Letter N°2:Two Curious Integrals

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ABSTRACT. We recall two curious integrals.

keywords: definite integrals, number Pi, series.

I. Two Integrals .

$$\pi^2 = 4\left(\ln\left(1+\sqrt{2}\right)\right)^2 - 16\int_{\ln\sqrt{1+\sqrt{2}}}^{\infty} \ln(\tanh(x))\,dx \tag{1}$$

$$\pi^2 = -4\left(\ln\left(1+\sqrt{2}\right)\right)^2 - 16\int_{0}^{\ln\sqrt{1+\sqrt{2}}} \ln(\tanh(x))\,dx \tag{2}$$

II. Some Formulas .

$$\pi^2 = -8\int_0^{\infty} \ln(\tanh(x))\,dx \tag{3}$$

$$\pi^2 = 4\left(\ln\left(1+\sqrt{2}\right)\right)^2 + 16\sum_{n=1}^{\infty} \frac{\left(1+\sqrt{2}\right)^{-(2n-1)}}{(2n-1)^2} \tag{4}$$

$$\pi^2 = -4\left(\ln\left(1+\sqrt{2}\right)\right)^2 + 16\sum_{n=1}^{\infty} \frac{1-\left(1+\sqrt{2}\right)^{-(2n-1)}}{(2n-1)^2} \tag{5}$$

$$\pi^2 = 8\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \tag{6}$$

$$\pi^2 = -4\left(\ln\left(1+\sqrt{2}\right)\right)^2 + 8\ln\left(1+\sqrt{2}\right)\left(1-\ln\ln\left(\sqrt{1+\sqrt{2}}\right)\right) - 8\sum_{n=1}^{\infty} \frac{(-1)^n\left(2^{2n-1}-1\right)B_n\left(\ln\left(1+\sqrt{2}\right)\right)^{2n+1}}{n\,(2n+1)!} \tag{7}$$

$$B_n = \left\{\frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \frac{1}{30}, \frac{5}{66}, \frac{691}{2730}, \ldots\right\} \tag{8}$$

$$\pi^2 = -8\int_0^a \ln(\tanh(x))\,dx + 8\ln\left(\frac{2}{1+\tanh(a)}\right) + 8\sum_{n=1}^{\infty}\frac{2^n}{n+1}\left\{\ln\left(\frac{2}{1+\tanh(a)}\right) - \sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k-1}}{k}\left(1-\left(\frac{1+\tanh(a)}{2}\right)^k\right)\right\} \quad , \; a > 0 \tag{9}$$

$$\pi^2 = -8\int_0^a \ln(\tanh(x))\,dx + 4\sum_{n=1}^{\infty}\frac{(1-\tanh(a))^n}{n^2}F\left(1, n, n+1, \frac{1-\tanh(a)}{2}\right) \; , \; a > 0 \tag{10}$$

$$\pi^2 = -8\int_0^a \ln(\tanh(x))\,dx + 4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}\left(\frac{1}{\tanh(a)}-1\right)^n F\left(1, n, n+1, \frac{1}{2}-\frac{1}{2\tanh(a)}\right) \; , \; a > \frac{\ln(3)}{2} \tag{11}$$

$$\pi^2 = 8\int_0^{\infty}\frac{x\,e^{-x}}{1-e^{-2x}}\,dx = 4\int_0^{\infty}\frac{x}{\sinh(x)}\,dx \tag{12}$$

$$\pi^2 = 4 + 4\sum_{n=1}^{\infty} n\ln\left(\tanh\left(\frac{n+1}{2}\right)\coth\left(\frac{n}{2}\right)\right) + 8\sum_{n=1}^{\infty}\frac{(-1)^n\,(2^{2n-1}-1)\,B_n}{(2n+1)!} + 4\int_0^1\sum_{n=1}^{\infty}\frac{x}{\sinh(x+n)}\,dx \tag{13}$$

Remarks :

■ B_n are the Bernoulli numbers, $\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 3.141592\ldots$, and $F(a, b, c, x)$ is the Gauss hypergeometric function.

III. References .

[1] Gradshteyn, I. S., and Ryzhik, I.M.: Table of Integrals, Series and Products. 7th ed., edited by Alan Jeffrey and Daniel Zwillinger, Academic Press, 2007.

[2] Olver, F.W.J., et al: NIST Handbook of Mathematical Functions. Cambridge University Press, 2010.

[3] Spanier, J., and Oldham, K.B.: An Atlas of Functions. Hemisphere Publishing, 1987.