Born's Reciprocal Relativity Theory, Curved Phase Space, Finsler Geometry and the Cosmological Constant

Carlos Castro Perelman
Center for Theoretical Studies of Physical Systems
Clark Atlanta University, Atlanta, Georgia. 30314,
Ronin Institute, 127 Haddon Pl., NJ. 07043
perelmanc@hotmail.com

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Abstract

A brief introduction of the history of Born's Reciprocal Relativity Theory, Hopf algebraic deformations of the Poincare algebra, de Sitter algebra, and noncommutative spacetimes paves the road for the exploration of gravity in *curved* phase spaces within the context of the Finsler geometry of the cotangent bundle T^*M of spacetime. A scalar-gravity model is duly studied, and exact nontrivial analytical solutions for the metric and nonlinear connection are found that obey the generalized gravitational field equations, in addition to satisfying the *zero* torsion conditions for *all* of the torsion components. The *curved* base spacetime manifold and internal momentum space both turn out to be (Anti) de Sitter type. The most salient feature is that the solutions capture the very early inflationary and very-late-time de Sitter phases of the Universe. A *regularization* of the 8-dim phase space action leads naturally to an extremely small effective cosmological constant Λ_{eff}, and which in turn, furnishes an extremely small value for the underlying four-dim spacetime cosmological constant Λ, as a direct result of a *correlation* between Λ_{eff} and Λ resulting from the field equations. The rich structure of Finsler geometry deserves to be explore further since it can shine some light into Quantum Gravity, and lead to interesting cosmological phenomenology.

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1 Introduction

Most of the work devoted to Quantum Gravity has been focused on the geometry of spacetime rather than phase space per se. Moyal [2] anticipated the importance of phase space and introduced the noncommutative star product $A(x,p) * B(x,p)$ (Moyal-Weyl-Wigner-Groenewold product) of functions in phase space which spawned the Moyal-Fedosov deformation quantization program. A thorough study of the geometry of phase space can be found in [3]. The first indication that phase space should play a role in Quantum Gravity was raised by [1]. The principle of Born's reciprocal relativity [1] was proposed long ago based on the idea that coordinates and momenta should be unified on the same footing, and consequently, if there is a limiting speed (temporal derivative of the position coordinates) in Nature there should be a maximal force [4] as well, since force is the temporal derivative of the momentum. A *maximal* speed limit (speed of light) must be accompanied with a *maximal* proper force (which is also compatible with a *maximal* and *minimal* length duality).

The generalized velocity and acceleration boosts (and rotations) transformations of the $flat$ $8D$ Phase space, where $x^i, t, E, p^i; i = 1,2,3$ are *all* boosted (rotated) into each-other, were given by [7] based on the group $U(1,3)$ and which is the Born version of the Lorentz group $SO(1,3)$. The $U(1,3) = SU(1,3) \times U(1)$ group transformations leave invariant the symplectic 2-form $\Omega = -\ dt \wedge dp_0 + \delta_{ij} dx^i \wedge dp^j; i,j = 1,2,3$, and also the following Born-Green line interval in the $flat$ $8D$ phase-space $(d\omega)^2 = \eta_{\mu\nu} dx^\mu dx^\nu + b^{-2} \eta_{\mu\nu} dp^\mu dp^\nu$. Factoring out the spacetime proper time $d\tau^2 = \eta_{\mu\nu} dx^\mu dx^\nu$ leaves $(d\omega)^2 = (d\tau)^2 (1 - \frac{F^2}{b^2})$, where $-F^2 < 0$ is the spacelike proper-force squared $(dp_\mu / d\tau)(dp^\mu / d\tau) < 0$ associated to a timelike interval $(d\tau)^2 > 0$. The Born constant b is the maximal proper force which can be postulated to be the Planck mass-squared M_P^2 in the units $\hbar = c = 1$.

A study of the many novel consequences of Born's reciprocal relativity theory (BRRT) can be found in [5], in particular the relativity of locality. Given a local event in a given reference frame represented by the intersection of two worldlines associated with two particles of equal mass, but $different$ energies and momenta, there is an accelerated frame of reference with sufficient acceleration-rapidity parameter such that no intersection of the worldlines occurs. Besides relativity of *locality*, we also may have relativity of *chronology*. One observer will describe as a physical event to be the one defined by the intersection of two worldlines taken place in his (her) $future$, while an accelerated observer will describe it as an intersection of two worldlines taken place in his (her) *past* [5].

Relative locality [8], [13] in a very different context originated from some interpretational issues connected to the possibility that energy-momentum space be curved, as for example in doubly special relativity (DSR) [9] , some models of noncommutative geometry [11] and 3D quantum gravity [14]. It is better understood now that the Planck-scale modifications of the particle dispersion relations can be encoded in the nontrivial geometrical properties of momentum space [8]. When both spacetime curvature and Planck-scale deformations of momentum space are present, it is expected that the nontrivial geometry of momentum space and spacetime get intertwined. The interplay between spacetime curvature and non-trivial momentum space effects was essential in the notion of "relative locality" and in the deepening of the relativity principle [8].

The theory is based on the assumption that physics takes place in phase space and there is no invariant global projection that gives a description of physical processes in spacetime. Therefore, local observers can construct descriptions of particles interacting in spacetime, but different observers construct different spacetimes, which correspond to different foliations of phase space. So, the notion of locality becomes observer dependent, whence the name of the theory.

This formulation of relative locality is very $different$ than ours despite the fact that both rely on the geometry of phase-spaces. Our results above are based on the nontrivial transformation properties of the phase space coordinates under force/acceleration boost transformations which *mix* spacetime coordinates with energy-momentum coordinates. Whereas the formulations [8], [9], [11], [14] rely on the geometry of *curved* phase-spaces, and the use of Hopf algebras leading to a deformed Poincare algebra, modified dispersion relations, a coproduct of momenta, and a coproduct of Lorentz generators.

We recall that DSR introduces in special relativity a new fundamental scale with the dimension of mass (usually identified with the Planck mass) in addition to the speed of light. The new scale gives rise to $deformations$ of the action of the Lorentz group on phase space, and consequently of the dispersion law of particles, of the addition law of momenta, and so on. Although doubly special relativity is mainly concerned with energy-momentum space, it is often realized in terms of *noncommutative* geometries that postulate a noncommutative structure of spacetime with a fundamental length scale of the order of the Planck length, and are in some sense dual to the DSR approach. It is important to emphasize that a maximal proper force does not necessarily imply a minimum length. Setting $F = mc^2/L = b$, as a maximal proper force, one could have the scenario where $m \to 0, L \to 0$ such that $(m/L)c^2 = b$, and consequently there is no minimal length but there is a maximal proper force.

The energy-momentum space geometry defined in [8] has been investigated in a specific instance in [15], where it has been applied to the case of the κ-Poincare model [11], one of the favorite realization of DSR. This is a model of noncommutative geometry displaying a $deformed$ action of the Lorentz group on spacetime, whose energy-momentum space can be identified with a *curved* hyperboloid embedded in a 5-dimensional flat space [16].

The theory of relative locality refines this picture, by introducing some additional structures in the geometry of energy-momentum space, related to the properties of the $deformed$ addition law of momenta, due to the coproduct of momenta associated with the Hopf algebraic structure. The authors [13] investigated a different example of noncommutative geometry, namely the Snyder model [17] and its generalizations [18]. The distinctive property of this class of models is the preservation of the *linear* action of the Lorentz algebra on spacetime. This implies that the leading-order corrections to the composition law of the momenta must be cubic in the momenta, rather than quadratic. Moreover, the composition law is not only *noncommutative* but also *nonassocative*.

A new proposal [19] for the notion of spacetime in a relativistic generalization of special relativity based on a modification of the composition law of momenta was presented. Locality of interactions is the principle which defines the spacetime structure for a system of particles. The main result [19] has been to show that it is possible to define a noncommutative spacetime for particles participating in an interaction in such a way that the interaction is seen as local for every observer. There exists then a large freedom to introduce a noncommutative spacetime in a relativistic theory beyond Special Relativity (SR) in a way compatible with the locality of interactions. An interesting particular case is the one in which the new spacetime of the two-particle system is such that the coordinates of one of the particles depend only on its own momentum.

Quantum groups, non-commutative Lorentzian spacetimes and curved momentum spaces were analyzed further by [12]. Most importantly, (Anti) de Sitter non-commutative spacetimes and curved momentum spaces in $(1+1)$ and $(2+1)$ dimensions arising from the κ-deformed quantum group symmetries. The generalization of these results to the physically relevant $(3+1)$-dimensional deformation was also discussed.

The aim of this work is to explore gravity in curved phase spaces within the context of the Finsler geometry of the cotangent bundle of spacetime. We study a scalar-gravity model and find exact nontrivial analytical solutions for the metric and nonlinear connection that obey the generalized gravitational field equations, in addition to satisfying the *zero* torsion conditions for *all* of the torsion components. The *curved* base spacetime manifold and momentum space both turn out to be (Anti) de Sitter type. The most salient feature is that the solutions capture the very early inflationary and very-late-time de Sitter phases of the Universe. A *regularization* of the 8-dim phase space action leads naturally to an extremely small effective cosmological constant Λ_{eff}, and which in turn, furnishes an extremely small value for the underlying four-dim spacetime cosmological constant Λ, as a direct result of a *correlation* between Λ_{eff} and Λ resulting from the field equations. Therefore, the rich structure of Finsler geometry deserves to be explore further since it can lead to interesting cosmological phenomenology [21], and shine some light into Quantum Gravity.

2 Curved Phase Space and Finsler Geometry

To explore the geometry behind a maximal proper force and/or maximal acceleration in more general *curved* phase spaces (cotangent bundles), we shall follow next the description by [20], [22] where one may study in detail the geometry of Lagrange-Finsler and Hamilton-Cartan spaces and their higher order (jet bundles) generalizations. For other references on Finsler geometry see [21].

In the case of the cotangent space of a d-dim manifold T^*M_d the Sasaki-

Finsler metric can be rewritten in the *block diagonal* form as

$$(d\omega)^2 \ = \ g_{ij}(x^k, p_a) \ dx^i d \ x^j \ + \ h^{ab}(x^k, p_c) \ \delta p_a \ \delta p_b \ =$$
$$g_{ij}(x^k, p_a) \ dx^i d \ x^j \ + \ h_{ab}(x^k, p_c) \ \delta p^a \ \delta p^b \tag{1}$$

The indices range is $i, j, k = 0, 1, 2, 3, ...d-1$; $a, b, c = 0, 1, 2, 3, ...d-1$, and the standard coordinate basis frame has been replaced by the following anholonomic frames (non-coordinate basis)

$$\delta_i \ = \ \delta/\delta x^i \ = \ \partial_{x^i} \ + \ N_{ia} \ \partial^a \ = \ \partial_{x^i} \ + \ N_{ia} \ \partial_{p_a}; \ \ \partial^a \ \equiv \ \partial_{p_a} \ = \ \frac{\partial}{\partial p_a} \tag{2}$$

The signature is chosen to be Lorentzian $(-, +, +, +, \cdots, +)$ for both g_{ij} and h_{ab}. It is important to emphasize that one does *not* have two times because the energy coordinate is not time. One should note the *key* position of the indices that allows us to distinguish between derivatives with respect to x^i and those with respect to p_a. The dual basis of $(\delta_i = \delta/\delta x^i; \partial^a = \partial/\partial p_a)$ is

$$dx^i, \ \ \delta p_a \ = \ dp_a \ - \ N_{ja} \ dx^j, \ \ \delta p^a \ = \ dp^a \ - \ N^a_j \ dx^j \tag{3}$$

where the N–coefficients define a nonlinear connection, N–connection structure.

An N-linear connection D on T^*M allows to construct covariant derivatives which are compatible with the structure induced by the nonlinear connection that preserve the horizontal-vertical split of the cotangent bundle. Thus, an N-linear connection D on T^*M can be uniquely represented in the adapted basis in the following form

$$D_{\delta_j}(\delta_i) \ = \ H^k_{ij} \ \delta_k; \ \ D_{\delta_j}(\partial^a) \ = \ - \ H^a_{bj} \ \partial^b; \tag{4a}$$

$$D_{\partial^a}(\delta_i) \ = \ C^{ka}_i \ \delta_k; \ \ D_{\partial^a}(\partial^b) \ = \ - \ C^{ba}_c \ \partial^c \tag{4b}$$

where $H^k_{ij}(x,p), H^a_{bj}(x,p), C^{ka}_i(x,p), C^{ba}_c(x,p)$ are the connection coefficients. Our notation for the derivatives is

$$\partial^a = \partial/\partial p_a, \ \ \partial_i \ = \ \partial_{x^i}, \ \ \delta_i \ = \ \delta/\delta x^i \ = \ \partial_{x^i} \ + \ N_{ia} \ \partial^a \tag{4c}$$

The N–connection structures can be naturally defined on (pseudo) Riemannian spacetimes and one can relate them with some anholonomic frame fields (vielbeins) satisfying the relations $\delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha = W^\gamma_{\alpha\beta} \delta_\gamma$. The only nontrivial (nonvanishing) nonholonomy coefficients are

$$W_{ija} \ = \ R_{ija}; \ \ W^a_{\ jb} \ = \ \partial^a N_{jb} \ = \ - \ W_j{}^a{}_b \tag{5a}$$

and

$$R_{ija} \ = \ \delta_j N_{ia} \ - \ \delta_i N_{ja} \tag{5b}$$

is the nonlinear connection curvature (N–curvature).

Imposing a zero nonmetricity condition of $g_{ij}(x,p), h^{ab}(x,p)$ along the horizontal and vertical directions, respectively, gives

$$D_i g_{jk} \;=\; \delta_i \; g_{gk} \;\; - H^l_{ij} \; g_{lk} \;\; - H^l_{ik} \; g_{jl} \;\; = \;\; 0, \tag{6a}$$

$$D^a h^{bc} \;\; = \;\; \partial^a \; h^{bc} \;\; + \;\; C^{ab}_d \; h^{dc} \;\; + \;\; C^{ac}_d \; h^{bd} \;\; = \;\; 0 \tag{6b}$$

Performig a cyclic permutation of the indices in eqs-(6a,6b), followed by linear combination of the equations obtained yields the irreducible (horizontal, vertical) h-v-components for the connection coefficients

$$H^i{}_{jk} \;\; = \;\; \frac{1}{2} g^{in} \left(\delta_k g_{nj} + \delta_j g_{nk} - \delta_n g_{jk} \right) \tag{7}$$

$$C^{ab}_c \;\; = \;\; - \;\; \frac{1}{2} \; h_{cd} \;\; \left(\; \partial^b h^{ad} \;\; + \;\; \partial^a h^{bd} \;\; - \;\; \partial^d h^{ab} \; \right) \tag{8}$$

The additional conditions $D_i h^{ab} = 0, D^a g_{ij} = 0$, yield the *mixed* components of the connection coefficients

$$H^b_{ja} \;\; = \;\; \partial^b N_{ja} \;\; + \;\; \frac{1}{2} \; h^{bc} \;\; \left(\; \delta_j h_{ac} \;\; - \;\; h_{ad} \; \partial^d N_{jc} \;\; - \;\; h_{cd} \; \partial^d N_{ja} \; \right) \tag{9}$$

and

$$C^{ja}_i \;\; = \;\; \frac{1}{2} \; g^{jk} \; \partial^a g_{ik} \tag{10}$$

For any N-linear connection D with the above coefficients the torsion 2-forms are

$$\Omega^i \;\; = \;\; \frac{1}{2} T^i_{jk} \; dx^j \wedge dx^k \;\; + \;\; C^{ia}_j \; dx^j \wedge \delta p_a \tag{11a}$$

$$\Omega_a \;\; = \;\; \frac{1}{2} R_{jka} \; dx^j \wedge dx^k \;\; + \;\; P^b_{aj} \; dx^j \wedge \delta p_b \;\; + \;\; \frac{1}{2} S^{bc}_a \; \delta p_b \wedge \delta p_c \tag{11b}$$

and the curvature 2-forms are

$$\Omega^i_j \;\; = \;\; \frac{1}{2} R^i_{jkm} \; dx^k \wedge dx^m \;\; + \;\; P^{ia}_{jk} \; dx^k \wedge \delta p_a \;\; + \;\; \frac{1}{2} S^{iab}_j \; \delta p_a \wedge \delta p_b \tag{12}$$

$$\Omega^a_b \;\; = \;\; \frac{1}{2} R^a_{bkm} \; dx^k \wedge dx^m \;\; + \;\; P^{ac}_{bk} \; dx^k \wedge \delta p_c \;\; + \;\; \frac{1}{2} S^{acd}_b \; \delta p_c \wedge \delta p_d \tag{13}$$

where one must recall that the dual basis of $\delta_i = \delta/\delta x^i, \; \partial^a = \partial/\partial p_a$ is given by $dx^i, \; \delta p_a = dp_a - N_{ja} dx^j$.

The distinguished torsion tensors are

$$T^i_{jk} \;\; = \;\; H^i_{jk} \;\; - \;\; H^i_{kj}; \;\;\; S^{ab}_c \;\; = \;\; C^{ab}_c \;\; - \;\; C^{ba}_c; \;\;\; T^{ia}_j \;\; = \;\; C^{ia}_j \;\; = \;\; - \;\; T^{ia}{}_j$$

$$P_b{}^a{}_j \;\; = \;\; H^a_{bj} \;\; - \;\; \partial^a N_{jb}, \;\;\; P_b{}^a{}_j \;\; = \;\; - \;\; P_{bj}{}^a$$

$$R_{ija} \;=\; \frac{\delta N_{ja}}{\delta x^i} \;-\; \frac{\delta N_{ia}}{\delta x^j} \tag{14}$$

The distinguished tensors of the curvature are

$$R^i_{kjh} \;=\; \delta_h H^i_{kj} \;-\; \delta_j H^i_{kh} \;+\; H^l_{kj}\; H^i_{lh} \;-\; H^l_{kh}\; H^i_{lj} \;-\; C^{ia}_k\; R_{jha} \tag{15}$$

$$P^{ab}_{cj} \;=\; \partial^a\; H^b_{cj} \;+\; C^{ad}_c\; P^b_{dj} \;-\; \left(\delta_j\; C^{ab}_c \;+\; H^b_{dj}\; C^{da}_c \;+\; H^a_{dj}\; C^{bd}_c \;-\; H^d_{cj}\; C^{ab}_d\right) \tag{16}$$

$$P^{ak}_{ij} \;=\; \partial^a\; H^k_{ij} \;+\; C^{al}_i\; T^k_{lj} \;-\; \left(\delta_j\; C^{ak}_i \;+\; H^a_{bj}\; C^{bk}_i \;+\; H^k_{lj}\; C^{al}_i \;-\; H^l_{ij}\; C^{ak}_l\right) \tag{17}$$

$$S^{abc}_d \;=\; \partial^c\; C^{ab}_d \;-\; \partial^b\; C^{ac}_d \;+\; C^{eb}_d\; C^{ac}_e \;-\; C^{ec}_d\; C^{ab}_e; \tag{18}$$

$$S^{ibc}_j \;=\; \partial^c C^{bi}_j \;-\; \partial^b C^{ci}_j \;+\; C^{bh}_j\; C^{ci}_h \;-\; C^{ch}_j\; C^{bi}_h \tag{19}$$

$$R^a_{bjk} \;=\; \delta_k H^a_{bj} \;-\; \delta_j H^a_{bk} \;+\; H^c_{bj}\; H^a_{ck} \;-\; H^c_{bk}\; H^a_{cj} \;-\; C^{ca}_b\; R_{jkc} \tag{20}$$

Adopting the units where $\hbar = c = G = 1$ such that the Planck mass and length squared are respectively $M_P^2 = 1, L_P^2 = 1$; given $g^{AB} \equiv g^{ij}, h^{ab}$, and the definitions $\partial_A \Phi(x,p) \equiv \delta_i \Phi(x,p), \partial_a \Phi(x,p)$, where the ordinary ∂_a and elongated derivatives δ_i defined by eq-(2) act on $\Phi(x,p)$, one may construct the simplest gravity-scalar field action of the form[1]

$$\mathcal{S} \;=\; \mathcal{S}_G \;+\; \mathcal{S}_M \;=\; \frac{1}{2\kappa} \int d^4x\; d^4p\; \sqrt{|\det g_{AB}|}\; \left(g^{ij}\; R_{(ij)} \;+\; h_{ab}\; S^{(ab)} \right) \;-$$
$$\int d^4x\; d^4p\; \sqrt{|\det g_{AB}|}\; \left(\frac{1}{2}\; g^{AB}\; \partial_A \Phi\; \partial_B \Phi \;+\; V(\Phi) \right) \tag{21}$$

The determinant factorizes $det(g_{AB}) = det(g_{ij}) det(h_{ab})$ in an anhololomic basis adapted to the nonlinear connection (the metric assumes the block diagonal form (1)). κ is the gravitational coupling constant. If the phase space action action (21) is dimensionless, after reintroducing the physical constants that were set to unity, gives $\kappa = 8\pi \to (8\pi G/c^4)(M_p c)^4$.

After a very laborious procedure the authors [21] have shown that variation of the action (21)

$$\frac{\delta \mathcal{S}}{\delta g_{ij}} \;=\; 0, \quad \frac{\delta \mathcal{S}}{\delta h_{ab}} \;=\; 0, \quad \frac{\delta \mathcal{S}}{\delta N_{ia}} \;=\; 0, \quad \frac{\delta \mathcal{S}}{\delta \Phi} \;=\; 0 \tag{22}$$

[1] $d^4x\; d^4p = dx^0 \wedge dx^1 \wedge \cdots \wedge \delta p_0 \wedge \delta p_1 \wedge \cdots = dx^0 \wedge dx^1 \wedge \cdots \wedge dp_0 \wedge dp_1 \wedge \cdots$

leads to the following field equations

$$R_{(ij)}(x,p) \ - \frac{1}{2} \ g_{ij}(x,p) \ (R+S) \ + \ R_{k(ia} \ C^{ka}_{j)} \ = \ T_{ij} \tag{23}$$

$$S_{(ab)}(x,p) \ - \frac{1}{2} \ h_{ab}(x,p) \ (R+S) \ = \ T_{ab} \tag{24}$$

$$g^{ik} \ \partial^a H^j_{kj} \ - \ g^{kl} \ \partial^a H^i_{kl} \ = \ T^{ia} \tag{25}$$

where

$$R_{kh} \ = \ R^i_{kjh} \ \delta^j_i, \quad R \ = \ g^{kh} \ R_{(kh)} \quad S^{ac} \ = \ S^{abc}_d \ \delta^d_b, \quad S \ = \ h_{ac} \ S^{(ac)} \tag{26}$$

after symmetrizing the indices accordingly and denoted by (). The components of the stress energy tensor are defined as

$$T_{ij} \ = \ - \ \frac{2}{\sqrt{|detG_{AB}|}} \ \frac{\delta(\sqrt{|detG_{AB}|}L_M)}{\delta g^{ij}}, \ T_{ab} \ = \ - \ \frac{2}{\sqrt{|detG_{AB}|}} \ \frac{\delta(\sqrt{|detG_{AB}|}L_M)}{\delta h^{ab}} \tag{27}$$

$$T^{ia} \ = \ - \ \frac{2}{\sqrt{|detG_{AB}|}} \ \frac{\delta(\sqrt{|detG_{AB}|}L_M)}{\delta N_{ia}} \tag{28}$$

and given by

$$T_{ij} \ = \ (\delta_i \Phi(x,p)) \ (\delta_j \Phi(x,p)) - g_{ij} \ \left(\frac{1}{2} \ g^{AB} (\partial_A \Phi(x,p)) \ (\partial_B \Phi(x,p)) \ + \ V(\Phi) \right) \tag{29}$$

$$T_{ab} \ = \ (\partial_a \Phi(x,p)) \ (\partial_b \Phi(x,p)) - h_{ab} \ \left(\frac{1}{2} \ g^{AB} (\partial_A \Phi(x,p)) \ (\partial_B \Phi(x,p)) \ + \ V(\Phi) \right) \tag{30}$$

$$T^{ia} \ = \ g^{ij} \ \delta_j \Phi(x,p) \ \partial^a \Phi(x,p) \tag{31}$$

One must include also the equation of motion for the scalar field $\Phi(x,p)$, which is a generalization of the d'Alambert equation,

$$g^{ij} \ D_i D_j \Phi \ + \ h^{ab} \ D_a D_b \Phi \ - \ \frac{\partial V(\Phi)}{\partial \Phi} \ = \ 0 \tag{32}$$

$$D_i D_j \Phi \ = \ \delta_i \delta_j \Phi \ - \ H^k_{ij} \ \delta_k \Phi, \quad D_a D_b \Phi \ = \ \partial_a \partial_b \Phi \ - \ C^c_{ab} \ \partial_c \Phi \tag{33}$$

The system of *coupled* nonlinear differential equations (23,24,25,32) leading to the solutions for $g_{ij}(x,p), h_{ab}(x,p), N_{ai}(x,p), \Phi(x,p)$ are highly nontrivial. The scalar field $\Phi(x,p)$ curves both spacetime and momentum space. The equations have almost a similar form to the Einstein gravitational field equation with the difference of the extra term $R_{k(ia} \ C^{ka}_{j)}$ in eq-(23).

Many authors *choose* the nonlinear connection depending on the physical context rather than including the last equation (25) obtained from the variation

$\frac{\delta S}{\delta N_{ia}} = 0$. For example, the authors [21] investigated the cosmological bounce realization in the framework of generalized modified gravities arising from Finsler and Finsler-like geometries. They chose a specific nonlinear connection in the modified Friedman equations that satisfied the general cosmological bounce conditions and thus induced the bounce.

Instead of choosing the nonlinear connection by hand, and *eliminating* eq-(25) in the process, one could also impose the zero torsion condition

$$P_b{}^a{}_j \ = \ H^a_{bj} \ - \ \partial^a N_{jb} \ = \ 0 \ \Rightarrow$$

$$N_{jb}(x,p) \ = \ \int \ H^a_{bj}(h,N) \ dp_a \ + \ f_{jb}(x) \tag{34}$$

yielding an integro-differential equation for N_{jb}. The connection $H^a_{bj}(h,N)$ defined in eq-(9) is a function of h_{ab} and N_{bj}, and $f_{jb}(x)$ are arbitrary integration functions. Hence, instead of using eq-(25) obtained from a variation with respect to a dynamical nonlinear connection N_{ja}, eq-(34) determines, in principle, the nonlinear connection $N_{jb}(x,p)$ in terms of $h_{ab}(x,p)$, and the integration functions $f_{jb}(x)$.

Let us find solutions to these equations in the case when $\Phi(x,p) = \Phi_o =$ constant, $V(\Phi) = V_o =$ constant; the horizontal metric solely depends on x : $g_{ij}(x)$, and the vertical metric solely depends on $p : h_{ab}(p)$. In doing so one gets for the connection components the following

$$C^{ka}_j \ = \ 0, \ \ H^i_{jk} \ = \ \Gamma^i_{jk}(x), \ \ C^a_{bc} \ = \ \Gamma^a_{bc}(p) \tag{35}$$

where $\Gamma^i_{jk}(x), \Gamma^a_{bc}(p)$ are the ordinary Levi-Civita (Christoffel) connections written in terms of $g_{ij}(x), h_{ab}(p)$, respectively. Eq-(25) is identically satisfied since $T^{ia} = 0$ for constant Φ and $H^i_{jk}(x)$ depend on x only. From eq-(35) one then finds that the nonlinear connection decouples from the field equations (23,24) leading to

$$R_{ij}(x) \ - \frac{1}{2} \ g_{ij}(x) \ (R(x) + S(p)) \ = \ - \ \kappa \ g_{ij}(x) \ V_o \tag{36}$$

$$S_{ab}(p) \ - \frac{1}{2} \ h_{ab}(p) \ (R(x) + S(p)) \ = \ - \ \kappa \ h_{ab}(p) \ V_o \tag{37}$$

Let us find solutions to eqs-(36,37) which are (Anti) de Sitter like. These solutions can be generalized to other dimensions than $d + d = 4 + 4$. In a d-dim base spacetime one has $R_{ij} = \frac{2\Lambda_1}{d-2} g_{ij}$, and $R = \frac{2d}{d-2}\Lambda_1$ with $\Lambda_1 > 0$ for the d-dim de Sitter space dS_d, and $\Lambda_1 < 0$ for anti de Sitter AdS_d. Similar expressions hold for the internal d-dim momentum space Ricci and scalar curvatures $S_{ab} = \frac{2\Lambda_2}{d-2} h_{ab}$ and $S = \frac{2d}{d-2}\Lambda_2$. Taking the trace of eqs-(36,37), where the indices range is now given by $i, j = 0, 1, 2, \cdots, d-1$, and $a, b = 0, 1, 2, \cdots, d-1$, leads to

$$R \ (1 - \frac{d}{2}) \ - \frac{1}{2} \ d \ S \ = \ - \ \Lambda_{eff} \ d \ \equiv \ - \ \kappa V_o \ d \tag{38}$$

$$S \ (1 - \frac{d}{2}) \ - \frac{1}{2} \ d \ R \ = \ - \ \Lambda_{eff} \ d \ \equiv \ - \ \kappa V_o \ d \tag{39}$$

Upon using the expressions for the Ricci scalar curvatures R, S in terms of Λ_1, Λ_2 respectively, and inserting them into eqs-(38, 39), one arrives at

$$\Lambda_1 \ = \ \Lambda_2 \ = \ \Lambda \ = \ \frac{d(d-2)}{2(d-1)} \ \kappa V_0 \ = \ \frac{d(d-2)}{2(d-1)} \ \Lambda_{eff} \qquad (40a)$$

The cosmological constant is given in terms of the (Anti) de Sitter throat size L as

$$\Lambda \ = \ \pm \ \frac{(d-2)(d-1)}{2L^2} \ \Rightarrow \ \kappa V_0 \ = \ \pm \ \frac{(d-1)^2}{d} \ \frac{1}{L^2} \qquad (40b)$$

and the scalar curvatures (positive for de Sitter, negative for Anti deSitter) are

$$R \ = \ S \ = \ \frac{d^2}{d-1} \ \kappa V_0 \ = \ \pm \ \frac{d(d-1)}{L^2} \qquad (40c)$$

One should note that when $d = 2 \Rightarrow \Lambda = 0$ but the effective cosmological constant $\Lambda_{eff} \equiv \kappa V_0 = \pm \frac{1}{2L^2} \neq 0$. So we have a situation where Λ_{eff} could be extremely large, like in the order of $M_P^2 = L_P^{-2}$ for a Planck-sized throat size L, while $\Lambda = 0$. The quantity Λ_{eff} is associated with gravity in the $four$-dim phase space (cotangent bundle) while $\Lambda = 0$ is associated with the two-dim base manifold (spacetime) and the internal two-dim momentum space. The two-dim (Anti) de Sitter metrics are conformally flat with (negative) constant positive scalar curvature $\pm \frac{2}{L^2}$.

In particular, the solutions to eqs-(36,37) in $d + d = 4 + 4$ dimensions are given by

$$g_{tt} \ = \ - \ (1 - \frac{\Lambda}{3} \ r^2), \ \ g_{rr} \ = \ (1 - \frac{\Lambda}{3} \ r^2)^{-1}, \ \ g_{\theta\theta} \ = \ r^2, \ \ g_{\phi\phi} \ = \ r^2 \ sin^2\theta \quad (41)$$

$$h_{EE} \ = \ - \ (1 \ - \ \frac{\Lambda}{3} \ p_r^2), \ \ h_{p_r p_r} \ = \ (1 \ - \ \frac{\Lambda}{3} \ p_r^2)^{-1},$$
$$h_{p_\theta p_\theta} \ = \ p_r^2, \ \ h_{p_\phi p_\phi} \ = \ p_r^2 \ sin^2 p_\theta \qquad (42)$$

with $x^i = (t, r, \theta, \phi)$; $p_a = (E, p_r, p_\theta, p_\phi)$, and $\Lambda = \pm \frac{3}{L^2}$. The above solutions are given in static global coordinates that cover all of (Anti) de Sitter space. The units are taken such that the Planck length (mass) are set to unity. One can reintroduce the physical constants in eqs-(41, 42) if one wishes so all expressions have the correct physical units.

Because in this simple case the nonlinear connection N_{jb} has decoupled from the field equations, we can obtain it by imposing the zero torsion condition $P^a_{bk} = 0$ in eq-(14). Instead of solving the integro-differential equation (34) it is far simpler to choose the ansatz $N_{jk}(x,p) = N_{jk}(x)$ leading to

$$P_{bk}{}^a \ = \ H^a_{bk} \ - \ \partial^a N_{kb}(x) \ = \ 0 \ \Rightarrow \ H^a_{bk} \ = \ 0 \ \Rightarrow$$
$$h^{ac} \ \delta_k h_{bc} \ = \ h^{ac} \ (\partial_k \ + \ N_{kd} \ \partial^d) \ h_{bc} \ = \ h^{ac} \ N_{kd} \ \partial^d \ h_{bc} \ = \ 0 \qquad (43)$$

because the internal space metric $h_{ac}(p)$ (42) does *not* depend on x. As a reminder, $\partial_k = (\partial/\partial x^k)$; and $\partial^a = (\partial/\partial p_a)$.

A solution to (43) can be found by setting

$$N_{k0}(x) \neq 0, \quad N_{k1}(x) = N_{k2}(x) = N_{k3}(x) = 0 \tag{44}$$

since the metric $h_{ac}(p)$ (42) does *not* depend on the energy. One can further restrict the expression for $N_{k0}(x)$ by setting the remaining torsion R_{ija} in eq-(14) to zero when $N_{jk}(x,p) = N_{jk}(x)$

$$R_{ija} = \delta_j N_{ia}(x) - \delta_i N_{ja} = \partial_j N_{ia}(x) - \partial_i N_{ja}(x) = 0 \Rightarrow$$

$$N_{ia}(x) = \partial_i N_a(x), \quad N_{ja}(x) = \partial_j N_a(x) \Rightarrow \partial_{[i}\partial_{j]} N_a(x) = 0 \tag{45}$$

From eqs-(44,45) one learns that the nonholonomic functions $N_a(x)$ are

$$N_0(x) \neq 0, \quad N_1(x) = N_2(x) = N_3(x) = 0 \tag{46}$$

so that the nonvanishing nonlinear connection coefficients $N_{k0}(x) = \partial_k N_0(x)$ are given in terms of *one* nonholonomic function $N_0(x)$. The spherical symmetry requires $N_0(x) = N_0(r)$ for an arbitrary function of r. Therefore, the only nonvanishing nonlinear connection coefficient $N_{r0} = \partial_r N_0(r)$ is given in terms of one nonholonomic function $N_0(r)$. Thus, to conclude, the solutions (41,42,45,46) above yield *zero* torsion for *all* of the torsion components of eq-(14), and obey the field equations (23,24,25,32) when $\Phi(x,p)$ and $V(\Phi)$ are constants.

Concluding, the Sasaki-Finsler metric corresponding to the above solutions yields the infinitesimal interval

$$(d\omega)^2 = g_{ij}(x^k)\ dx^i\ dx^j - 2\ h^{EE}(p_c)\ N_{rE}(r)\ dr\ dE +$$

$$h^{EE}(p_c)\ N_{rE}(r)\ N_{rE}(r)\ (dr)^2 + h^{ab}(p_c)\ dp_a\ dp_b \tag{47}$$

One must note the presence of the key off-diagonal term in eq-(47) due to the nonlinear connection coefficient $\partial_r N_0(r) = N_{r0}(r) \equiv N_{rE}(r)$. It also modifies the spacetime metric via the extra term $h^{EE}(p_c)(N_{rE}(r))^2(dr)^2$. The base manifold $g_{ij}(x)$ and internal metric $h_{ab}(p)$ are (Anti) de Sitter-like as displayed in eqs-(41, 42). Thus, the cotangent bundle metric is parametrized by a family of arbitrary functions $N_0(r)$.

Our findings associated to the geometry of the cotangent bundle are different from those in [21] that were based on Finsler-like geometries where the zero torsion conditions were *not* imposed; the internal metric $h_{ab}(x,y) = h_{ab}(x)$ was chosen to be diagonal and independent of the internal fiber coordinates y; the nonholonomic function defined by $\partial_{y^a} N_0^a(x^i, y^a) = N_0(t)$ was specifically chosen to depend on time only, and to satisfy the general cosmological bounce conditions. At early times they found that one can acquire an exponential de Sitter solution.

Besides these differences we found exact (Anti) de Sitter solutions in eqs-(41,42). There are many different expressions to describe the de Sitter metric depending on the coordinates being used. A flat slicing of the four-dim de Sitter space is given by a FLRW metric with zero spatial curvature parameter

$$(ds)^2_x \ = \ - \ (dt)^2 \ + \ e^{2H_o t} \ ((dx)^2 \ + \ (dy)^2 \ + \ (dz)^2) \tag{48}$$

The internal de Sitter-like metric $h_{ab}(p)$ for the momentum variables (with $\hbar = c = G = 1$) is

$$(ds)^2_p \ = \ - \ (dE)^2 \ + \ e^{2H_o E} \ ((dp_x)^2 \ + \ (dp_y)^2 \ + \ (dp_z)^2) \tag{49}$$

where $\frac{4}{3}\kappa V_0 = \Lambda = 3H_o^2$ which follows from (40) when $d = 4$. Thus, instead of using expressions in eqs-(41,42) we could have written the metrics in the form provided by eqs-(48,49).

In this case, the nonvanishing nonlinear connection coefficient can be chosen to be $N_{tp_x}(t) = \partial_t N_{p_x}(t)$ where $N_{p_x}(t)$ is the nonholonomic function of time. The Sasaki-Finsler metric (1) will have an off-diagonal $-2h^{p_x p_x}(E) N_{tp_x}(t) dt dp_x$ term, and an additional $h^{p_x p_x}(E)(N_{tp_x}(t))^2 (dt)^2$ term which will $modify$ the underlying four-dim spacetime de Sitter metric (48). The energy dependence of $h^{p_x p_x}(E) = e^{-2H_o E}$ reflects the Lorentz-violating character of the kinematics in Finsler geometry. Such property is called *dynamic anisotropy*, and as such it has many relevant cosmological applications as described in [21]. At very large energies, say in the very early universe, $lim_{E\rightarrow\infty} \ h^{p_x p_x}(E) = e^{-2H_o E} \rightarrow 0$ (we choose $N_{tp_x}(t = 0) \neq \infty$) and one recovers the four-dim spacetime de Sitter metric (48). At very late stages, very low energies, we can have $N_{tp_x}(\infty) \rightarrow 0$, and once again one recovers a four-dim spacetime de Sitter metric (48). In both limits, the off-diagonal term $-2e^{-2H_o E} N_{tp_x}(t) dt dp_x \rightarrow 0$.

To conclude, choosing the function $N(t) = N_{tp_x}(t) = \partial_t N_{p_x}(t)$ judiciously, so that $N(t = 0) \neq \infty, \ N(t = \infty) = 0$, the most salient feature of the Sasaki-Finsler metric

$$\begin{aligned}(d\omega)^2 \ = \ & - \ (dt)^2 \ + \ e^{2H_o t} \ ((dx)^2 \ + \ (dy)^2 \ + \ (dz)^2) \ - \\ & (dE)^2 \ + \ e^{2H_o E} \ ((dp_x)^2 \ + \ (dp_y)^2 \ + \ (dp_z)^2) \ + \\ & e^{-2H_o E} \ N(t)^2 \ (dt)^2 \ - \ 2 \ e^{-2H_o E} \ N(t) \ dt \ dp_x \end{aligned} \tag{47'}$$

is that it captures both the very early inflationary and very-late-times de Sitter phases of the four-dim Universe.

The on-shell value of the 8-dim cotangent space (phase space) action for the solutions found in eqs-(41,42) when Φ and $V(\Phi)$ are constant, is

$$\mathcal{S} \ \sim \ \frac{1}{2\kappa} \ \kappa V_0 \ \int \ d^4x \ d^4p \ \sqrt{|det g_{ij}(x)|} \ \sqrt{|det h_{ab}(p)|} \ \sim \ V_0 \ \Omega^8 \tag{50}$$

where $\Omega^8 \equiv \Omega^4_x \ \Omega^4_p$ is the 8-dim phase space proper hyper-volume. Since the proper four-volumes of the de Sitter domains $diverge$, the action (50) diverges unless one takes the double-scaling limit $V_0 \rightarrow 0, \ \Omega^8 \rightarrow \infty$, such that the product $V_0 \ \Omega^8$ is finite. Thus a *regularization* of the phase space action (50) leads naturally to an extremely small effective cosmological constant $\Lambda_{eff} = \kappa V_0$, and which in turn, furnishes an extremely small value for the four-dim spacetime cosmological constant Λ as a direct result of eq-(40a). It is the *correlation* between Λ_{eff} and Λ displayed by eq-(40a), and resulting from the field equations, combined with the regularization procedure, which forces Λ to be extremely small. The latter regularization by itself is not enough.

One can estimate the extremely low value for V_0 after equating the momentum space integral in eq-(50) $V_0 \int d^4p \sqrt{|deth_{ab}(p)|} = V_0 \Omega_p^4$ to the observed vacuum energy density ρ_{vac} by setting the cutoff of the momentum four-volume domain Ω_p^4 to be M_P^4. Hence, if $V_0 M_P^4 = \rho_{vac} = (3/8\pi G R_H^2) \sim 10^{-120} M_P^4$, yields $V_0 \sim 10^{-120}$. The constant V_0 in eq-(50) is dimensionless in the system of units where $[px] = l^0$. And, as such, it *cannot* be equated to an energy density.

More precisely, one cannot use the value of the huge vacuum energy density $(\Lambda/8\pi G) = \rho_P = M_P^4$, obtained from the regularization of the zero point energy in QFT, to regularize the spacetime integral $\rho_P \int d^4x \sqrt{|detg_{ij}(x)|}$ since both factors are already huge. However, one can regularize the four-volume of the spacetime integral such that the product $\rho_{vac} \int d^4x \sqrt{|detg_{ij}(x)|} = \rho_{vac} \Omega_x^4$ is finite, since one factor (ρ_{vac}) is extremely small, and the other (Ω_x^4) is extremely large.

To sum up, it is the key dimensionless (and extremely small) factor of V_0 in $V_0 M_P^4 = \rho_{vac}$ originating from the 8-dim phase space action which makes all the difference. If the scalar field $\Phi(x,p) = \varphi(p)$ depends on the four-momentum only, V_0 can be set equal to the average value of a complicated oscillating potential $V(\varphi(p))$ whose magnitude ranges between $\pm M_P^4$, and given as $< V >= \Omega_p^{-4} \int d^4p \sqrt{|deth_{ab}(p)|} V(\varphi(p))$. In this scenario, the average $< V >= V_0$ can be close to zero.

Consequently, this key finding may cast some light into the resolution of the cosmological constant problem. Functional regularization group methods of the effective action (FRGE) are also very promising [23], [24], [25]. Instead of working on the geometry/gravity of phase space, they rely on the energy-momentum scale k dependence of the effective average action Γ_k to study the Wilsonian flow as a function of the scale k (a coordinate invariant $k = \sqrt{|k_\mu k^\mu|}$). In particular, the corrections to the classical scalar potential due to the quantum fluctuations lead to an effective scalar potential with a running cosmological constant which vanishes in the $t \to \infty$ limit [23]. Rainbow metrics in DSR [10] also depend on the energy, however this approach is very different from the Finsler geometry of the cotangent bundle.

To finalize we add some concluding remarks. To find other exact analytical solutions than those found in this work after setting Φ and $V(\Phi)$ to a constant is a daunting task. Solutions to the vacuum field equations in $2+2$ dimensions have been found by [20]. More general actions can be proposed. Like adding the squared and derivatives of torsion terms; curvature squared terms; $f(R), f(S)$ types of actions, etc ... Furthermore, if the metric g_{AB} has also *antisymmetric* components one may include the other curvature tensors (19,20) in the action.

Defining

$$R_{[jk]} \ = \ R^a_{bjk} \ \delta^b_a, \quad S^{[bc]} \ = \ S^{ibc}_j \ \delta^j_i \tag{51}$$

one may add the terms

$$g^{[jk]} \ R_{[jk]} \ + \ h_{[ab]} \ S^{[ab]} \tag{52}$$

to the action. Caution must be taken in inverting the metric because $g^{[ij]} \neq (g_{[ij]})^{-1}$, but instead it is a function of both g_{ij} *and* $g_{[ij]}$. Therefore, a variation with respect to the metric g_{AB} will $modify$ the original field equations (23,24,25), and it will add extra equations due to the variation of $\mathcal{S}_G + \mathcal{S}_M$ with respect to the antisymmetric components of the metric. Because the nonlinear connection does *not* transform as a tensor under local coordinate transformations of the base manifold [22], [21] one *cannot* use N_{ai} to contract the indices of the remaining two curvature tensors

$${}^{(1)}P^a_j \ = \ \delta^c_b \ P^{ab}_{cj}, \quad {}^{(2)}P^a_j \ = \ \delta^i_k \ P^{ak}_{ij} \tag{53}$$

as follows

$$g^{ij} \ N_{ai} \ (\ {}^{(1)}P^a_j \ + \ {}^{(2)}P^a_j \ ) \tag{54}$$

and include the terms of eq-(54) in the action.

A local transformation of the base spacetime manifold coordinates

$$x'^i \ = \ x'^i(x^0, x^1, \cdots, x^{d-1}), \quad det|| \frac{\partial x'^i}{\partial x^j} || \ \neq 0 \tag{55a}$$

leads to a transformation of the internal fiber momentum coordinates of the form

$$p'_a \ = \ \frac{\partial x^b}{\partial x'^a} \ p_b \ = \ \mathcal{M}^b_a \ p_b, \quad a, b = 0, 1, \cdots, d-1, \quad x^a \ = \ \delta^a_i \ x^i \tag{55b}$$

Since the elongated differential $\delta p_a = dp_a - N_{ai} dx^i$ must transform covariantly $\delta p'_a = \mathcal{M}^b_a \ \delta p_b$, one can deduce the inhomogeneous transformation property of the nonlinear connection

$$N'_{ai} \ = \ \frac{\partial x^b}{\partial x'^a} \ N_{bj} \ \frac{\partial x^j}{\partial x'^i} \ + \ \frac{\partial x'^b}{\partial x^c} \ \frac{\partial^2 x^c}{\partial x'^a \partial x'^i} \ p'_b \tag{56}$$

One should note that the transformations (55a,55b) are very $different$ from the most general coordinate transformations of a (curved) $2d$-dim manifold $Z'^A = Z'^A(Z^B), A, B = 1, 2, \cdots, 2d$ where the new coordinates are functions of *all* the original coordinates, and all the components of the metric tensor g_{AB} transform covariantly. This is what occurs in the local $U(1, d-1)$ transformations associated with Born's Reciprocal Relativity theory in phase spaces [7] that mix the spacetime coordinates with the energy-momentum ones.

A Born's Reciprocal complex gravitational theory (and its deformation) was constructed by [6] based on a $U(1,3)$ gauge theory formulation of complex gravity. Because the Weyl unitary trick allows to convert the pseudo unitary group

$U(1,3)$ into $U(2,2) = SU(2,2) \times U(1)$, and the latter $SU(2,2)$ is the conformal group in four-dimensions it is warranted to explore further Born's Reciprocal Relativity within the context of conformal gravity in $4D$. Having found (Anti) de Sitter solutions in this work within the framework of Finsler gravity in the cotangent bundle is very encouraging due to its appeal behind the AdS/CFT correspondence.

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