Number Pi , Bernoulli Numbers

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ABSTRACT. This note presents some formulas for Pi.

Los Numeros de Bernoulli

Los números de Bernoulli B_n , $n = 1, 2, 3, ...$, se definen por las series

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} B_n x^{2n}}{(2n)!} \quad , \quad |x| < 2\pi \tag{1}$$

$$1 - \frac{x}{2} \cot\left(\frac{x}{2}\right) = \sum_{n=1}^{\infty} \frac{B_n x^{2n}}{(2n)!} \quad , \quad |x| < \pi \tag{2}$$

algunos números de Bernoulli son

$$B_n = \left\{\frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \frac{1}{30}, \frac{5}{66}, \frac{691}{2730}, \frac{7}{6}, ...\right\} \tag{3}$$

una fórmula explicita es

$$B_n = \sum_{k=2}^{2n+1} \frac{(-1)^{n+k}}{k} \binom{2n+1}{k} \sum_{m=1}^{k-1} m^{2n} \quad , \quad n = 1, 2, 3, ... \tag{4}$$

El Numero Pi

El número Pi se define por la serie

$$\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \tag{5}$$

algunas representaciones alternativas son

$$\pi = 4 \int_0^1 \frac{1}{1 + x^2} \, dx \tag{6}$$

$$\pi = 2 \int_0^1 \frac{1}{\sqrt{1 - x^2}} \, dx \tag{7}$$

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}}\ \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\ \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\ \ldots \tag{8}$$

$$\frac{2}{\pi} = \prod_{n=1}^{\infty}\left(1 - \frac{1}{(2\,n)^2}\right) \tag{9}$$

$$\frac{1}{\pi} = \sum_{n=0}^{\infty}\binom{2\,n}{n}^3 \frac{42\,n+5}{2^{12\,n+4}} \tag{10}$$

Formulas

Entry 1. Si $z = 1.292695719373\ldots$, $\cos z = e^{-z}$, entonces

$$\pi = 2\,z + \sum_{n=1}^{\infty}\frac{(-1)^{n-1} B_{2\,n-1}\, 2^{2\,n}\, z^{4\,n-2}}{(2\,n-1)\,(4\,n-2)!} \tag{11}$$

Entry 2. Si $z = 0.909409711174\ldots$, $\left(1+\frac{1}{\sqrt{3}}\right)\cos z - \left(1-\frac{1}{\sqrt{3}}\right)\sin z = \left(1+\frac{1}{\sqrt{3}}\right)e^{-z}$, entonces

$$\pi = 3\,z + \frac{3}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1} B_{2\,n-1}\, 2^{2\,n}\, z^{4\,n-2}}{(2\,n-1)\,(4\,n-2)!} \tag{12}$$

Entry 3. Si $z = 0.703032924612\ldots$, $\sqrt{2}\,\cos z - \left(2-\sqrt{2}\right)\sin z = \sqrt{2}\,e^{-z}$, entonces

$$\pi = 4\,z + 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1} B_{2\,n-1}\, 2^{2\,n}\, z^{4\,n-2}}{(2\,n-1)\,(4\,n-2)!} \tag{13}$$

Entry 4. Si $z = 0.484479802662\ldots$, $\left(3-\sqrt{3}\right)\cos z - \left(\sqrt{3}-1\right)\sin z = \left(3-\sqrt{3}\right)e^{-z}$, entonces

$$\pi = 6\,z + 3\sum_{n=1}^{\infty}\frac{(-1)^{n-1} B_{2\,n-1}\, 2^{2\,n}\, z^{4\,n-2}}{(2\,n-1)\,(4\,n-2)!} \tag{14}$$

Referencias

1. Beckmann, P.: A History of π , 3rd ed. New York: Dorset Press, 1989.
2. Blatner, D.: The Joy of π , New York: Walker, 1997.
3. Olver, F.W.J. et al.: NIST Handbook of Mathematical Functions. Cambridge University Press,2010.