Essential Problems on the Origins of Mathematics; Division by Zero Calculus and New World

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Abstract: Based on the preprint survey paper ([25]), we will give a viewpoint of the division by zero calculus from the origins of mathematics that are the essences of mathematics. The contents in this paper seem to be serious for our mathematics and for our world history with the materials in [25]. So, the author hopes that the related mathematicians, mathematical scientists and others check and consider the topics from various viewpoints.

Key Words: Division by zero, division by zero calculus, differential equation, analysis, infinity, discontinuous, point at infinity, Laurent expansion, conformal mapping, stereographic projection, Riemann sphere, horn torus, elementary geometry, zero and infinity, $1/0 = 0/0 = z/0 = \tan(\pi/2) = \log 0 = 0, (z^n)/n = \log z$ for $n = 0$ and $e^{(1/z)} = 1$ for $z = 0$.

AMS Mathematics Subject Classifications: 00A05, 00A09, 42B20, 30E20.

1 The origins of mathematics

As the origins of mathematics we will be able to consider Euclidean geometry (Euclid: 325 BC?-265 BC?) and the law of four arithmetic operations by introducing 0 in Brāhmasphuṭasiddhānta (AD 628) by Brahmagupta (598 -668 ?). Indeed, we will study their mathematics in the elementary school. We note that the two fundamentals may be unified by the introduction of coordinates by the idea of René Descartes (1596 - 1650). Of course, his idea is the basis of calculus and complex analysis; that is, of analysis.

For Euclidian geometry we can not see and we can not consider the point at infinity. We consider there that the spaces are extended endlessly (unboundedly) in the uniform way.

In complex analysis, we consider the Riemann sphere by the stereographic projection mapping of the complex plane to a sphere and the point at infinity may be realized as the north pole of the sphere. This idea is the long common sense in modern mathematics. However, the division by zero will introduce a new space idea for the point at infinity; that is the point at infinity is represented by zero. Therefore, the Riemann sphere should be changed to the horn torus model that the point at infinity and zero point is attaching ([2]).

Meanwhile, Brahmagupta could not consider the division by zero in a general situation, however, he defined as $0/0 = 0$. We have been, however, considering that his definition $0/0 = 0$ is wrong for over 1300 years, but, we found that his definition is right and suitable.

From the viewpoint of algebra, we can easily obtain a field structure containing the division by zero as the Yamada field, however, in order to realize the division by zero clearly, we will need the concept of division by zero calculus using the Laurent expansion for analytic functions. This concept says that we can consider analytic functions even at singular points. This will show a new world and the values at singular points have meanings. The division by zero calculus will give a great impact to analysis and geometry globally. We will give the essence of the impacts by using the related references, simply.

This paper may be considered as a short version of the paper ([26]) from the viewpoint of the initial part of the division by zero calculus, not from the viewpoint of applications and great impacts to our mathematics, because many people do not still have interests for the division by zero.

2 Backgrounds and short history

In order to state the contents in a self-contained manner, we first state the essences of division by zero and the division by zero calculus.

For the long history of division by zero, see [1, 18]. S. K. Sen and R. P. Agarwal [27] quite recently referred to our paper [4] in connection with division by zero, however, their understandings on the paper seem to be not suitable (not right) and their ideas on the division by zero seem to be traditional, indeed, they stated as the conclusion of the introduction of the book in the following way:

"Thou shalt not divide by zero" remains valid eternally.

However, in [22] we stated simply that based on the division by zero calculus

We Can Divide the Numbers and Analytic Functions by Zero with a Natural Sense.

For the long tradition on the division by zero, people may not be accepted our new results against many clear evidences on our division by zero, however, a physicist stated as follows:

Here is how I see the problem with prohibition on division by zero, which is the biggest scandal in modern mathematics as you rightly pointed out (2017.10.14.08:55).

The common sense on the division by zero with the long and mysterious history is wrong and our basic idea on the space around the point at infinity is also wrong since Euclid. On the gradient or on differential coefficients we have a great missing since $\tan(\pi/2) = 0$. Our mathematics is also wrong in elementary mathematics on the division by zero. In this paper, in a new and definite sense, we will show and give various applications of the division by zero $0/0 = 1/0 = z/0 = 0$ with the related references. In particular, we will introduce several fundamental concepts in calculus, Euclidean geometry, analytic geometry, complex analysis and differential equations. We will see new properties on the Laurent expansion, singularity, derivative, extension of solutions of differential equations beyond analytical and isolated singularities, and reduction problems of differential equations. On Euclidean geometry and analytic geometry, we will find new fields by the concept of the division by zero. We will give many concrete properties in mathematical sciences from

the viewpoint of the division by zero. We will know that the division by zero is our elementary and fundamental mathematics.

The contents in ([25]) are as follows.

1. Introduction.
2. Division by zero.
3. Division by zero calculus.
4. We can divide the numbers and analytic functions by zero.
5. General division and usual division.
6. Division by zero calculus.
7. Derivatives of functions.
8. Differential equations.
9. Euclidean spaces and division by zero calculus.
10. Analytic functions and division by zero calculus.
11. The Descartes circle theorem.
12. Horn torus models and division by zero calculus – a new world.

3 Division by zero

The division by zero with the mysterious and long history was indeed trivial and clear as in the followings.

By the concept of the Moore-Penrose generalized solution of the fundamental equation $ax = b$, the division by zero was trivial and clear as $b/0 = 0$ in the **generalized fraction** that is defined by the generalized solution of the equation $ax = b$. Here, the generalized solution is always uniquely determined and the theory is very classical. See [4] for example.

Recall the uniqueness theorem by S. Takahasi on the division by zero. See [4, 30]:

Proposition 2.1 *Let F be a function from $\mathbf{C} \times \mathbf{C}$ to $\mathbf{C}$ such that*

$$F(a,b)F(c,d) = F(ac,bd)$$

for all

$$a, b, c, d \in \mathbf{C}$$

and

$$F(a,b) = \frac{a}{b}, \quad a, b \in \mathbf{C}, b \neq 0.$$

Then, we obtain, for any $a \in \mathbf{C}$

$$F(a, 0) = 0.$$

In the long mysterious history of the division by zero, this proposition seems to be decisive.

Following Proposition 2.1, we should **define**

$$F(b, 0) = \frac{b}{0} = 0,$$

and we should consider that for the mapping

$$W = f(z) = \frac{1}{z}, \tag{3.1}$$

the image $f(0)$ of $z = 0$ is $W = 0$ (**should be defined from the form**). This fact seems to be a curious one in connection with our well-established popular image for the point at infinity on the Riemann sphere. As the representation of the point at infinity on the Riemann sphere by the zero $z = 0$, we will see some delicate relations between 0 and ∞ which show a strong discontinuity at the point of infinity on the Riemann sphere. We did not consider any value of the elementary function $W = 1/z$ at the origin $z = 0$, because we did not consider the division by zero $1/0$ in a good way. Many and many people consider its value at the origin by limiting like $+\infty$ and $-\infty$ or by the point at infinity as ∞. However, their basic idea comes from **continuity** with the common sense or based on the basic idea of Aristotele. – However, as the division by zero we will consider its value of the function $W = 1/z$ as zero at $z = 0$. We will see that this new definition is valid widely in mathematics and mathematical sciences, see ([8, 10]) for example. Therefore, the division by zero will give great impact to calculus, Euclidean geometry, analytic geometry, complex analysis and the theory of differential equations at an undergraduate level and furthermore to our basic idea for the space and universe.

In particular, note that the function $W = 1/z$ is continuous when the function is considered as a mapping on the horn torus into the horn torus. See ([25]).

The simple field structure containing division by zero was established by M. Yamada ([7]) in a natural way. For a simple introduction, H. Okumura [9] discovered the very simple essence that:

To divide by zero is to multiply by zero.

For the operator properties of the generalized fractions, see [30].

4 Division by zero calculus

As the number system containing the division by zero, the Yamada field structure is perfect. However, for applications of the division by zero to **functions**, we need the concept of the division by zero calculus for the sake of unique determination of the results and for other reasons.

We will introduce the division by zero calculus. For any Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{-1} C_n(z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n(z-a)^n, \tag{4.1}$$

we **define** the identity

$$f(a) = C_0. \tag{4.2}$$

Apart from the motivation, we define the division by zero calculus by (4.2). With this assumption, we can obtain many new results and new ideas. However, for this assumption we have to check the results obtained whether they are reasonable or not. By this idea, we can avoid any logical problems. – In this point, the division by zero calculus may be considered **as a fundamental assumption like an axiom.**

In addition, we will refer to an interesting viewpoint of the division by zero calculus.

Recall the Cauchy integral formula for an analytic function $f(z)$; for an analytic function $f(z)$ around $z = a$ and for a smooth simple Jordan closed curve γ enclosing one time the point a, we have

$$f(a) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(z)}{z-a} dz.$$

Even when the function $f(z)$ has any singularity at the point a, we assume that this formula is valid as the division by zero calculus.

We **define** the value of the function $f(z)$ at the singular point $z = a$ with the above Cauchy integral.

5 We can divide the numbers and analytic functions by zero

In the division by zero like $1/0, 0/0$ the important problem was on their definitions. We will give our interpretation.

Based on the division by zero calculus, the meaning (definition) of

$$\frac{1}{0} = 0$$

is given by $f(0) = 0$ by means of the division by zero calculus for the function $f(z) = 1/z$. Similarly, the definition

$$\frac{0}{0} = 0$$

is given by $f(0) = 0$ by means of the division by zero calculus for the function $f(z) = 0/z$.

In the division by zero, the essential problem was in the sense of the division by zero (**definition**) $z/0$. Many confusions and simple history of division by zero may be looked in [15].

In order to give the precise meaning of division by zero, we will give a simple and affirmative answer, for a famous rule that we are not permitted to divide the numbers and functions by zero. In our mathematics, **prohibition** is a famous word for the division by zero.

For any analytic function $f(z)$ around the origin $z = 0$ that is permitted to have any singularity at $z = 0$ (of course, any constant function is permitted), we can consider the value, by the division by zero calculus

$$\frac{f(z)}{z^n} \tag{5.1}$$

at the point $z = 0$, for any positive integer n. This will mean that from **the form** we can consider it as follows:

$$\frac{f(z)}{z^n} \mid_{z=0} . \tag{5.2}$$

For example,

$$\frac{e^x}{x^n} \mid_{x=0} = \frac{1}{n!}.$$

This is the definition of our division by zero (general fraction). In this sense, we can divide the numbers and analytic functions by zero. For $z \neq 0$, $\frac{f(z)}{z^n}$ means the usual division of the function $f(z)$ by z^n.

Surprisingly enough, Brahmāgupta said 0/0=0 in 628, thirteen hundred years ago. However, our world history shows that his result is wrong and we have still in confusions on the division by zero.

However, his result and idea was right.

For many applications, see the original survey paper ([25]) and the references in this paper.

In order to show the importance of our division by zero and division by zero calculus we obtained many applications and many examples over 1000 items from an elementary mathematics. However, with the results stated in the references, we think the importance of our division by zero may be definitely stated already and clearly.

Of course, mathematics is developing on some axioms; basic assumptions. For the division by zero calculus, we will need the basic assumption for the definition. We gave many and many motivations and applications, already.

6 Remarks

As a history of mathematics, European (containing the USA) people did not like to consider ZERO and VOID for long years, meanwhile people of India will have a long history and deep culture for them. These feelings seem to be still valid with the problem of division by zero.

7 Conclusions

The contents in this paper seem to be serious for our mathematics and for our world history with the materials in [25]. So, the author expects that the related mathematicians, mathematical scientists and others check and consider the topics from various viewpoints.

We stated for the estimation of mathematics in [19] as follows: Mathematics is a collection of relations and, good results are fundamental, beautiful, and give good impacts to human beings.

With this estimation, we stated that the Euler formula

$$e^{\pi i} = -1$$

is the best result in mathematics in details in:

No.81, May 2012 (pdf 432kb) www.jams.or.jp/kaiho/kaiho-81.pdf

Now, how will be our results:

$$\frac{1}{0} = \frac{0}{0} = \frac{z}{0} = \tan(\pi/2) = \log 0 = 0,$$

$$\frac{z^n}{n}|_{n=0} = \log z$$

and

$$e^{(1/z)}|_{z=0} = 1$$

?

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