PROPERTIES OF QUADRATIC ANTICOMMUTATIVE HYPERCOMPLEX NUMBER SYSTEMS

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1. Abstract

Hypercomplex numbers are, roughly speaking, numbers of the form $x_1+i_1x_2++i_nx_{n+1}$ such that $x_1+i_1x_2++i_nx_{n+1}=y_1+i_1y_2++i_ny_{n+1}$ if and only if $x_j=y_j$ for all $j\in 1,2,,n$. I define a quadratic anticommutative hypercomplex numbers as hypercomplex numbers $x_1+i_1x_2++i_nx_{n+1}$ such that ${i_j}^2=p_j$ for all j (where p_j is a real number) and $i_ji_k=-i_ki_j$ for all k not equal to j. These numbers have some interesting properties. In particular, in this paper I prove a generalized form of the Demoivres formula for these numbers, and determine certain conditions required for a function on a Quadratic Anticommutative Hypercomplex plane to be analytic; including generalizations of the Cauchy-Riemann equations.

2. Introduction

In this paper I demonstrate some of the properties of a kind of hypercomplex number systems that I will refer to as Quadratic Anticommutative Hypercomplex number systems. These number systems are defined by two specific properties, namely, for an arbitrary number a_0 + a_1i_1 + ...+a_ni_n in a Quadratic Anticommutative Hypercomplex number system, $i_j^2\in R$ for all j and $k_jk_k=-i_ki_j$ for all $k\neq j$. These numbers have some nice properties, the most important one being that $|a_0+a_1i_1+...+a_ni_n|^2=a_0^2-p_1a_1^2-...-p_{n-1}a_{n-1}^2$, which is helpful in simplifying many calculations. On the other hand, the fact that these numbers are anticommutative results in them having some fascinating and unusual properties. Although Quadratic Anticommutative Hypercomplex number systems are, as far as I am aware, an unstudied catagory of hypercomplex number systems, some specific Quadratic

Anticommutative Hypercomplex number systems are well-known, such as the quaternions, the complex numbers, and the split-complex numbers. Indeed, I discovered Quadratic Anticommutative Hypercomplex number systems as a way to generalize some of the results Harkin and Harkin [HH04] described in their examination of generalized complex numbers to an arbitrary number of dimensions. Here, the generalized complex numbers are simply Quadratic Anticommutative Hypercomplex numbers of dimension 2.

I will describe two main results, one being a generalization of Demoivre's formula, the other being a result related to differentiation involving generalized Cauchy-Riemann equations.

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3. Definitions.

We will start with the necessary definitions. The first definition is based on the definition for hypercomplex number systems given in [KS].

Def:HCNS **Definition 3.1.** A **hypercomplex number system** of dimension $n+1$ is defined as a set of numbers of the form $a_0 + a_1 i_1 + ... + a_n i_n$, where $a_x \in \mathbb{R}$ for all $x \in 0, 1, ..., n$, and $a_0 + a_1 i_1 + ... + a_n i_n = b_0 + b_1 i_1 + ... + b_n i_n$ if and only if $a_x = b_x$ for all $x \in 0, 1, 2, ..., n$ [?]. Addition and subtraction are defined on these numbers as:

$$(a_0 + a_1 i_1 + ... + a_n i_n) + (b_0 + b_1 i_1 + ... + b_n i_n) = (a_0 + b_0) + (a_1 + b_1) i_1 + ... + (a_n + b_n) i_n$$

and

$$(a_0 + a_1 i_1 + ... + a_n i_n) - (b_0 + b_1 i_1 + ... + b_n i_n) = (a_0 - b_0) + (a_1 - b_1) i_1 + ... + (a_n - b_n) i_n,$$

respectively. Multiplication follows the distributive law and the rule $(a_x i_x) * (b_y i_y) = a_x b_y * (i_x i_y)$ and is defined by a multiplication table of n rows and n columns, where each for each $c, d \in 0, 1, 2, ..., n$:

$i_c i_d = p_{cd,0} + p_{cd,1} i_1 + ... + p_{cd,n} i_n$, where $p_{cd,x} \in \mathbb{R}$ for all $x \in 0, 1, 2, ..., n$.

In other words, the product of any two imaginary units is a hypercomplex number. For the complex numbers, the multiplication table has only one entry, $i_0 i_0 = -1$. The quaternions are a better example, their multiplication table is:

-1	i_1	$-i_2$
$-i_1$	-1	i_0
i_2	$-i_0$	-1

The second definition is based on the definition for $cosp(\phi)$ and $sinp(\phi)$ given in [HH04].

Def:CPSP **Definition 3.2.** Let $p \in \mathbb{R}$. Then $cosp(\phi)$ and $sinp(\phi)$ be defined as follows:

If $p < 0$, then $cosp(\phi) = cos(\phi\sqrt{-p})$ and $sinp(\phi) = \frac{1}{\sqrt{-p}} sin(\phi\sqrt{-p}$

If $p = 0$, then $cosp(\phi) = 1$ and $sinp(\phi) = \phi$.

If $p > 0$, then $cosp(\phi) = cosh(\phi\sqrt{p})$ and $sinp(\phi) = \frac{1}{\sqrt{p}} sinh(\phi\sqrt{p})$.

ef:modulus **Definition 3.3.** Let $x = a_0 + a_1 i_1 + ... + a_n i_n$ be an element of a hypercomplex number system of dimension n+1. Then, the **conjugate** of x is defined as $\bar{x} = a_0$ - $a_1 i_1$ - ...-$a_n i_n$. The **modulus** of x is defined as $|x| = +\sqrt{x\bar{x}}$.

The next two definitions are my own. The first is simply for convenience, as it is easier to name the type of hypercomplex number systems I am dealing with in this paper than to restate their properties in every theorem and lemma.

Def:QAHNS **Definition 3.4.** A Hypercomplex Number System of dimension n is a **Quadratic Anticommutative Hypercomplex Number System** if for all $x, y \in 1, 2, ..., n-1$ where $x \neq y$, $i_x i_y = -i_y i_x$ and $i_x^2 = p_x$ where $p_x \in \mathbb{R}$, for all $x \in 1, 2, ..., n-1$. A number in this system is referred to as a Quadratic Anticommutative Hypercomplex Number.

Because Quadratic Anticommutative Hypercomplex numbers are not always commutative, $\frac{b\bar{a}}{|a|^2}$ does not always equal $\frac{\bar{a}b}{|a|^2}$. Thus, we must define left and right fractions seperately.

Def:RLF **Definition 3.5.** The *right fraction* of the hypercomplex number a over the hypercomplex number b, notated $\frac{a}{b}\triangleright$, is defined as $\frac{a}{b}\triangleright = \frac{a\bar{b}}{|b|^2}$.

The *left fraction* of the hypercomplex number a over the hypercomplex number b, notated $\triangleleft\frac{a}{b}$, is defined as $\triangleleft\frac{a}{b} = \frac{\bar{b}a}{|b|^2}$

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4. Associativity of Addition and the Modulus of Quadratic Anticommutative Hypercomplex numbers

There are two theorems that will be useful throughout this paper, so I will prove them first. The first of these is not a theorem I developed myself; rather it is stated to be fact in one of my sources KS [KS].

Lem:AA **Theorem 4.1.** *Associativity of addition holds for all hypercomplex number systems.*

Proof. Let $a = a_0 + a_1 i_1 + ... + a_n i_n$ and $b = b_0 + b_1 i_1 + ... + b_n i_n$ be hypercomplex numbers of a hypercomplex number system C_{n+1}. Then, by the definition of addition on hypercomplex numbers, we have:

$$((a_0 + a_1 i_1 + ... + a_n i_n) + (b_0 + b_1 i_1 + ... + b_n i_n)) + (c_0 + c_1 i_1 + ... + c_n i_n)$$

$$= (a_0 + b_0) + (a_1 + b_1)i_1 + ... + (a_n + b_n)i_n + (c_0 + c_1 i_1 + ... + c_n i_n)$$

$$= (a_0 + b_0 + c_0) + (a_1 + b_1 + c_1)i_1 + ... + (a_n + b_n + c_n)i_n \tag{4.1}$$

$$= (a_0 + (b_0 + c_0)) + (a_1 + (b_1 + c_1))i_1 + ... + (a_n + (b_n + c_n))i_n$$

$$= (a_0 + a_1 i_1 + ... + a_n i_n) + ((b_0 + b_1 i_1 + ... + b_n i_n) + (c_0 + c_1 i_1 + ... + c_n i_n))$$

□

Note that the Associativity of *multiplication* does not necessarily hold for hypercomplex number systems, and it does not necessarily hold for the subset of hypercomplex number systems we are referring to as Quadratic Anticommutative Hypercomplex number systems either. However, because the coefficients of Quadratic Anticommutative Hypercomplex numbers are real, and thus associative under multiplication, this is not so great an obstacle in proofs as one might assume.

The second theorem, which allows one to quickly determine the modulus of any Quadratic Anticommutative Hypercomplex number, and is my own–along with all the theorems following this one.

MQAH **Theorem 4.2.** *Let* C_n *be a quadratic anticommutative hypercomplex number system of dimension* n *with constants* $p_1, p_2, ..., p_{n-1}$. *Then* $|z|^2 = a_0^2 - p_1a_1^2 - ... - p_{n-1}a_{n-1}^2$ *for all* $z \in C_n$, *where* $z = a_0 + a_1i_1 + ... + a_{n-1}i_{n-1}$.

Proof. We will prove this theorem through induction. The base case is simply the generalized complex numbers. The modulus of a given generalized complex number z $= a_0 + a_1i_1$ of a system $C_{2,p}$ is stated in HH [HH04] to be $|z| = \sqrt{a_0^2 - pa_1^2}$. This is not proven in the original paper, but it is easy to do so using the rules given for addition and multiplication of hypercomplex numbers and the identity $z\bar{z} = |z|^2$:

$$|a_0 + a_0i_1|^2 = (a_0 + a_1i_1)(a_0 - a_1i_1) = a_0^2 - a_0a_1i_1 + a_1a_0i_1 - a_1^2i_1^2 = a_0^2 - a_1^2p = a_0^2 - p_0a_1^2$$

Now, assume that for any system C_n for which $i_xi_y =$ -i_yi_x and $i_x^2 = p_x$ for all $x, y \in 1, ..., n-1$ and $p_x \in \mathbb{R}$ for all $x \in 1, ..., n-1$, $|z|^2 = a_0^2 - p_1a_1^2 - ... - p_{n-1}a_{n-1}^2$ for all $z \in C_n$, where $z = a_0 + a_1i_1 + ... + a_{n-1}i_{n-1}$. Then, consider a system C_{n+1} for which i_xi_y = -i_yi_x and $i_x^2 = p_x$ for all $x, y \in 1, ..., n$ and $p_x \in \mathbb{R}$ for all $x \in 1, ..., n$. Again using the identity $z\bar{z} = |z|$, we have:

$|z| = z\bar{z} = (a_0 + a_1i_1 + ... + a_{n-1}i_{n-1} + a_ni_n)(a_0 - a_1i_1 - ... - a_{n-1}i_{n-1} - a_ni_n)$

By Lemma 1.1, associativity of addition holds on all hypercomplex number systems, so:

$$(a_0 + a_1i_1 + ... + a_{n-1}i_{n-1} + a_ni_n)(a_0 - a_1i_1 - ... - a_{n-1}i_{n-1} - a_ni_n)$$

$$= ((a_0 + a_1i_1 + ... + a_{n-1}i_{n-1}) + a_ni_n)((a_0 - a_1i_1 - ... - a_{n-1}i_{n-1}) - a_ni_n)$$

$$= (a_0 + a_1i_1 + ... + a_{n-1}i_{n-1})(a_0 - a_1i_1 - ... - a_{n-1}i_{n-1}) - (a_0 + a_1i_1 + ... + a_{n-1}i_{n-1})a_ni_n$$

$$+a_ni_n(a_0 - a_1i_1 - ... - a_{n-1}i_{n-1}) - a_n^2i_n^2$$

But we know from our inductive hypothesis that:

$$(a_0 + a_1 i_1 + ... + a_{n-1} i_{n-1})(a_0 - a_1 i_1 - ... - a_{n-1} i_{n-1}) = a_0^2 - p_1 a_1^2 - ... - p_{n-1} a_{n-1}^2,$$

so we have:

$$\begin{aligned}
&(a_0 + a_1 i_1 + ... + a_{n-1} i_{n-1})(a_0 - a_1 i_1 - ... - a_{n-1} i_{n-1}) - (a_0 + a_1 i_1 + ... + a_{n-1} i_{n-1}) a_n i_n \\
&+ a_n i_n (a_0 - a_1 i_1 - ... - a_{n-1} i_{n-1}) - a_n^2 i_n^2 \\
&= a_0^2 - p_1 a_1^2 - ... - p_{n-1} a_{n-1}^2 - (a_0 + a_1 i_1 + ... + a_{n-1} i_{n-1}) a_n i_n + a_n i_n (a_0 - a_1 i_1 - ... - a_{n-1} i_{n-1}) - a_n^2 i_n^2 \\
&= a_0^2 - p_1 a_1^2 - ... - p_{n-1} a_{n-1}^2 - a_n^2 p_n \\
&= a_0^2 - p_1 a_1^2 - ... - p_{n-1} a_{n-1}^2 - p_n a_n^2
\end{aligned}$$

□

Now, we can begin the main material of this paper. In order to prove the generalized Demoivre's formula for quadratic anticommutative hypercomplex number systems, we will have to prove a more generalized form of the angle addition formula, and derive the coordinate system that we will need in the proof.

5. The Generalized Angle Addition Formula

Proposition 5.1. *I will prove that*

$$\text{sinp}(a+b) = \text{sinp}(a)\text{cosp}(b) + \text{sinp}(b)\text{cosp}(a) \tag{5.1}$$

and

$$\text{cosp}(a+b) = \text{cosp}(a)\text{cosp}(b) + p\,\text{sinp}(a)\text{sinp}(b) \tag{5.2}$$

for all $p, a, b \in \mathbb{R}$.

Proof. We have three cases.

Case 1: assume $p < 0$. By the angle addition formula, $sin(a+b) = sin(a)cos(b)+cos(a)sin(b)$ and $cos(a + b) = cos(a)cos(b) - sin(a)sin(b)$. Thus:

$$
\begin{aligned}
&\operatorname{sinp}(a)\operatorname{cosp}(b) + \operatorname{sinp}(b)\operatorname{cosp}(a) \\
&= \frac{1}{\sqrt{-p}} \sin(a\sqrt{-p})cos(b\sqrt{-p}) + \frac{1}{\sqrt{-p}} \sin(b\sqrt{-p}) \cos(a\sqrt{-p}) \\
&= \frac{1}{\sqrt{-p}} \left[\sin(a\sqrt{-p}) \cos(b\sqrt{-p}) + \sin(b\sqrt{-p}) \cos(a\sqrt{-p})\right] \\
&= \frac{1}{\sqrt{-p}} \sin((a+b)\sqrt{-p}) \\
&= \operatorname{sinp}(a+b)
\end{aligned} \tag{5.3}
$$

and

(5.4)

$$
\begin{aligned}
cosp(a)cosp(b) + psinp(a)sinp(b) &= cos(a\sqrt{-p})cos(b\sqrt{-p}) + p\frac{1}{\sqrt{-p}}sin(a\sqrt{-p})\frac{1}{\sqrt{-p}}sin(b\sqrt{-p}) \\
&= cos(a\sqrt{-p})cos(b\sqrt{-p}) + p\frac{1}{(-p)}sin(a\sqrt{-p})sin(b\sqrt{-p}) \\
&= cos(a\sqrt{-p})cos(b\sqrt{-p}) - sin(a\sqrt{-p})sin(b\sqrt{-p}) \\
&= cos((a+b)\sqrt{-p}) \\
&= cosp(a+b)
\end{aligned}
$$

Case 2: Assume $p = 0$. Then:

$sinp(a)cosp(b) + sinp(b)cosp(a) = a + b = sinp(a + b)$

and

$cosp(a)cosp(b) + psinp(a)sinp(b) = cosp(a)cosp(b) = 1 = cosp(a + b)$

Case 3: Assume $p > 0$. By definition,

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

and

$$\cosh(x) = \frac{e^x + e^{-x}}{2}.$$

So:

(5.5)

$$\begin{aligned}
\sinh(a)\cosh(b) + \sinh(b)\cosh(a) &= \frac{(e^a - e^{-a})(e^b + e^{-b})}{4} + \frac{(e^b - e^{-b})(e^a + e^{-a})}{4} \\
&= \frac{e^{a+b} + e^{a-b} - e^{b-a} - e^{-a-b}}{4} + \frac{e^{b+a} + e^{b-a} - e^{-b+a} - e^{-b-a}}{4} \\
&= \frac{e^{a+b} + e^{-a-ab}}{2} \\
&= \sinh(a+b)
\end{aligned}$$

and

(5.6)

$$\begin{aligned}
\cosh(a)\cosh(b) + \sinh(a)\sinh(b) &= \frac{(e^a + e^{-a})(e^b + e^{-b})}{4} + \frac{(e^a - e^{-a})(e^b - e^{-b})}{4} \\
&= \frac{e^{a+b} + e^{a-b} + e^{-a-b} + e^{-a-b}}{4} + \frac{e^{a+b} - e^{a-b} - e^{-a+b} + e^{-a-b}}{4} \\
&= \frac{e^{a+b} + e^{-a-b}}{2} \\
&= \cosh(a+b)
\end{aligned}$$

Thus:

(5.7)

$$\begin{aligned} sinp(a)cosp(b) + sinp(b)cosp(a) &= \frac{1}{\sqrt{p}}sinh(a\sqrt{p})cosh(b\sqrt{p}) + \frac{1}{\sqrt{p}}sinh(b\sqrt{p})cosh(a\sqrt{p}) \\ &= \frac{1}{\sqrt{p}}[sinh(a\sqrt{p})cosh(b\sqrt{p}) + sinh(b\sqrt{p})cosh(a\sqrt{p})] \\ &= \frac{1}{\sqrt{p}}sinh((a+b)\sqrt{p}) \\ &= sinp(a+b) \end{aligned}$$

and

(5.8)

$$\begin{aligned} cosp(a)cosp(b) + psinp(a)sinp(b) &= cosh(a\sqrt{p})cosh(b\sqrt{p}) + p\frac{1}{\sqrt{p}}sinh(a\sqrt{p})\frac{1}{\sqrt{p}}sinh(b\sqrt{p}) \\ &= cosh(a\sqrt{p})cosh(b\sqrt{p}) + p\frac{1}{p}sinh(a\sqrt{p})sinh(b\sqrt{p}) \\ &= cosh(a\sqrt{p})cosh(b\sqrt{p}) + sinh(a\sqrt{p})sinh(b\sqrt{p}) \\ &= cosh((a+b)\sqrt{p}) \\ &= cosp(a+b) \end{aligned}$$

□

6. Derivation of Coordinates

Proposition 6.1. *Let* $u = x_1 + i_1x_2 + i_2x_3 + ...i_nx_{n+1}$ *be a quadratic anticommutative hypercomplex number such that* $x_j \in \mathbb{R}$ *and* $i_j^2 = p_j \neq 0$ *for all* $j \in 1, 2, ..., n+1$*. Then:*

$x_1 = rcosp_1(\phi_1)$

$x_2 = rsinp_1(\phi_1)cos\frac{-p_2}{p_1}(\phi_2)$

$x_3 = rsinp_1(\phi_1)sin\frac{-p_2}{p_1}(\phi_2)cos\frac{-p_3}{p_2}(\phi_3)$

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$x_n = rsinp_1(\phi_1)sin\frac{-p_2}{p_1}(\phi)...sin\frac{-p_{n-1}}{p_{n-2}}(\phi_{n-1})cos\frac{-p_n}{p_{n-1}}(\phi_n)$

$x_{n+1} = rsinp_1(\phi_1)sin\frac{-p_2}{p_1}(\phi)...sin\frac{-p_{n-1}}{p_{n-2}}(\phi_{n-1})sin\frac{-p_n}{p_{n-1}}(\phi_n)$

for some $r \in \mathbb{R}$*,* $\phi_1, \phi_2, ..., \phi_n \in \mathbb{R}$

Proof. By Theorem 4.2, for a given

$u = x_1 + i_1x_2 + i_2x_3 + ...i_nx_{n+1}$,

$r^2 = x_1^2 - p_1x_2^2 - p_2x_3^2 - ... - p_nx_{n+1}^2$, where $r^2 \in \mathbb{R}$.

Now, because each x_j is real, we can write each x_j as a multiple of any nonzero real number. Thus, we can write:

$x_1 = rz_1, x_2 = ra_1z_2, x_3 = ra_1a_2z_3, ..., x_n = ra_1a_2...a_{n-1}z_n, x_{n+1} = ra_1a_2...a_{n-1}a_n$,

where each a_j and z_j is some real number. We can then write u as:

$u = rz_1 + i_1ra_1z_2 + i_2a_1a_2z_3 + ... + i_{n-1}ra_1a_2...a_{n-1}z_n + i_nra_1a_2...a_{n-1}a_n$,

where $x_j, z_j \in \mathbb{R}$ for all j.

First, I will show that $z_1 = cosp_1(\phi_1)$ and $a_1 = sinp_1(\phi_1)$. Let $n = 1$. Then:

$r^2 = r^2z_1^2 - r^2p_1a_1^2$

$1 = z_1^2 - p_1a_1^2$

This is the form of a point on the unit circle for a the generalized complex number system corresponding to p_1 by Theorem 4.2 (a unit circle is defined by $|z|^2 = 1$). Thus, $z_1 = cosp_1(\phi_1)$ and $a_1 = sinp_1(\phi_1)$ for some ϕ_1.

Now, I will show that $z_j = cos\frac{-p_j}{p_{j-1}}(\phi_j)$ and $a_j = sin\frac{-p_j}{p_{j-1}}(\phi_j)$ for all $1 < j < n$.

$r^2 = r^2z_1^2 - r^2p_1a_1^2z_2^2 - r^2p_2a_1^2a_2^2z_3^2 - ... - r^2p_{n-1}a_1^2a_2^2...a_{n-1}^2z_n^2 - r^2p_na_1^2a_2^2...a_n^2z_{n+1}^2 - r^2p_{n+1}a_1^2a_2^2...a_{n+1}^2$

$= r^2z_1^2 - r^2p_1a_1^2z_2^2 - r^2p_2a_1^2a_2^2z_3^2 - ... - r^2p_{n-1}a_1^2a_2^2...a_{n-1}^2z_n^2 - r^2p_nz_1^2a_2^2...a_{n-1}^2a_n^2 + r^2p_nz_1^2a_2^2...a_{n-1}^2a_n^2 - r^2p_na_1^2a_2^2...a_n^2z_{n+1}^2 - r^2p_{n+1}a_1^2a_2^2...a_{n+1}^2$

$= r^2 + r^2p_nz_1^2a_2^2...a_{n-1}^2a_n^2 - r^2p_na_1^2a_2^2...a_n^2z_{n+1}^2 - r^2p_{n+1}a_1^2a_2^2...a_{n+1}^2$

Thus:

$$\begin{aligned} r^2 &= r^2 + r^2p_nz_1^2a_2^2...a_{n-1}^2a_n^2 - r^2p_na_1^2a_2^2...a_n^2z_{n+1}^2 - r^2p_{n+1}a_1^2a_2^2...a_{n+1}^2 \\ -r^2p_nz_1^2a_2^2...a_{n-1}^2a_n^2 &= -r^2p_na_1^2a_2^2...a_n^2z_{n+1}^2 - r^2p_{n+1}a_1^2a_2^2...a_{n+1}^2 \\ 1 &= z_{n+1}^2 + \frac{p_{n+1}}{p_n}a_{n+1}^2 \\ 1 &= z_{n+1}^2 - \frac{-p_{n+1}}{p_n}a_{n+1}^2 \end{aligned} \tag{6.1}$$

By Theorem MQAH 4.2 This is the unit circle for the system of generalized complex numbers with $p = \frac{-p_{n+1}}{p_n}$. Thus, $z_{n+1} = cos\frac{-p_{n+1}}{p_n}(\phi_{n+1})$, $a_{n+1} = sin\frac{-p_{n+1}}{p_n}(\phi_{n+1})$. If we plug these into the expressions for each x_j, we get:

$x_1 = rcosp_1(\phi_1)$

$x_2 = rsinp_1(\phi_1)cos\frac{-p_2}{p_1}(\phi_2)$

$x_3 = rsinp_1(\phi_1)sin\frac{-p_2}{p_1}(\phi_2)cos\frac{-p_3}{p_2}(\phi_3)$

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.

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$x_n = rsinp_1(\phi_1)sin\frac{-p_2}{p_1}(\phi_2)...sin\frac{-p_{n-1}}{p_{n-2}}(\phi_{n-1})cos\frac{-p_n}{p_{n-1}}(\phi_n)$

$x_{n+1} = rsinp_1(\phi_1)sin\frac{-p_2}{p_1}(\phi_2)...sin\frac{-p_{n-1}}{p_{n-2}}(\phi_{n-1})sin\frac{-p_n}{p_{n-1}}(\phi_n)$

□

Thus, we can use these coordinates to write any quadratic anticommutative hypercomplex number as:

$y = rcosp_1(\phi_1) + i_1 rsinp_1(\phi_1)cos\frac{-p_2}{p_1}(\phi_2) + i_2 rsinp_1(\phi_1)sin\frac{-p_2}{p_1}(\phi_2)cos\frac{-p_3}{p_2}(\phi_3)$
$+ ... + i_{n-1} rsinp_1(\phi_1)sin\frac{-p_2}{p_1}(\phi_2)...sin\frac{-p_{n-1}}{p_{n-2}}(\phi_{n-1})cos\frac{-p_n}{p_{n-1}}(\phi_n)$
$+ i_n rsinp_1(\phi_1)sin\frac{-p_2}{p_1}(\phi_1)...sin\frac{-p_n}{p_{n-1}}(\phi_n)$
We can use summation and product notation to write this more concisely:
$$y = r[cosp_1(\phi_1)+sinp_1(\phi_1)\sum_{k=1}^{n-1} i_k[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1})+i_n sinp_1(\phi_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)]$$

Example 6.2. *In the case of a quadratic anticommutative hypercomplex number system where $p_x^2 = -1$ for all $x \in 1, ..., n$, these coordinates are similar to the n-spherical coordinates, with the only difference being the multiplication by r.*

7. Generalized De Moivre's Formula for Quadratic Anticommutative Hypercomplex Number Systems

First, I will prove a lemma which I will need to use in the proof of the Generalized De Moivre's formula.

Lem:Mod2 **Lemma 7.1.** *Let $y = x_1 + i_1 x_2 + ... + i_n x_{n+1}$ be a number in a quadratic anticommutative hypercomplex number system of degree $n+1$, $n \geq 1$, and assume $p_k = i_k^2 \neq 0$ for all $k \in 0, 1, ..., n$. Then, we can write y in the coordinate system described above:*
$$y = r[cosp_1(\phi_1)+sinp_1(\phi_1)\sum_{k=1}^{n-1} i_k[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1})+i_n sinp_1(\phi_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)]$$
If we define x as:
$$x = \sum_{k=1}^{n-1} i_k[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1}) + i_n \prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)$$
Then $x^2 = p_1$.

Proof. I will prove this through induction.

Base Case: If $n = 1$, then:

$$x = i_1 cos\frac{-p_2}{p_1}(\phi_2) + i_2 sin\frac{-p_2}{p_1}(\phi_2)cos\frac{-p_3}{p_2}(\phi_3)$$

and

$$x^2 = [i_1 cos\frac{-p_2}{p_1}(\phi_2) + i_2 sin\frac{-p_2}{p_1}(\phi_2) cos\frac{-p_3}{p_2}(\phi_3)]^2$$
$$= p_1 cos\frac{-p_2}{p_1}(\phi_2)^2 + p_2 sin\frac{-p_2}{p_1}(\phi_2)^2 cos\frac{-p_3}{p_2}(\phi_3)^2$$
$$= p_1[cos\frac{-p_2}{p_1}(\phi_2)^2 - \frac{-p_2}{p_1} sin\frac{-p_2}{p_1}(\phi_2)^2 cos\frac{-p_3}{p_2}(\phi_3)^2]$$
$$= p_1$$

Inductive Step: Assume that
$x_n^2 = [\sum_{k=1}^{n-1} i_k[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1}) + i_n \prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)]^2 = p_1$
(this is our Inductive Hypothesis) Then:

(7.1)

$$x_{n+1}^2 = [\sum_{k=1}^{n} i_k[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1}) + i_{n+1}\prod_{j=2}^{n+1} sin\frac{-p_j}{p_{j-1}}(\phi_j)]^2$$
$$= [\sum_{k=1}^{n-1} i_k[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1}) + i_n\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j) - i_n\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)$$
$$+ i_n\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)cos\frac{-p_{n+1}}{p_n}(\phi_{n+1}) + i_{n+1}\prod_{j=2}^{n+1} sin\frac{-p_j}{p_{j-1}}(\phi_j)]^2$$
$$= [x_n - i_n\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j) + i_n\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)cos\frac{-p_{n+1}}{p_n}(\phi_{n+1}) + i_{n+1}\prod_{j=2}^{n+1} sin\frac{-p_j}{p_{j-1}}(\phi_j)]^2$$
$$= [x_n + \prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)[i_n(-1 + cos\frac{-p_{n+1}}{p_n}(\phi_n)) + i_{n+1} sin\frac{-p_{n+1}}{p_n}(\phi_{n+1})]]^2$$
$$= x_n^2 + x_n\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)[i_n(-1 + cos\frac{-p_{n+1}}{p_n}(\phi_n)) + i_{n+1} sin\frac{-p_{n+1}}{p_n}(\phi_{n+1})]$$
$$+ \prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)[i_n(-1 + cos\frac{-p_{n+1}}{p_n}(\phi_n)) + i_{n+1} sin\frac{-p_{n+1}}{p_n}(\phi_{n+1})]x_n$$
$$+ [\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)]^2[i_n(-1 + cos\frac{-p_{n+1}}{p_n}(\phi_n)) + i_{n+1} sin\frac{-p_{n+1}}{p_n}(\phi_{n+1})]^2$$

(7.2)

$$
\begin{aligned}
&= x_n^2 + x_n \prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)[i_n(-1 + cos\frac{-p_{n+1}}{p_n}(\phi_n)) + i_{n+1}sin\frac{-p_{n+1}}{p_n}(\phi_{n+1})] \\
&\quad + \prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)[i_n(-1 + cos\frac{-p_{n+1}}{p_n}(\phi_n)) + i_{n+1}sin\frac{-p_{n+1}}{p_n}(\phi_{n+1})]x_n \\
&\quad + \prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)^2[p_n(-1 + cos\frac{-p_{n+1}}{p_n}(\phi_n))^2 \\
&\quad + i_n i_{n+1}(-1 + cos\frac{-p_{n+1}}{p_n}(\phi_n))sin\frac{-p_{n+1}}{p_n}(\phi_{n+1}) \\
&\quad + i_{n+1} i_n(-1 + cos\frac{-p_{n+1}}{p_n}(\phi_n))sin\frac{-p_{n+1}}{p_n}(\phi_{n+1}) + p_{n+1}sin\frac{-p_{n+1}}{p_n}(\phi_{n+1})^2] \\
&= x_n^2 + x_n \prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)[i_n(-1 + cos\frac{-p_{n+1}}{p_n}(\phi_n)) + i_{n+1}sin\frac{-p_{n+1}}{p_n}(\phi_{n+1})] \\
&\quad + \prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)[i_n(-1 + cos\frac{-p_{n+1}}{p_n}(\phi_n)) + i_{n+1}sin\frac{-p_{n+1}}{p_n}(\phi_{n+1})]x_n \\
&\quad + \prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)^2[p_n(-1 + cos\frac{-p_{n+1}}{p_n}(\phi_n))^2 + p_{n+1}sin\frac{-p_{n+1}}{p_n}(\phi_{n+1})^2] \\
&= x_n^2 + x_n \prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)[i_n(-1 + cos\frac{-p_{n+1}}{p_n}(\phi_n)) + i_{n+1}sin\frac{-p_{n+1}}{p_n}(\phi_{n+1})] \\
&\quad + \prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)[i_n(-1 + cos\frac{-p_{n+1}}{p_n}(\phi_n)) + i_{n+1}sin\frac{-p_{n+1}}{p_n}(\phi_{n+1})]x_n \\
&\quad + \prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)^2[p_n(1 - 2cos\frac{-p_{n+1}}{p_n}(\phi_n) + cos\frac{-p_{n+1}}{p_n}(\phi_n)^2) + p_{n+1}sin\frac{-p_{n+1}}{p_n}(\phi_{n+1})^2] \\
&= x_n^2 + x_n \prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)[i_n(-1 + cos\frac{-p_{n+1}}{p_n}(\phi_n)) + i_{n+1}sin\frac{-p_{n+1}}{p_n}(\phi_{n+1})] \\
&\quad + \prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)[i_n(-1 + cos\frac{-p_{n+1}}{p_n}(\phi_n)) + i_{n+1}sin\frac{-p_{n+1}}{p_n}(\phi_{n+1})]x_n \\
&\quad + \prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)^2[p_n(1 - 2cos\frac{-p_{n+1}}{p_n}(\phi_n)) + p_n[cos\frac{-p_{n+1}}{p_n}(\phi_n)^2 - \frac{-p_{n+1}}{p_n}sin\frac{-p_{n+1}}{p_n}(\phi_{n+1})^2)]] \\
&= x_n^2 + x_n \prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)[i_n(-1 + cos\frac{-p_{n+1}}{p_n}(\phi_n)) + i_{n+1}sin\frac{-p_{n+1}}{p_n}(\phi_{n+1})] \\
&\quad + \prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)[i_n(-1 + cos\frac{-p_{n+1}}{p_n}(\phi_n)) + i_{n+1}sin\frac{-p_{n+1}}{p_n}(\phi_{n+1})]x_n \\
&\quad + \prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)^2[p_n(1 - 2cos\frac{-p_{n+1}}{p_n}(\phi_n)) + p_n]
\end{aligned}
$$

$$= x_n^2 + x_n \prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)[i_n(-1 + cos\frac{-p_{n+1}}{p_n}(\phi_n)) + i_{n+1}sin\frac{-p_{n+1}}{p_n}(\phi_{n+1})]$$

$$+ \prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)[i_n(-1 + cos\frac{-p_{n+1}}{p_n}(\phi_n)) + i_{n+1}sin\frac{-p_{n+1}}{p_n}(\phi_{n+1})]x_n \tag{7.3}$$

$$+ \prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)^2[2p_n - 2p_n cos\frac{-p_{n+1}}{p_n}(\phi_n)]$$

Now, because $i_x i_y = -i_y i_x$ for all $x, y \in 1, 2, ..., n$, $x \neq y$, the only term in the expansion of the expression: $x_n \prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)[i_n(-1 + cos\frac{-p_{n+1}}{p_n}(\phi_n)) + i_{n+1}sin\frac{-p_{n+1}}{p_n}(\phi_{n+1})]$ $+ \prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)[i_n(-1 + cos\frac{-p_{n+1}}{p_n}(\phi_n)) + i_{n+1}sin\frac{-p_{n+1}}{p_n}(\phi_{n+1})]x_n$ that doesn't cancel is the $i_n^2 = p_n$ term: $2p_n \prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)^2(-1 + cos\frac{-p_{n+1}}{p_n}(\phi_n))$ Thus:

$$x_n^2 \quad + x_n \prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)[i_n(-1 + cos\frac{-p_{n+1}}{p_n}(\phi_n)) + i_{n+1}sin\frac{-p_{n+1}}{p_n}(\phi_{n+1})]$$

$$+ \prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)[i_n(-1 + cos\frac{-p_{n+1}}{p_n}(\phi_n)) + i_{n+1}sin\frac{-p_{n+1}}{p_n}(\phi_{n+1})]x_n$$

$$+ \prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)^2[2p_n - 2p_n cos\frac{-p_{n+1}}{p_n}(\phi_n)] \tag{7.4}$$

$$= x_n^2 + 2p_n \prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)^2(-1 + cos\frac{-p_{n+1}}{p_n}(\phi_n))$$

$$+ \prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)^2[2p_n - 2p_n cos\frac{-p_{n+1}}{p_n}(\phi_n)]$$

$$= x_n^2$$

And $x_n^2 = p_n$ by the inductive hypothesis. □

Now, I will prove the main theorem

Theorem 7.2. *Let $y = x_1 + i_1x_2 + ... + i_nx_{n+1}$ be a number in a quadratic anticommutative hypercomplex number system of degree $n+1$ such that $p_k = i_k^2 \neq 0$ for all $k \in 1, 2, ..., n$. We can write y in the coordinate system described above:*

(7.5)

$$y = r[cosp_1(\phi_1)+sinp_1(\phi_1)\sum_{k=1}^{n-1} i_k[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1})+i_n sinp_1(\phi_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)]$$

Then,

(7.6)

$$y^m = r^m[cosp_1(m\phi_1)+sinp_1(m\phi_1)[\sum_{k=1}^{n-1} i_k[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1})+i_n(\phi_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)]$$

for all $m \in \mathbb{Z}^+$.

Proof. Base Case:

(7.7)

$$y^2 = [r[cosp_1(\phi_1) + \sum_{k=1}^{n-1} i_k sinp_1(\phi_1)[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1}) + i_n sinp_1(\phi_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)]]^2$$

$$= r^2[cosp_1(\phi_1) + \sum_{k=1}^{n-1} i_k sinp_1(\phi_1)[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1}) + i_n sinp_1(\phi_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)]^2$$

$$= r^2[cosp_1(\phi_1)^2$$

$$+ 2cosp_1(\phi_1)[\sum_{k=1}^{n-1} i_k sinp_1(\phi_1)[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1}) + i_n sinp_1(\phi_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)]$$

$$+ [\sum_{k=1}^{n-1} i_k sinp_1(\phi_1)[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1}) + i_n sinp_1(\phi_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)]^2]$$

$$= r^2[cosp_1(\phi_1)^2$$

$$+ 2cosp_1(\phi_1)[\sum_{k=1}^{n-1} i_k sinp_1(\phi_1)[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1}) + i_n sinp_1(\phi_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)]$$

$$+ sinp_1(\phi_1)^2[\sum_{k=1}^{n-1} i_k[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1}) + i_n\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)]^2]$$

Which, by Lemma Lem:Mod2 7.1, is equal to:

(7.8)

$$= r^2[cosp_1(\phi_1)^2$$

$$+ 2cosp_1(\phi_1)[\sum_{k=1}^{n-1} i_k sinp_1(\phi_1)[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1}) + i_n sinp_1(\phi_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)]$$

$$+ p_1 sinp_1(\phi_1)^2]$$

$$= r^2[cosp_1(\phi_1)^2 + p_1 sinp_1(\phi_1)^2$$

$$+ 2cosp_1(\phi_1)[\sum_{k=1}^{n-1} i_k sinp_1(\phi_1)[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1}) + i_n sinp_1(\phi_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)]]$$

$$= r^2[cosp_1(\phi_1)^2 + p_1 sinp_1(\phi_1)^2$$

$$+ 2cosp_1(\phi_1)sinp_1(\phi_1)[\sum_{k=1}^{n-1} i_k[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1}) + i_n\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)]]$$

Which, by the generalized angle addition formula, is equal to:

(7.9)

$$= r^2[cosp_1(2\phi_1) + sinp_1(2\phi_1)[\sum_{k=1}^{n-1} i_k[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1}) + i_n(\phi_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)]$$

Inductive Step: Assume that

$$y^m = r^m[cosp_1(m\phi_1)+sinp_1(m\phi_1)[\sum_{k=1}^{n-1} i_k[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1})+i_n(\phi_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)]$$

This is our inductive hypothesis. Using that Hypothesis:

(7.10)

$$y^{m+1} = y^m y$$

$$= r^m[cosp_1(m\phi_1) + sinp_1(m\phi_1)[\sum_{k=1}^{n-1} i_k[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1}) + i_n(\phi_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)]]$$

$$* [r[cosp_1(\phi_1) + \sum_{k=1}^{n-1} i_k sinp_1(\phi_1)[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1}) + i_n sinp_1(\phi_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)]]$$

$$= r^{m+1}[cosp_1(m\phi_1)cosp_1(\phi)$$

$$+ 2sinp_1(m\phi_1)cosp_1(\phi_1)[\sum_{k=1}^{n-1} i_k[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1}) + i_n(\phi_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)]$$

$$+ sinp_1(m\phi_1)sinp_1(\phi_1)[\sum_{k=1}^{n-1} i_k[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1}) + i_n(\phi_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)]^2]$$

Which, by Lemma Lem:Mod2 7.1, equals:

(7.11)

$$= r^{m+1}[cosp_1(m\phi_1)cosp_1(\phi)$$

$$+ 2sinp_1(m\phi_1)cosp_1(\phi_1)[\sum_{k=1}^{n-1} i_k[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1}) + i_n(\phi_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)]$$

$$+ p_1 sinp_1(m\phi_1)sinp_1(\phi_1)]$$

$$= r^{m+1}[cosp_1(m\phi_1)cosp_1) + p_1 sinp_1(m\phi_1)sinp_1(\phi_1)$$

$$+ 2sinp_1(m\phi_1)cosp_1(\phi_1)[\sum_{k=1}^{n-1} i_k[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1}) + i_n(\phi_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)]]$$

Which, by the generalized angle addition formula, is equal to:

(7.12)

$$= r^{m+1}[cosp_1((m+1)\phi_1)$$

$$+ sinp_1((m+1)\phi_1)[\sum_{k=1}^{n-1} i_k[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1}) + i_n(\phi_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)]]$$

□

8. Proof of Generalized de Moivre's Formula for Negative Integers

We will begin with a Lemma which is necessary to prove the Generalized de Moivre's formula for quadratic anticommutative hypercomplex numbers in the case where the number is raised to a negative power.

Lem:nsin **Lemma 8.1.** *Let $p \in \mathbb{R}$ and let $\phi \in \{\phi \mid \exists\, x = cosp(\phi), y = sinp(\phi)\}$. Then, $cosp(\phi) = cosp(-\phi)$ and $sinp(-\phi) = -sinp(\phi)$.*

Proof. We have three cases.

Case 1: Assume $p < 0$. Then, $cosp(\phi) = cos(\phi\sqrt{-p}) = cos(-\phi\sqrt{-p}) = cosp(-\phi)$ and $sinp(-\phi) = \frac{1}{\sqrt{-p}}sin(-\phi\sqrt{-p}) = -\frac{1}{\sqrt{-p}}sin(\phi\sqrt{-p}) = -sinp(\phi)$.

Case 2: Assume $p = 0$. Then, $cosp(\phi) = 1 = cosp(-\phi)$ and $sinp(-\phi) = -\phi = -sinp(\phi)$.

Case 3: Assume $p > 0$. Then,

$cosp(-phi) = cosh(-\phi\sqrt{p}) = \frac{e^{-\phi\sqrt{p}}+e^{\phi\sqrt{p}}}{2} = \frac{e^{\phi\sqrt{p}}+e^{-\phi\sqrt{p}}}{2} = cosh(\phi\sqrt{p}) = cosp(\phi)$ and

$sinp(-\phi) = \frac{1}{\sqrt{p}}sinh(-\phi\sqrt{p}) = \frac{e^{-\phi\sqrt{p}}-e^{\phi\sqrt{p}}}{2\sqrt{p}} = -\frac{-e^{-\phi\sqrt{p}}+e^{\phi\sqrt{p}}}{2\sqrt{p}} = -\frac{1}{\sqrt{p}}sinh(\phi\sqrt{p}) = -sinp(\phi)$

□

Theorem 8.2. *Let y be a number in a quadratic anticommutative hypercomplex number system of degree $n+1$. We can write y in the coordinate system described above:*

$$y = r[cosp_1(\phi_1)+\sum_{k=1}^{n-1} i_k sinp_1(\phi_1)[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1})+i_n sinp_1(\phi_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)]$$

Then,

$$y^m = r^m[cosp_1(m\phi_1)+sinp_1(m\phi_1)[\sum_{k=1}^{n-1} i_k[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1})+i_n(\phi_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)]]$$

for all $m \in \mathbb{Z}^-$

Proof. Because we have already shown that the equation

$$[r[cosp_1(\phi_1)+\sum_{k=1}^{n-1} i_k sinp_1(\phi_1)[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1})+i_n sinp_1(\phi_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)]]^m$$

$$= r^m[cosp_1(m\phi_1)+sinp_1(m\phi_1)[\sum_{k=1}^{n-1} i_k[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1})+i_n(\phi_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)]]$$

holds for all $m \in \mathbb{Z}^+$, we just need to show that

$$[r[cosp_1(\phi_1)+\sum_{k=1}^{n-1} i_k sinp_1(\phi_1)[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1})+i_n sinp_1(\phi_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)]]^{-1}$$

$$= \frac{1}{r}[cosp_1(-\phi_1)+sinp_1(-\phi_1)[\sum_{k=1}^{n-1} i_k[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1})+i_n(\phi_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)]]$$

Now, let

$$u = \sum_{k=1}^{n-1} i_k[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1}) + i_n(\phi_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)$$

Then, we can simplify the expression:

$$\tfrac{1}{r}[cosp_1(\phi_1) + sinp_1(\phi_1)[\sum_{k=1}^{n-1} i_k[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1}) + i_n(\phi_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)]]^{-1}$$

to

$$\tfrac{1}{r}[cosp_1(\phi_1) + sinp_1(\phi_1)u]^{-1} = \frac{1}{r[cosp_1(\phi_1) + sinp_1(\phi_1)u]}$$

Which equals: $\dfrac{cosp_1(\phi_1) - sinp_1(\phi_1)u}{r[cosp_1(\phi_1) + sinp_1(\phi_1)u][cosp_1(\phi_1) - sinp_1(\phi_1)u]} = \dfrac{cosp_1(\phi_1) - sinp_1(\phi_1)u}{r[cosp_1(\phi_1)^2 - sinp_1(\phi_1)^2u^2]}$

Now, by Lemma Lem:Mod2 7.1, $u^2 = p_1$, so:

$$\frac{cosp_1(\phi_1) - sinp_1(\phi_1)u}{r[cosp_1(\phi_1)^2 - sinp_1(\phi_1)^2u^2]} = \frac{cosp_1(\phi_1) - sinp_1(\phi_1)u}{r[cosp_1(\phi_1)^2 - sinp_1(\phi_1)^2p_1]} = \frac{cosp_1(\phi_1) - sinp_1(\phi_1)u}{r}$$

Which, by Lemma Lem:nsin 8.1, is equal to $\dfrac{cosp_1(-\phi_1) + sinp_1(-\phi_1)u}{r}$ ☐

9. Roots

From the Generalized De Moivre's formula, you can prove a result about the roots of quadratic anticommutative hypercomplex numbers.

Theorem 9.1. *Let*

$$y = r[cosp_1(\phi_1)+\sum_{k=1}^{n-1} i_k sinp_1(\phi_1)[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1})+i_n sinp_1(\phi_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)]$$

be a number in a quadratic anticommutative hypercomplex number system. Then, if $cosp_1(m\theta_1) = cosp_1(\phi_1)$ *and* $sinp_1(m\theta_1) = sinp_1(\phi_1)$*, where* $m \in \mathbb{Z}$*, then*

$$y = [r^{1-m}[cosp_1(\theta_1)+sinp_1(\theta_1)[\sum_{k=1}^{n-1} i_k[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1})+i_n(\phi_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)]]]^m$$

Proof. By the Generalized De Moivre's formula:

$$[r^{1-m}[cosp_1(\theta_1)+sinp_1(\theta_1)[\sum_{k=1}^{n-1} i_k[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1})+i_n(\phi_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)]]]^m$$

$$= r[cosp_1(m\theta_1)+sinp_1(m\theta_1)[\sum_{k=1}^{n-1} i_k[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1})+i_n(\phi_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)]]$$

$$= r[cosp_1(\phi_1) + sinp_1(\phi_1)[\sum_{k=1}^{n-1} i_k[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1}) + i_n(\phi_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)]]$$

$$= y$$ □

It turns out that (in the case of a quadradic anticommutative hypercomplex number system where $p_k \neq 0$ *for all* k*), all roots of a hypercomplex number in that system are of this form. To show this, we will first have to prove a lemma.*

Lemma 9.2. *Let* $p \in \mathbb{R}$*,* $p \neq 0$*. Suppose* $cosp(\phi) = cosp(m\phi)$*, where* $m \in \mathbb{Z}$*. Then,* $sinp(\phi) = \pm sinp(m\phi)$*.*

Proof. We have two cases:

Case 1: If $p < 0$, then $cosp(\phi) = cos(\phi\sqrt{-p})$ and $sinp(\phi) = \frac{1}{\sqrt{-p}}sin(\phi\sqrt{-p})$ by definition Def:CPSP 3.2. Since $cos^2(m\phi\sqrt{-p}) + sin^2(m\phi\sqrt{-p}) = 1$,$cos^2(\phi\sqrt{-p}) + sin^2(\phi\sqrt{-p}) = 1$, and $cos(\phi\sqrt{-p}) = cos(m\phi\sqrt{-p})$ by hypothesis:

$sin^2(m\phi\sqrt{-p}) = 1 - cos^2(m\phi\sqrt{-p}) = 1 - cos^2(\phi\sqrt{-p}) = sin^2(\phi\sqrt{-p})$.

Since $sin^2(\phi\sqrt{-p}) = sin^2(m\phi\sqrt{-p})$, $sin(\phi\sqrt{-p}) = \pm sin(m\phi\sqrt{-p})$.

Thus, $\frac{1}{\sqrt{-p}}sin(\phi\sqrt{-p}) = \pm\frac{1}{\sqrt{-p}}sin(m\phi\sqrt{-p})$ and $sinp(\phi) = \pm sinp(m\phi)$.

Case 2: If $p > 0$, then $cosp(\phi) = cosh(\phi\sqrt{p})$ and $sinp(\phi) = \frac{1}{\sqrt{p}}sinh(\phi\sqrt{p})$ by definition Def:CPSP 3.2. Because $cosh(m\phi\sqrt{p})^2 - sinh(m\phi\sqrt{p})^2 = 1$,$cosh(\phi\sqrt{p})^2 - sinh(\phi\sqrt{p})^2 = 1$, and

$cosh(\phi\sqrt{p}) = cosh(m\phi\sqrt{p})$ by hypothesis:

$sinh(m\phi\sqrt{p})^2 = cosh(m\phi\sqrt{p})^2 - 1 = cosh(\phi\sqrt{p})^2 - 1 = sinh(\phi\sqrt{p})^2$.

Since $sinh(m\phi\sqrt{p})^2 = sinh(\phi\sqrt{p})^2$, $sinh(m\phi\sqrt{p}) = \pm sinh(\phi\sqrt{p})$.

Thus, $\frac{1}{\sqrt{p}}sinh(\phi\sqrt{p}) = \pm\frac{1}{\sqrt{p}}sinh(m\phi\sqrt{p})$, and $sinp(\phi) = \pm sin(m\phi)$.

□

Theorem 9.3. *Let*

$$y = r[cosp_1(\phi_1) + sinp_1(\phi_1)[\sum_{k=1}^{n-1} i_k[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1}) + i_n(\phi_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)]]$$

be a hypercomplex number. Because y^m is a quadratic anticommutative hypercomplex number for $m \in \mathbb{Z}$, there must exist $\theta_1, \theta_2, ..., \theta_n$ such that

$$y^m = r^m[cosp_1(\theta_1)+sinp_1(\theta_1)[\sum_{k=1}^{n-1} i_k[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\theta_j)]cos\frac{-p_{k+1}}{p_k}(\theta_{k+1})+i_n(\theta_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\theta_j)]]$$

If these conditions hold, then

$cosp_1(\theta_1) = cosp_1(m\phi_1)$, $sinp_1(\theta_1) = \pm sinp_1(m\phi_1)$, and

$$y = r[cosp_1(\phi_1) \pm sinp_1(\phi_1)[\sum_{k=1}^{n-1} i_k[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\theta_j)]cos\frac{-p_{k+1}}{p_k}(\theta_{k+1}) + i_n(\theta_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\theta_j)]]$$

Proof. By the generalized De Moivre's formula,

$$y^m = r^m[cosp_1(m\phi_1)+sinp_1(m\phi_1)[\sum_{k=1}^{n-1} i_k[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1})+i_n(\phi_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)]]$$

So, by the definition of hypercomplex number, since

$$y^m = r^m[cosp_1(\theta_1)+sinp_1(\theta_1)[\sum_{k=1}^{n-1} i_k[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\theta_j)]cos\frac{-p_{k+1}}{p_k}(\theta_{k+1})+i_n(\theta_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\theta_j)]]$$

it must be true that $r^m cosp_1(\theta_1) = r^m cosp_1(m\phi_1)$. Since $r^m cosp_1(\theta_1) = r^m cosp_1(m\phi_1)$, $cosp_1(\theta_1) = cosp_1(m\phi_1)$ and since $cosp_1(\theta_1) = cosp_1(m\phi_1)$ and $p_1 \neq 0$, $sinp_1(\theta_1) = \pm sinp_1(m\phi_1)$

by the preceding lemma. This means that:

(9.1)

$$y^m = cosp_1(\theta_1) \pm sinp_1(\theta_1)[\sum_{k=1}^{n-1} i_k[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1}) + i_n(\phi_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)]$$

$$= cosp_1(\theta_1) \pm sinp_1(\theta_1)[\sum_{k=1}^{n-1} i_k[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\theta_j)]cos\frac{-p_{k+1}}{p_k}(\theta_{k+1}) + i_n(\theta_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\theta_j)]$$

and thus

$$[\sum_{k=1}^{n-1} i_k[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\phi_j)]cos\frac{-p_{k+1}}{p_k}(\phi_{k+1}) + i_n(\phi_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\phi_j)]$$

(9.2)
$$=$$

$$\pm[\sum_{k=1}^{n-1} i_k[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\theta_j)]cos\frac{-p_{k+1}}{p_k}(\theta_{k+1}) + i_n(\theta_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\theta_j)]]$$

Thus,

$$y = r[cosp_1(\phi_1) \pm sinp_1(\phi_1)[\sum_{k=1}^{n-1} i_k[\prod_{j=2}^{k} sin\frac{-p_j}{p_{j-1}}(\theta_j)]cos\frac{-p_{k+1}}{p_k}(\theta_{k+1}) + i_n(\theta_1)\prod_{j=2}^{n} sin\frac{-p_j}{p_{j-1}}(\theta_j)]]$$

□

This means that if $p_k \neq 0$ for all k, the roots of y^m depend solely on the number of ϕ_1's such that $cosp_1(m\phi_1) = cosp_1(\theta_1)$. That is, the number of roots depends on the value of p_1 and θ_1 but not on any of the other p or θ values.

10. Derivatives in Quadratic anticommutative Hypercomplex number systems

When defining derivatives for quadratic anticommutative Hypercomplex number systems, because division is involved, we must seperately define right and left derivatives. This definition is based on the definitions of differentiability and derivitives on the complex plane in [KS]*,* *extended to apply to quadratic anticommutative Hypercomplex planes.*

RLD **Definition 10.1.** Let H be a quadratic anticommutative hypercomplex number system, and let $f(z)$ be a function from the open set $D \in H$ to H. $f(z)$ is *right differentiable* at a point $z_0 \in D$ if and only if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \triangleright$$

exists. We will refer to this limit as the *right derivative* of $f(z)$ at z_0, symbolized $f^{\triangleleft}(z_0)$ Similarly, $f(z)$ is *left differentiable* at a point $z_0 \in D$ if and only if the limit

$$\lim_{z \to z_0} \triangleleft \frac{f(z) - f(z_0)}{z - z_0}$$

exists. We will refer to this limit as the *left derivative* of $f(z)$ at z_0, symbolized as $f^{\triangleright}(z_0)$.

Next, we will define left and right analyticity based on their equivalent in complex analysis. Similarly,these definitions are based on the definition of analyticity in KS [KS].

RLA **Definition 10.2.** If $f^{\triangleleft}(z)$ exists for all z in an open set around z_0, then we say that $f(z)$ is *right analytic* at z_0. If $f^{\triangleright}(z)$ exists for all z in an open set around z_0, then we say that $f(z)$ is *left analytic* at z_0.

Theorem 10.3. *Let $f(x)$ be a function on a quadratic anticommutative hypercomplex plane of dimension n, $n > 1$, such that:*

$$f(z) = u(z) + i_1 v_1(z) + ... + i_n v_n(z) = u(z) + \sum_{k=1}^{n} i_k v_k(z)$$

Where $u(z) \in \mathbb{R}$ and $v_j \in \mathbb{R}$ for all $j \in 1, 2, ..., n$. Let z be a point on this hypercomplex plane, where $z_0 = x_0 + i_1 y_{0,1} + ... + i_n y_{0,n} = x_0 + \sum_{k=1}^{n} i_k y_{0,k}$. If $f(z)$ is both right and left analytic at z_0, then

$$\frac{\partial u(z_0)}{\partial y_b} = \frac{\partial v_b(z_0)}{\partial x} = 0 \textit{ for all } b \in 0,1,...,n$$

(10.1)

$$\frac{\partial v_b(z_0)}{\partial y_b} = \frac{\partial u(z_0)}{\partial x} \textit{ for all } b \in 0,1,...,n \textit{ and}$$

$$\frac{i_b(\sum_{k=1}^{b-1} i_k \partial v_k(z_0) + \sum_{k=b+1}^{n} i_k \partial v_k(z_0))}{\partial y_b} = \frac{(\sum_{k=1}^{b-1} i_k \partial v_k(z_0) + \sum_{k=b+1}^{n} i_k \partial v_k(z_0)) i_b}{\partial y_b} = 0$$

Proof. First, I will show that

$$\frac{i_b(\sum_{k=1}^{b-1} i_k \partial v_k(z_0) + \sum_{k=b+1}^{n} i_k \partial v_k(z_0))}{\partial y_b} = \frac{(\sum_{k=1}^{b-1} i_k \partial v_k(z_0) + \sum_{k=b+1}^{n} i_k \partial v_k(z_0)) i_b}{\partial y_b} = 0$$

Proof:

Because $f(z)$ is analytic, the derivatives $f^{\triangleright}(z_0)$ and $f^{\triangleleft}(z_0)$ are defined by definition. Now, if we approach z_0 in the x direction (that is to say, we set $z = x + i_1 y_{0,1} + ... + i_n y_{0,n}$), then by definition the right derivative can be represented as

(10.2)

$$\lim_{(x+\sum_{k=1}^{n} i_k y_{0,k}) \to (x_0 + \sum_{k=1}^{n} i_k y_{0,k})} \frac{f(x + \sum_{k=1}^{n} i_k y_{0,k}) - f(x_0 + \sum_{k=1}^{n} i_k y_{0,k})}{x + \sum_{k=1}^{n} i_k y_{0,k} - (x_0 + \sum_{k=1}^{n} i_k y_{0,k})} \triangleright$$

$$= \lim_{x \to x_0} \frac{u(x + \sum_{k=1}^{n} i_k y_{0,k}) + \sum_{k=1}^{n} i_k v_k(x + \sum_{k=1}^{n} i_k y_{0,k}) - [u(x_0 + \sum_{k=1}^{n} i_k y_{0,k}) + \sum_{k=1}^{n} i_k v_k(x_0 + \sum_{k=1}^{n} i_k y_{0,k})]}{x - x_0} \triangleright$$

$$= \frac{\partial u(z_0) + \sum_{k=1}^{n} i_k \partial v_k(z_0)}{\partial x} \triangleright$$

Similarly, the left derivative when approached in the x direction becomes

$$\triangleleft \frac{\partial u(z_0) + \sum_{k=1}^{n} i_k \partial v_k(z_0)}{\partial x}$$

A similar proof shows that when approaching from the y_b direction, the left and right derivatives become

$$\triangleleft\frac{\partial u(z_0)+\sum_{k=1}^{n} i_k\partial v_k(z_0)}{i_b\partial y_b}$$

and

$$\frac{\partial u(z_0)+\sum_{k=1}^{n} i_k\partial v_k(z_0)}{i_b\partial y_b}\triangleright$$

Now, because $f(z)$ is right analytic at z_0, $f^{\triangleright}(z_0)$ is defined and it must have the same value no matter what direction z approaches z_0. Thus

$$\frac{\partial u(z_0)+\sum_{k=1}^{n} i_k\partial v_k(z_0)}{i_b\partial y_b}\triangleright=\frac{\partial u(z_0)+\sum_{k=1}^{n} i_k\partial v_k(z_0)}{i_a\partial y_a}\triangleright=\frac{\partial u(z_0)+\sum_{k=1}^{n} i_k\partial v_k(z_0)}{\partial x}\triangleright$$

for all $a, b \in 1, 2, ..., n$. Similarly, because $f(z)$ is left analytic at z_0, $f^{\triangleleft}(z_0)$ is defined and must have the same value no matter what direction z approaches z_0 from, so

$$\triangleleft\frac{\partial u(z_0)+\sum_{k=1}^{n} i_k\partial v_k(z_0)}{i_b\partial y_b}=\triangleleft\frac{\partial u(z_0)+\sum_{k=1}^{n} i_k\partial v_k(z_0)}{i_a\partial y_a}=\triangleleft\frac{\partial u(z_0)+\sum_{k=1}^{n} i_k\partial v_k(z_0)}{\partial x}$$

for all $a, b \in 1, 2, ..., n$. Notice that because ∂x is a real number

$$\frac{\partial u(z_0)+\sum_{k=1}^{n} i_k\partial v_k(z_0)}{\partial x}\triangleright=\triangleleft\frac{\partial u(z_0)+\sum_{k=1}^{n} i_k\partial v_k(z_0)}{\partial x}=\frac{\partial u(z_0)+\sum_{k=1}^{n} i_k\partial v_k(z_0)}{\partial x}$$

This means that

$$\triangleleft\frac{\partial u(z_0)+\sum_{k=1}^{n} i_k\partial v_k(z_0)}{i_b\partial y_b}=\frac{\partial u(z_0)+\sum_{k=1}^{n} i_k\partial v_k(z_0)}{\partial x}=\frac{\partial u(z_0)+\sum_{k=1}^{n} i_k\partial v_k(z_0)}{i_b\partial y_b}\triangleright$$

for all $b \in 1, 2, ..., n$.

Now, by the definitions of right and left fraction

$$\triangleleft\frac{\partial u(z_0)+\sum_{k=1}^{n} i_k\partial v_k(z_0)}{i_b\partial y_b}=\frac{(\partial u(z_0)+\sum_{k=1}^{n} i_k\partial v_k(z_0))i_b}{p_b\partial y_b}$$

and

$$\frac{\partial u(z_0)+\sum_{k=1}^{n} i_k\partial v_k(z_0)}{i_b\partial y_b}\triangleright=\frac{i_b(\partial u(z_0)+\sum_{k=1}^{n} i_k\partial v_k(z_0))}{p_b\partial y_b}$$

for all $b \in 1, 2, ..., n$. So

$$\frac{(\partial u(z_0) + \sum_{k=1}^{n} i_k \partial v_k(z_0)) i_b}{p_b \partial y_b} = \frac{i_b(\partial u(z_0) + \sum_{k=1}^{n} i_k \partial v_k(z_0))}{p_b \partial y_b}$$
for all $b \in 1, 2, ..., n$

Because $\frac{\partial u(z_0)}{p_b \partial y_b}$ is real (as $u(z_0)$ and y_b are both real), $i_b \frac{\partial u(z)}{p_b \partial y_b} = \frac{\partial u(z_0)}{p_b \partial y_b} i_b$. Thus

$$\frac{(\sum_{k=1}^{n} i_k \partial v_k(z_0)) i_b}{p_b \partial y_b} = \frac{i_b(\sum_{k=1}^{n} i_k \partial v_k(z_0))}{p_b \partial y_b}$$

$$\frac{(\sum_{k=1}^{b-1} i_k \partial v_k(z_0) + i_b \partial v_b(z_0) + \sum_{k=b+1}^{n} i_k \partial v_k(z_0)) i_b}{p_b \partial y_b} = \frac{i_b(\sum_{k=1}^{b-1} i_k \partial v_k(z_0) + i_b \partial v_b(z_0) + \sum_{k=b+1}^{n} i_k \partial v_k(z_0))}{p_b \partial y_b}$$

$$\frac{(\sum_{k=1}^{b-1} i_k \partial v_k(z_0) + \sum_{k=b+1}^{n} i_k \partial v_k(z_0)) i_b}{p_b \partial y_b} = \frac{i_b(\sum_{k=1}^{b-1} i_k \partial v_k(z_0) + \sum_{k=b+1}^{n} i_k \partial v_k(z_0))}{p_b \partial y_b}$$

Now, because each $\frac{\partial v_k(z_0)}{p_b \partial y_b}$ is real, and $i_b i_k = -i_k i_b$ for all $b \neq k$

$$\frac{(\sum_{k=1}^{b-1} i_k \partial v_k(z_0) + \sum_{k=b+1}^{n} i_k \partial v_k(z_0)) i_b}{p_b \partial y_b} = -\frac{(\sum_{k=1}^{b-1} i_k \partial v_k(z_0) + \sum_{k=b+1}^{n} i_k \partial v_k(z_0)) i_b}{p_b \partial y_b}$$

$$2\frac{(\sum_{k=1}^{b-1} i_k \partial v_k(z_0) + \sum_{k=b+1}^{n} i_k \partial v_k(z_0)) i_b}{p_b \partial y_b} = 0$$

$$\frac{(\sum_{k=1}^{b-1} i_k \partial v_k(z_0) + \sum_{k=b+1}^{n} i_k \partial v_k(z_0)) i_b}{p_b \partial y_b} = 0$$

Now I will show that

$\frac{\partial u(z_0)}{\partial y_b} = \frac{\partial v_b(z_0)}{\partial x_0} = 0$ for all $b \in 0, 1, ..., n$

$\frac{\partial v_b(z_0)}{\partial y_b} = \frac{\partial u(z_0)}{\partial x}$ for all $b \in 0, 1, ..., n$

Proof: As shown in the previous proof

$$\triangleleft \frac{\partial u(z_0) + \sum_{k=1}^{n} i_k \partial v_k(z_0)}{i_b \partial y_b} = \frac{\partial u(z_0) + \sum_{k=1}^{n} i_k \partial v_k(z_0)}{\partial x}$$

and

$$\triangleleft \frac{\partial u(z_0) + \sum_{k=1}^{n} i_k \partial v_k(z_0)}{i_b \partial y_b} = \frac{(\partial u(z_0) + \sum_{k=1}^{n} i_k \partial v_k(z_0)) i_b}{p_b \partial y_b}$$

Thus,

$$\frac{(\partial u(z_0) + \sum_{k=1}^{n} i_k \partial v_k(z_0)) i_b}{p_b \partial y_b} = \frac{\partial u(z_0) + \sum_{k=1}^{n} i_k \partial v_k(z_0)}{\partial x}$$

Because

$$\frac{(\sum_{k=1}^{b-1} i_k \partial v_k(z_0) + \sum_{k=b+1}^{n} i_k \partial v_k(z_0)) i_b}{p_b \partial y_b} = 0,$$

we can simplify the previous equation to

$$\frac{\partial u(z_0) i_b + i_b \partial v_b(z_0) i_b}{p_b \partial y_b} = \frac{\partial u(z_0) + \sum_{k=1}^{n} i_k \partial v_k(z_0)}{\partial x}$$

$$\frac{\partial u(z_0) i_b + p_b \partial v_b(z_0)}{p_b \partial y_b} = \frac{\partial u(z_0) + \sum_{k=1}^{n} i_k \partial v_k(z_0)}{\partial x}$$

By the definition of hypercomplex numbers,

$$\frac{\partial v_b(z_0)}{\partial y_b} = \frac{\partial u(z_0)}{\partial x}$$

$$\frac{\partial u(z_0)}{\partial y_b} = \frac{\partial v_b(z_0)}{\partial x}$$

and

$$\frac{\partial v_k(z_0)}{\partial x} = 0 \text{ for all } k \in 1, 2, ..., n, k \neq b$$

However, because b can be any integer in the range $1, 2, ..., n$, it must be true for two distinct numbers $b, c \in 1, 2, ..., n, b \neq c$. Thus

$$\frac{\partial v_k(z_0)}{\partial x} = 0 \text{ for all } k \in 1, 2, ..., n, k \neq b$$

and

$$\frac{\partial v_k(z_0)}{\partial x} = 0 \text{ for all } k \in 1, 2, ..., n, k \neq c$$

So, $\frac{\partial v_b(z_0)}{\partial x} = 0$ as well.

Thus, for all $b \in 1, 2, ..., n$

$$\frac{\partial v_b(z_0)}{\partial y_b} = \frac{\partial u(z_0)}{\partial x}$$

$$\frac{\partial u(z_0)}{\partial y_b} = \frac{\partial v_b(z_0)}{\partial x} = 0$$

□

Using this theorem, and the results concerning the Cauchy-Riemann equations for generalized complex numbers in the Harkin paper, the next theorem follows.

Theorem 10.4. *Let $f(x)$ be a function on a quadratic anticommutative hypercomplex plane of dimension n, such that:*

$$f(z) = u(z) + i_1 v_1(z) + ... + i_n v_n(z) = u(z) + \sum_{k=1}^{n} i_k v_k(z)$$

Where $u(z) \in \mathbb{R}$ and $v_j \in \mathbb{R}$ for all $j \in 1, 2, ..., n$. Let z be a point on this hypercomplex plane, where $z_0 = x_0 + i_1 y_{0,1} + ... + i_n y_{0,n} = x_0 + \sum_{k=1}^{n} i_k y_{0,k}$. If $f(z)$ is both right and left analytic at z_0, then

$$\begin{aligned} \frac{\partial u(z_0)}{\partial y_b} &= -\frac{\partial v_b(z_0)}{\partial x} \text{ for all } b \in 0, 1, ..., n \text{ and} \\ \frac{\partial v_b(z_0)}{\partial y_b} &= \frac{\partial u(z_0)}{\partial x} \text{ for all } b \in 0, 1, ..., n \end{aligned} \tag{10.3}$$

The equations numbered 9.3 are the generalized Cauchy-Riemann equations for Quadratic anticommutative hypercomplex numbers.

11. Further Research

The defining properties of Quadratic Anticommutative Hypercomplex Numbers have enabled me to prove several interesting theorems, but there are many more research questions to pursue regarding these numbers. Is it possible to prove a theorem similar to theorems 10.3 and 10.4 for functions that are right analytic but not left analytic, or vice versa? Can one prove an if and only if statement which determines whether or not a function on a Quadratic Anticommutative Hypercomplex number system is right analytic, left analytic, or both? Is it possible to define a generalized form of integration on these numbers, and do any of the theorems regarding integration in complex analysis extend to Quadratic Anticommutative Hypercomplex number systems? Ultimately, the theorems proved in this paper may only *scratch the surface of what Quadratic Anticommutative Hypercomplex number systems have to offer.*

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