Solid Strips Configurations

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Abstract

We introduce the idea of Solid Strip Configurations which is a way of construction 3-dimensional compact manifolds alternative to Δ-complexes and CW complexes. The proposed method is just an idea which we believe deserve further formal mathematical investigation.

Key Words: compact manifolds, manifold decomposition.

1 Introduction

Compact manifolds of dimension higher then 2 are very hard to study and classify. Starting from a method for the 2D case and focusing on 3D manifolds, we propose in this paper a method, alternative to Δ-complexes and CW complexes, to construct these manifolds which, if further developed, we believe may results very convenient.

2 Strip Configurations in 2-Dimensions

2.1 Main Definitions

A **Strip** is a 2-dimensional manifold with boundaries obtained by identifying 2 opposite edges of the 4 edges of a square. It can be done without a twist (Untwisted Strip) or with a twist (Mobius strip).

A **Strip Configuration** is a finite set of strips, crossing each other or not, such that it exist a compact 2-dimensional manifold in which the set of strips can be embedded. An example of two strings that do not form a string configuration is given in Fig. 1a. Once we embed the strips on such a manifold we are allowed to move the strips on the manifold at will. If a and b are two strips then we will use the notation $a \diamond b$ for the configuration obtained by making a and b crossing 1 time.

A non path connected strip configuration can always be changed in a path connected one according to the following procedure: 1-embed the strips in a compact two dimensional manifold; 2- bring two strips from two non path connected subset of the configuration close each other without changing the configuration of the two subset(see Fig. 1b); 3- overlap the two trips so that they cross in two points (see Fig. 1c).

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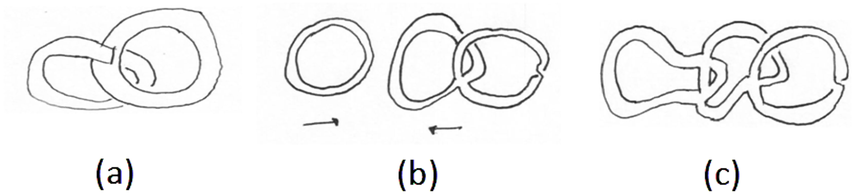

Figure 1: Definition of String Configuration

Note that the boundary of a string configuration is made of a finite number of **sub-boundaries** (i.e. non path connected parts) each of which being a circles (i.e. $\mathbf{S}^1$). The **Associated (Compact) Manifold** to a strip configuration is the compact manifold obtained by making the configuration path connected (if it is not) and identifying the boundary of a disk ($\mathbf{D}^2$) to each sub-boundaries of the strip configuration. We will use the notation $\Omega(A)$ for the associated manifold to the strip configuration A.

Two strip configurations are **Homeomorphic Associated Equivalent** if their associated manifolds are homeomorphic or, which is the same, if once embedded in the associated manifold one string configuration can be changed into the other by moving the strips on the manifold and deforming the manifold by means of continuous transformations. In the process each strip shall always keep its own identity even when it crosses other strips with continuous transformations meaning that a strip cannot be cut and glued to form other strips. Two strip configurations are **Homotopy Associated Equivalent** if their associated manifolds are homotopy equivalent.

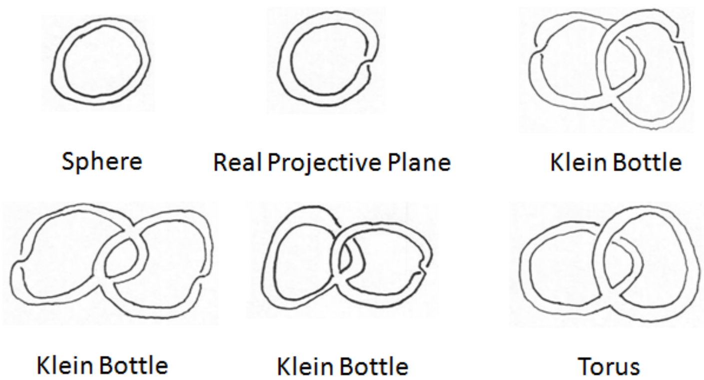

Figure 2: 1 and 2 Strip Configurations

In a strip configuration a string can be twisted n times (with $n \geq 0$) (if n is even then the string is homomorphic to an untwisted strip, if n is odd to a Mobius strip) and two strips can cross each other m times (with $m \geq 0$).

We want to give now some criteria for two strip configurations to be homeomorphic associated equivalent. Some of these criteria are not obvious and should be formally proved.

1. A non path connected strip configuration and the path connected one obtained from it using the procedure explained in the paragraph above

are equivalent.

2. An untwisted strip that does not cross any other strip can be removed from the configuration because this is equivalent to removing from the associated manifold a sphere which is sum connected to the manifold.

3. Given a strip configuration, this is equivalent to the same strip configuration where strips that are twisted an odd number of times are replaced by Mobius strips and strips that are twisted an even number of times are replaced with untwisted strips.

We note that the direct sum of 2-dimensional manifolds has a non path connected strip configuration given by the union of the two strip configurations of the two manifolds.

However, the above criteria are not enough and we want to evaluate equivalences by calculating topological invariants on the configurations. Strip configurations are very convenient from this point of view because the fundamental group of the associated manifold can be easily computed from its strip configuration using the van Kampen theorem.

To evaluate the fundamental group, the generators are given by the open maximal spanning graph obtained from the graph we get homotopyng each strips to a 1- dimensional space (i.e. we turn strips into lines) while the conditions to present the group can be evaluated on the strip configuration itself.

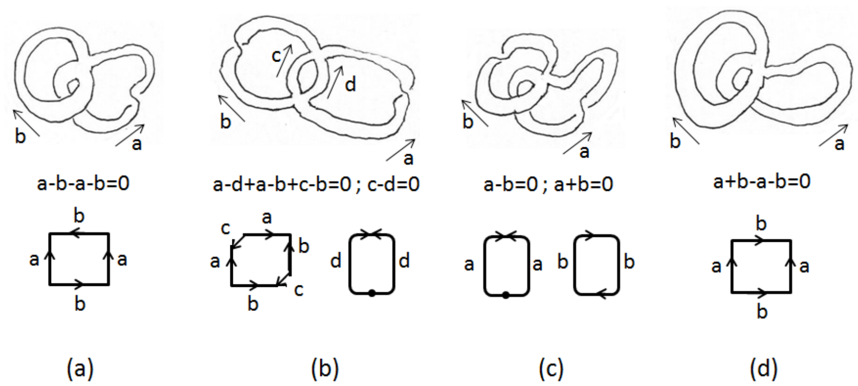

Figure 3: Strip Configurations Fundamental Groups

We will show this with some examples. In figure Fig. 3 we show some strips configurations with the generators used to have the free non commutative groups. The conditions to present the fundamental group of the associated manifold are drawn in a "polygonal picture" under each configuration. These conditions are obtained starting from a point and adding the generators (group are presented with an additive operation although unusual for non commutative groups) that we encounter on the boundary going all around till we get to the same point.

For case of Fig. 3a the condition lead to the group $\pi_1 = \mathbb{Z} \oplus \mathbb{Z}_2$ which is the group of the Klein bottle. For case 3b, we have $c = d$ which, with a simple algebraic manipulations give the condition presented in the two polygon under the configuration in the figure. These lead to the group $\pi_1 = \mathbb{Z} \oplus \mathbb{Z}_2$ which is the group of the Klein bottle. For case 3.c, from the two conditions we have that $a = b$ and therefore the two conditions became $a - a = 0$ and $b + b = 0$. Once again these lead to the group $\pi_1 = \mathbb{Z} \oplus \mathbb{Z}_2$ which is the group of the Klein bottle. Condition of Fig. 3d leads to the commutative free group on two generators which is the group of the torus $\pi_1 = \mathbb{Z}^2$.

We note that for cases of Fig. 3b and 3c we need to manipulate the conditions algebraically to permute the names of edges for the polygonal representation end this because in each polygon we want to have pairs of edges with the same name.

2.2 Represented 2D Manifolds

A question we may ask is how many compact 2D manifolds we can represent with strip configurations. We have the following proposition:

Proposition 2.1: *If a 2D compact manifold has a Δ-complex representation, then it has also a strip configuration representation.*

We will only sketch in an informal way the proof of the above proposition. Given a 2-compact manifold A and a Δ-complex X representing A and made of n 2-simplices, we can find all possible path connected subspaces X_i of X by taking proper path connected subset of the n simplices. The boundaries of these subspaces are circles (i.e. loops) or wedges of circles that can be split. With abuse of terminology we may say that these loops are orientable or not where for orientable we mean that a flat man living in the surface and walking on the loop would go back to the original point staying on the same side of the surface with respect of the loop. From the orientable loops we can get untwisted strips and from the non orientable we can get twisted strips in A. To get the strips from loops in A, the easiest way is to take a vector of small length δL on the loop orthogonal to it and laying in A and move the vector along the loop till it comes back to the original position and orientation (it takes to go around once for orientable loops and twice for non orientable). The vector will sweep a surface defining the strip. The above strips and the way they cross in X will form a redundant strip configuration which associated manifold is A. This already prove the proposition.

Moreover, we can partition the above strips ξ_i in classes C_j of strips that are are isotopic in A. By taking only one element ξ_j for each class we get finally the minimal strip configuration we where looking for.

We will show our argument with an example. Given the Klein bottle, its $\mathbf{H}_1$ homology group is $\mathbf{H}_1 = \mathbb{Z} + \mathbb{Z}_2$. This group tells us that in the strip configuration associated to the Klein bottle the maximum number of loops that cross in a non trivial way is 2 (i.e. rank of $\mathbf{H}_1$ plus 1) and that they cross non trivially at most in 1 point (i.e. rank of $\mathbf{H}_1$). We have also the $\mathbb{Z}_2$ term that tells us that in the strip configuration there is at least one loop that is not orientable. An analysis of the problem at hand shows indeed that two strips, one of which non orientable, that cross in a non trivial way have as associated manifold the Klein bottle, although we know that this is not the only configuration generating it.

3 Motivation for 3-Dimensions Strip Configurations

If we think for a moment to what we did in the previous paragraph we see that we represent 2D manifold starting from strip configurations or, another way to see it, we use 2D strips to probe a 2D space in a similar way homotopy theory does with loops. Given a strip configurations, this may not be embedded in $\mathbb{R}^2$ but it does exist a minimal (in away that may be made precise using the concept of associated space) 2D compact manifold where this strip configuration can be embedded. In other words a strip configuration defines a compact 2D manifolds in the same way a CW complex or a Δ-complex does.

This way to probe spaces has the advantage to see differences in some spaces that are homotopy equivalent. The most trivial example (although with boundaries) is the Mobius strip which is homotopy equivalent to $\mathbf{S}^1$. However, in this space obviously a smaller Mobius Strip can be embedded while the same cannot be done in the circle.

In the following sections we will try to show that a possible extension of the idea of a strip configuration to 3D manifolds may be possible.

4 Strip Configurations in 3-Dimensions

4.1 Main Definitions

In 2-dimensions we use 2-D strips obtained by identifying one couple of opposite edges of the two couples of edges of a square. In 3-dimensions we will use **Solids Strips** which are 3-D "strips" obtained by identifying two couples of opposite faces of the three couples of faces of a cube. This manifolds have been studied in the paper [1] where they are named "Solid Strips".

Moreover, to define any closed 3-manifold we will need to extend the class of solid strip with some **generalised solid strip** that will be defined in the paragraphs below.

The boundary of a solid trip is build by identifying the edges of two squares and what we get may form one or two sub-boundaries. The total homeomorphic configurations of Solid Strips are 21 (reported in Appendix A.1) but they may be further reduced to 15 Homology equivalent classes of solid strips with the same boundary and same homology groups (see [1]).

Solid strips are 3-manifolds or, another way we see it, they are **Thick Compact 2D Surfaces** where by that we mean that they are like surfaces expanded by a δL in the third dimension which, by sake of visualization for the reasoning that will follow, we may think to be small with respect to the surface itself when needed. Also the idea of a thick surfaces will be further clarified in the paragraphs below.

Broadly speaking, and taking into account the approximation that the following sentence has, some solid strips are thick surfaces that look like tori or Klein bottles because they are like pipes that are joint at their far ends in various way. In $\mathbb{R}^3$, torus intersect another closed surface in two ways. In one circle for what we will call a **non trivial intersection**, as shown in Fig 4a, or in two circles for what we will call a **trivial intersection**, as shown in fig 4b.

We may be a bit more precise saying that if we have two surfaces embedded in a 3 manifold x and they cross and if A is a compact subset of X which is homomorphic to a 3-disk, then if we can deform the surfaces in X such that we take all the point in which the two surfaces intersect in A and they are path then we say that the two surfaces have a trivial intersection. Solid strips cross (i.e. intersect) in the same way of surfaces although being tick surfaces they intersect in solid tori rather then circles. In a few word we may say that an intersection is trivial if it can be locally embedded in $\mathbb{R}^3$

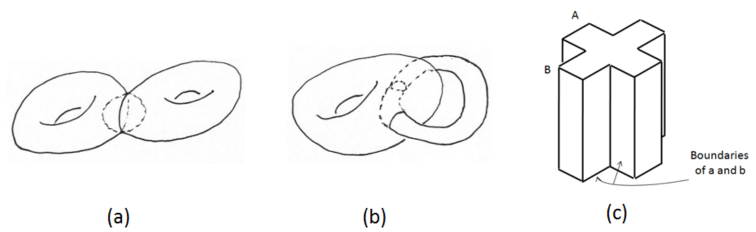

Figure 4: Crossing of two Tiles

We will call a **Solid Strip Configuration** a bunch of solid strips that cross each other (trivially or not) a finite number of times in the same way 2-strips cross forming the 2-strip configurations described in the paragraphs above. If a and b are two solid strip then we will use the notation $a \diamond b$ for the configuration obtained by making a and b crossing 1 time. The boundary of a solid strip configurations is formed by **Sub-Boundaries** exactly as in the 2D case.

In analogy with the 2D case, we will call the **Associated (Compact) Manifold** to a solid strip configurations the 3D compact manifolds that we get by filing the holes defined by its sub-boundaries (i.e. we attached manifolds to its boundaries till we get a compact space) in the "most simple" topological way where the meaning of the "most simple" will be clarified further on. In analogy with the 2D case we will use the notation $\Omega(A)$ for the associated manifold to the solid strip configuration A. We note explicitly that a non path connected strip configuration can be made connected using the same procedure we had for the 2D case.

4.2 Thick Surfaces

We want to use solid strip configurations to define 3D compact manifold. However, it turn out that the solid strip defined in [1] obtained by identifying opposite faces of a cube are not enough to define all possible 3D manifolds. We need to extend the class of solid strip with more objects, defined in this paragraph, that we will call thick surfaces or **generalised solid strip** (or yet, for brevity, just solid strips although here we word strip is totally unjustified). This will allow us to define a much broader class of compact 3D manifold (possibly all of them) using the idea of a solid strip configuration.

Let us consider n 3-simpices and the a Δ complex X composed of the n 3-simpices having in it $2n$ 2-simplices which are where one of the four faces of two and only two separate simpices are identified. X is a compact manifold. Let us consider a path connected subspace Y of X composed by a proper subset of the 3-simplices. Its boundary is a closed surface σ which, in general, is not a manifold.

From the surface σ we may define a 3-manifold with boundary ξ as follows. Given a small vector δL on σ and orthogonal to it in X (once X has been deformed so that the surface σ in it is smooth at least on the point we are at the moment). By moving δL on σ, the end of δL defines a second surface close to σ in X and the vector δL sweeps a volume which define our thick surface ξ. There are two possibilities. If σ is orientable, then σ will be part of the boundary of ξ. If σ is not orientable, then σ will be embedded in ξ but not part of its boundary. The space ξ is the generalised strip we where looking for.

We claim (although a formal proof should be given) that if the homology group $\mathbf{H}_2$ of X has rank k, then is is possible to find at most $k+1$ generalised strip that intersect in a non trivial way in k points. Moreover if the group $\mathbf{H}_2$ has torsion $\mathbb{Z}_2$ then at least one of these components are non orientable. There is no other possibility since $\mathbf{H}_2$ cannot have a torsion different from 0 or $\mathbb{Z}_2$ (although also this should be proved formally).

Finally, if we decompose a generalised strip ξ in components that cross in a non trivial way, we may further decomposed this components in sub-strips that cross each other with a trivial crossing and therefore can be separated in a region of $\Omega(\xi)$ locally homomorphic to $\mathbb{R}^3$. We will call **prime strips** these generalise solid strips that cannot be further decomposed and Γ the set of all possible generalised solid prime strips. Γ is very likely to be an infinite set.

4.3 Crossing of Solid Strips

Generalised solid strips may intersect in a trivial or in a non trivial way. The way generalised solid strips intersect deserve further study. In this paragraph however we want only to give an example of how to build the cell complex of some simple solid strips, of the type defined in [1], that cross in a non trivial way, just to give the reader the feeling of that it means.

Lets consider two solid strips of the type defined in [1]. We can get those solid strips by identifying opposite faces of cubes. Being tick surfaces, for sake of representation, we can imagine the above cubes to have one dimension smaller then the others so that they look like tiles. Fig 4c, shows a non trivial intersection between strips represented by the above mentioned tiles. This is a non trivial intersection because in $\mathbb{R}^3$, when we identify the up and down faces and two opposite faces of the tiles to get the relevant strips, we cannot avoid to have the tiles to cross a second time.

In higher dimensions we can identify the faces to get the strips without having a second intersection and two solid strips can cross (intersect) with a non trivial intersection any zero, odd or even number of times in perfect analogy with the 2D case. Another way to see that in higher dimensions two solid strips can cross only once, is from Fig 4b. If we take one of the two intersections and we move one of the two tori in the intersection along the 4^{th} dimensions, this will not intersect the other torus any more.

We have seen that we have a non trivial intersection when the two tiles A and B, from which we form the strips, intersect also on two of their faces see Fig. 4c. In the above figure we see that we can identify the vertical faces of the tiles as we want in order to get our strips but for the up and down faces, having them a square in common, we have some imitation in the way we can identify them.

Fig. 5a shows what the up and down face of the two strip (tiles) look like. In this case we cannot apply all the 8 symmetry of a square for identifying the up and down faces but only a subgroup of them which preserve the fact that the up faces of the two tiles are identified with the down faces of the same tile.

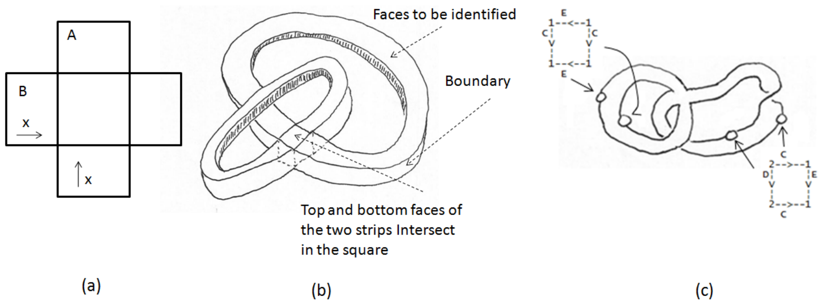

Figure 5: Crossing of two Solid Strips

In the case of strip of the type defined in [1], we can compress the up direction of each tile and draw solid strip configurations as in Fig 6b which give a good idea of what a non trivial intersection is. Although misleading, for sake of representation, if we ignore the up direction, we can draw solid strip configurations in a similar way as we do for the 2D case see Fig. 5c. In this representation the inside of the strip is represented as a surface but it is a 3D space and the boundaries of the configuration are represented as lines but they are surfaces. As for the 2D case, with the above representation, by following the boundaries of the strip configuration till we get back to the starting point, we identify sub-boundaries.

4.4 Associated Manifold

Given a strip configuration, we need to give a precise procedure to make its associated manifold. Given a strip configuration, its sub-boundaries are closed surfaces. For each sub-boundary in the configuration, we take a 2-Δ-Complex decomposition of the sub-boundary and we attach a 3 simplex to each 2-simpex of it. We identify the three faces of each of those 3-simplices each other following the same way the edges of the 2-simplices of the sub-boundary are. This will completely "fill the holes" of the solid strip configuration and will give us the compact manifold we where looking for.

4.5 Represented 3D Manifolds

In the previous section we have defined the associated manifold to a Solid Strip Configuration. We note explicitly that, in the 3D case, strip configurations may represent a large class of spaces. As for the 2D case we have:

Proposition 4.1: *If a 3D compact manifold has a Δ-complex representation, then it has also a solid strip representation.*

We will only sketch in an informal way the proof of the above proposition. Given a 3-compact manifold A and a Δ-complex X representing A and made of n 3-simplices, we can find all possible path connected subspaces X_i of X by taking proper path connected subset of the n simplices. The boundaries of these subspaces are closed surfaces σ_i. From these surfaces we can always find the relevant generalised strips ξ_i in X. The above generalised strips and the way they cross in X will form a redundant strip configuration which associated manifold is A. This already prove the proposition.

Moreover, we can partition the above strips ξ_i in classes C_j of strips that are are isotopic in A. By taking only one element ξ_j for each class we get finally the minimal strip configuration we where looking for. We note explicitly that the final minimal configuration may depends from the starting complex and the way carry on the procedure. This is because there may be more then one strip configuration which has the same associated manifold.

Appendix

A.1 Simple Solid Strips

This appendix contains the full set of solid strips equivalent class configurations. For more details and for the meaning of the $\xi(a_i, b_j)$ notation see [1].

$[\xi]$	Homology Class	ξ	$\partial\xi$	$\chi(\xi)$
1	1	$\xi(g_0, a_0)$	$\mathbf{T}^2 \sqcup \mathbf{T}^2$	0
2	2	$\xi(g_4, a_0)$, $\xi(g_0, a_4)$	$\mathbf{K} \sqcup \mathbf{K}$	0
3	3	$\xi(g_4, a_4)$	$\mathbf{RP}^2 \sqcup \mathbf{RP}^2$	1
4	4	$\xi(g_3, a_4)$, $\xi(g_2, a_4)$, $\xi(g_4, a_3)$, $\xi(g_4, a_2)$	$\mathbf{RP}^2 \vee \mathbf{RP}^2$	2
5	5	$\xi(g_3, a_3)$, $\xi(g_2, a_2)$	$\mathbf{X}_1 \vee \mathbf{X}_1$	2
6	6	$\xi(g_5, a_5)$	$\mathbf{S}^2$	1
7	7	$\xi(g_1, a_1)$	$\mathbf{T}^2$	0
8	7	$\xi(g_1, a_0)$, $\xi(g_0, a_1)$	$\mathbf{T}^2$	0
9	8	$\xi(g_5, a_0)$, $\xi(g_0, a_5)$	$\mathbf{T}^2$	0
10	9	$\xi(g_4, a_1)$, $\xi(g_1, a_4)$	$\mathbf{K}$	0
11	9	$\xi(g_5, a_1)$, $\xi(g_1, a_5)$	$\mathbf{K}$	0
12	10	$\xi(g_6, a_5)$, $\xi(g_7, a_5)$, $\xi(g_5, a_6)$, $\xi(g_5, a_7)$	$\mathbf{X}_1$	2
13	11	$\xi(g_6, a_6)$, $\xi(g_7, a_6)$, $\xi(g_6, a_7)$, $\xi(g_7, a_7)$	$\mathbf{X}_2$	1
14	12	$\xi(g_2, a_3)$, $\xi(g_3, a_2)$	$\mathbf{X}_2$	1
15	12	$\xi(g_3, a_1)$, $\xi(g_2, a_1)$, $\xi(g_1, a_3)$, $\xi(g_1, a_2)$	$\mathbf{X}_2$	1
16	12	$\xi(g_6, a_1)$, $\xi(g_7, a_1)$, $\xi(g_1, a_6)$, $\xi(g_1, a_7)$	$\mathbf{X}_2$	1
17	13	$\xi(g_3, a_0)$, $\xi(g_2, a_0)$, $\xi(g_0, a_3)$, $\xi(g_0, a_2)$	$\mathbf{Y}_1$	0
18	13	$\xi(g_6, a_0)$, $\xi(g_7, a_0)$, $\xi(g_0, a_6)$, $\xi(g_0, a_7)$	$\mathbf{Y}_1$	0
19	14	$\xi(g_6, a_4)$, $\xi(g_7, a_4)$, $\xi(g_4, a_6)$, $\xi(g_4, a_7)$	$\mathbf{Y}_1$	0
20	14	$\xi(g_5, a_3)$, $\xi(g_5, a_2)$, $\xi(g_3, a_5)$, $\xi(g_2, a_5)$	$\mathbf{Y}_1$	0
21	15	$\xi(g_6, a_3)$, $\xi(g_7, a_3)$, $\xi(g_6, a_2)$, $\xi(g_7, a_2)$, $\xi(g_3, a_6)$, $\xi(g_2, a_6)$, $\xi(g_3, a_7)$, $\xi(g_2, a_7)$	$\mathbf{Z}_1$	0
22	N/A	$\xi(g_5, a_4)$, $\xi(g_4, a_5)$	Not Feasible	N/A

Table A.1 : Solid Strips ξ with Strip Classes $[\xi]$, Boundaries $\partial\xi$ and the Euler Characteristics $\chi(\xi)$.

where:

- With the symbol $\sqcup$ (disjoint union) we mean two separate instances of a space which are not path connected.
- Space $\mathbf{X}_1$: is a 2-sphere where two separate points of the sphere are identified. This space has a point where the space is not locally homomorphic to $\mathbb{R}^2$ and therefore it is not a manifold.
- Space $\mathbf{X}_1 \vee \mathbf{X}_1$: is a wedge sum of two $\mathbf{X}_1$ spaces. This space has three points where the space is not locally homomorphic to $\mathbb{R}^2$ and therefore it is not a manifold.
- Space $\mathbf{X}_2$: is a 2-sphere where two couple of separate points of the sphere are identified. This space has two points where the space is not locally homomorphic to $\mathbb{R}^2$ and therefore it is not a manifold.
- Space $\mathbf{Y}_1$: is a 2-torus where two separate points of the torus are identified. This space has a point where the space is not locally homomorphic to $\mathbb{R}^2$ and therefore it is not a manifold.
- Space $\mathbf{Z}_1$: is a Klein Bottle where two separate points of the Klein Bottle are identified. This space has a point where the manifold is not locally homomorphic to $\mathbb{R}^2$ and therefore it is not a manifold.

References

[1] V. Nardozza. *Solid Strips*. http://vixra.org/abs/1910.0039 (2019)