Ramanujan's q-beta integral ([11], chs. 5,6).

Now let $z = e^{i\theta}$, $|\theta| < \pi$. Then

$$\lim_{q\to 1^-} \frac{(-z, -q/z; q)_\infty}{(-zq^x, -q^{1-x}/z; q)_\infty} = (1+z)^x (1+1/z)^{-x} = z^x.$$

Let $q \to 1^-$ with $0 < \alpha < 1$ fixed in equation (6). Then formally

$$\sum_{n=-\infty}^{\infty} \binom{a}{b+\alpha n} e^{i\theta n} = \int_{-\infty}^{\infty} \binom{a}{b+\alpha x} e^{i\theta x}\, dx, \qquad 0 < \alpha < 1. \tag{7}$$

The range of validity of (7) is $-\pi\alpha < \theta < \pi\alpha$ as in (9), and not $-\pi < \theta < \pi$. Continuing formal manipulations we obtain by using (7) and binomial theorem

$$\begin{aligned}\int_{-\infty}^{\infty} \binom{a}{b+\alpha x} e^{i\theta x}\, dx &= \frac{1}{\alpha} e^{-i\theta b/\alpha} \int_{-\infty}^{\infty} \binom{a}{x} e^{i\theta x/\alpha}\, dx \\ &= \frac{1}{\alpha} e^{-i\theta b/\alpha} \sum_{n=-\infty}^{\infty} \binom{a}{n} e^{i\theta n/\alpha} \\ &= \frac{1}{\alpha} e^{-i\theta b/\alpha} \sum_{n=0}^{\infty} \binom{a}{n} e^{i\theta n/\alpha} \\ &= \frac{1}{\alpha} e^{-i\theta b/\alpha} (1 + e^{i\theta/\alpha})^a, \quad -\pi\alpha < \theta < \pi\alpha. \end{aligned} \tag{8}$$

Finally (7) and (8) imply

$$\sum_{n=-\infty}^{\infty} \binom{a}{b+\alpha n} v^{b+\alpha n} = \frac{1}{\alpha}(1+v)^a, \quad |v| = 1,\ |\arg v| < \pi,\ 0 < \alpha \le 1, \tag{9}$$

which is T. Osler's generalization of binomial theorem [18]. According to Osler [18], the special case $\alpha = 1$ of (9) was first stated by Riemann [24]. It also follows from Ramanujan's ${}_1\psi_1$ sum in the limit $q \to 1^-$.

It should be noted that while (9) has a closed form, the series in Theorem 2 does not. If $p = q^2$,$z = 1$,$b = aq^2$, then one can prove that

$$\sum_{n=-\infty}^{\infty} (bq^n, p/aq^n; p)_\infty z^n q^{n(n-1)/2} = 2\left(qa, q/a; q^2\right)_\infty \sum_{n=-\infty}^{\infty} \frac{(-1/a)^n q^{n^2+n}}{1 - aq^{2n+1}}.$$

The sum on the RHS is proportional to Appell-Lerch sum $m(qa^2, q^2, q^2/a)$ in the notation of the paper [19]. In general Appell-Lerch sums do not have an infinite product representation. For example, by taking $a = q^{-1/2}$ in $m(qa^2, q^2, q^2/a)$ we get the sum of the type $m(1, q^2, z)$ which is related to mock theta function of order 2 (see formula (4.2) in [19]).

Corollary 2. *The series*

$$\sum_{n=-\infty}^{\infty} \frac{(bq^n, p/aq^n; p)_\infty}{(-zq^n, -q/zq^n; q)_\infty}, \qquad |p| < |q|$$

with p and q fixed depends only on b/z and az.

Theorem 3.

$$\int_{-\infty}^{\infty} \frac{(bq^x, aq^{-x}; p)_\infty}{(-q^x, -q^{1-x}; q)_\infty}\, e^{ixy}\, dx$$
$$= \frac{2\pi i/\log q}{\sinh \frac{\pi y}{\log q}} \frac{(-q, -q, e^{iy}, qe^{-iy}; q)_\infty}{(q, q, -e^{iy}, -qe^{-iy}; q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(bq^n, aq^{-n}; p)_\infty}{(-q^n, -q^{1-n}; q)_\infty}\, e^{iny}.$$

Proof. Consider the contour integral

$$\int_C \frac{(bq^z, aq^{-z}; p)_\infty}{(-q^z, -q^{1-z}; q)_\infty}\, e^{izy}\, dz$$

where C is rectangle with vertices at $(\pm R, 0)$, $(\pm R, -2\pi i/\log q)$. In view of asymptotics found in the proof of Lemma 1 integrals over the vertical segments vanish in the limit $R \to +\infty$. Integrals over the horizontal segments are convergent and related by a factor of $-e^{2\pi y/\log q}$. The integrand has simple poles at $z = n - \pi i/\log q$ with residues

$$-\frac{e^{\pi y/\log q}}{(q;q)_\infty^2 \log q} (-bq^n, -aq^{-n}; p)_\infty (-1)^n q^{n(n-1)/2} e^{iny}.$$

Application of the residue theorem yields

$$\int_{-\infty}^{\infty} \frac{(bq^x, aq^{-x}; p)_\infty}{(-q^x, -q^{1-x}; q)_\infty}\, e^{ixy}\, dx = \frac{\pi i/\log q}{(q;q)_\infty^2 \sinh \frac{\pi y}{\log q}} \sum_{n=-\infty}^{\infty} (-bq^n, -aq^{-n}; p)_\infty (-1)^n q^{n(n-1)/2} e^{iny}.$$

According to Corollary 2

$$\sum_{n=-\infty}^{\infty} (-bq^n, -aq^{-n}; p)_\infty (-1)^n q^{n(n-1)/2} e^{iny} = \frac{(e^{iy}, qe^{-iy}; q)_\infty}{(-e^{iy}, -qe^{-iy}; q)_\infty} \sum_{n=-\infty}^{\infty} (bq^n, aq^{-n}; p)_\infty q^{n(n-1)/2} e^{iny}.$$

To complete the proof observe that

$$\sum_{n=-\infty}^{\infty} (bq^n, aq^{-n}; p)_\infty q^{n(n-1)/2} e^{iny} = (-1, -q; q)_\infty \sum_{n=-\infty}^{\infty} \frac{(bq^n, aq^{-n}; p)_\infty}{(-q^n, -q^{1-n}; q)_\infty}\, e^{iny}$$

and $(-1, -q; q)_\infty = 2(-q; q)_\infty^2$. □

One can see from Theorem 3 that the function

$$g(x) = \frac{(bq^x, aq^{-x}; p)_\infty}{(-q^x, -q^{1-x}; q)_\infty}$$

is not band limited. However Fourier transform of $g(x)$ vanishes at frequencies $y = 2\pi m$, where $m \neq 0$ is an integer. Hence according to Poisson summation formula [20]

$$\sum_{n=-\infty}^{\infty} g(x) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} g(x) e^{-2\pi i n x} dx = \int_{-\infty}^{\infty} g(x) dx$$

in agreement with Corollary 1.

The fact that bilateral summation formulas in the theory of q-hypergeometric functions give examples of functions of the type (1) has been recognized in the literature.

Corollary 3. *Let $|p| < |q|$ and $m \in \mathbb{Z}$, then*

$$\int_{-\infty}^{\infty} \frac{(bq^x, aq^{-x}; p)_\infty}{(-q^x, -q^{1-x}; q)_\infty}\, q^{mx}\, dx = \sum_{n=-\infty}^{\infty} \frac{(bq^n, aq^{-n}; p)_\infty}{(-q^n, -q^{1-n}; q)_\infty}\, q^{mn}.$$

Proof. Resolve the $\frac{0}{0}$ ambiguity at the rhs of the formula of Theorem 2 using L'Hopital's Rule. □

Next we apply the method due to Bailey [22] to the identity in Theorem 2.

Theorem 4.

$$\sum_{n=-\infty}^{\infty} \big(b_1q^n, b_2q^n, a_1q^{-n}, a_2q^{-n}; p\big)_\infty z^n q^{n(n-1)} = z \sum_{n=-\infty}^{\infty} \big(b_1q^n/z, b_2q^n/z, a_1zq^{-n}, a_2zq^{-n}; p\big)_\infty z^{-n} q^{n(n-1)}.$$

Proof. Multiplying the equations

$$\sum_{n=-\infty}^{\infty} (b_1q^n, a_1q^{-n}; p)_\infty e^{i\theta n} q^{n(n-1)/2} = \frac{(-e^{i\theta}, -qe^{-i\theta}; q)_\infty}{\ln\frac{1}{q}} \int_0^\infty \frac{\big(b_1te^{-i\theta}, a_1e^{i\theta}/t; p\big)_\infty}{(-t, -q/t; q)_\infty}\frac{dt}{t},$$

$$\sum_{n=-\infty}^{\infty} (b_2q^n, a_2q^{-n}; p)_\infty e^{-i\theta n} z^n q^{n(n-1)/2} = \frac{(-ze^{-i\theta}, -qe^{i\theta}/z; q)_\infty}{\ln\frac{1}{q}} \int_0^\infty \frac{\big(b_2te^{i\theta}/z, a_2ze^{-i\theta}/t; p\big)_\infty}{(-t, -q/t; q)_\infty}\frac{dt}{t},$$

and integrating with respect to θ one obtains

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} \big(b_1q^n, b_2q^n, a_1q^{-n}, a_2q^{-n}; p\big)_\infty z^n q^{n(n-1)} \\
&= \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{(-e^{i\theta}, -qe^{-i\theta}; q)_\infty}{\ln\frac{1}{q}} \int_0^\infty \frac{\big(b_1t_1e^{-i\theta}, a_1e^{i\theta}/t_1; p\big)_\infty}{(-t_1, -q/t_1; q)_\infty}\frac{dt_1}{t_1} \\
&\times \frac{(-ze^{-i\theta}, -qe^{i\theta}/z; q)_\infty}{\ln\frac{1}{q}} \int_0^\infty \frac{\big(b_2t_2e^{i\theta}/z, a_2ze^{-i\theta}/t_2; p\big)_\infty}{(-t_2, -q/t_2; q)_\infty}\frac{dt_2}{t_2} \\
&= z\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{(-e^{-i\theta}, -qe^{i\theta}; q)_\infty}{\ln\frac{1}{q}} \int_0^\infty \frac{\big(b_2t_2e^{i\theta}/z, a_2ze^{-i\theta}/t_2; p\big)_\infty}{(-t_2, -q/t_2; q)_\infty}\frac{dt_2}{t_2} \\
&\times \frac{(-e^{i\theta}/z, -qze^{-i\theta}; q)_\infty}{\ln\frac{1}{q}} \int_0^\infty \frac{\big(b_1t_1e^{-i\theta}, a_1e^{i\theta}/t_1; p\big)_\infty}{(-t_1, -q/t_1; q)_\infty}\frac{dt_1}{t_1} \\
&= z\sum_{n=-\infty}^{\infty} \big(b_1q^n/z, b_2q^n/z, a_1zq^{-n}, a_2zq^{-n}; p\big)_\infty z^{-n} q^{n(n-1)}. \quad \square
\end{aligned}$$

Corollary 4. *Let $0 < q < 1$ and $0 < \alpha < 1$, then*

$$\sum_{n=-\infty}^{\infty} \begin{bmatrix} a_1 \\ b_1 + \alpha n \end{bmatrix}_p \begin{bmatrix} a_2 \\ b_2 + \alpha n \end{bmatrix}_p p^{\alpha n(n-1)+\theta n} = p^\theta \sum_{n=-\infty}^{\infty} \begin{bmatrix} a_1 \\ b_1 - \theta + \alpha n \end{bmatrix}_p \begin{bmatrix} a_2 \\ b_2 - \theta + \alpha n \end{bmatrix}_p p^{\alpha n(n-1)-\theta n}.$$

Theorem 2 can be generalized.

Theorem 5. *Let $q = p_1^{\alpha_1} = p_2^{\alpha_2}$ where $0 < \alpha_1 + \alpha_2 < 1$, then*

$$\sum_{n=-\infty}^{\infty} \begin{bmatrix} a_1 \\ b_1+\alpha n \end{bmatrix}_{p_1} \begin{bmatrix} a_2 \\ b_2+\alpha n \end{bmatrix}_{p_2} \frac{1}{(-zq^n, -q^{1-n}/z; q)_\infty}$$
$$= \int_{-\infty}^{\infty} \begin{bmatrix} a_1 \\ b_1+\alpha x \end{bmatrix}_{p_1} \begin{bmatrix} a_2 \\ b_2+\alpha x \end{bmatrix}_{p_2} \frac{dx}{(-zq^x, -q^{1-x}/z; q)_\infty}.$$

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