Study of the 3n+1 problem for a number which never « land » and others properties

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Abstract: In the present paper, we study the 3n+1 problem to know if there is a theoretical number such as the number never land on the cycle $\{4; 2; 1\}$ and grows continually.

We will specify the general formula such as the number of uneven steps $(3n + 1)$ and even steps $\left(\frac{n}{2}\right)$ are equals.

We want to know also the uneven number form $2^n + 2^0$ and its composition during n uneven $(3n + 1)$ and even steps $(\frac{n}{2})$.

The composition number $2^n + 2^0$ will be viewed in two ways :

1. When the powers which compose the numbers are expanded. We will recognize the Pascal's triangle.

2. When the powers which compose the numbers are agglutinate. We will recognize $(3^m)_2$.

Keywords: Collatz problem, 3n+1 problem, syracuse conjecture, pascal's triangle, binary forms.

1 Introduction

The $3n + 1$ problem (known as : Syracuse conjecture, Collatz conjecture) has been studied and presented on a number of occasions (Collatz, 1986, [1], Wirsching, 1998, [2], Lagarias, 1985, [3]).

Here the function :

Definition 1 :

$\forall\, n \in \mathbb{N}$, if n is uneven we applied $3n + 1$ otherwise $\frac{n}{2}$.

$$C(n) = \begin{cases} \frac{n}{2}\ if\ \ n\ mod\ \ 2 = 0 \\ 3n + 1\ \ otherwise \end{cases} \qquad (1)$$

$C(n)$ compressed :

$$C(n) = \begin{cases} \frac{n}{2} if\ \ n\ mod\ \ 2 = 0 \\ \frac{3n+1}{2}\ \ otherwise \end{cases} \quad (2)$$

The $3n+1$ problem states that for each $n \in \mathbb{N}^*, C^{(m)}(n) = 1$ (the function C is repeated m times to n result to 1.
Jean-Paul Delahaye, french specialist on the $3n+1$ problem published many papers about this conjecture, notably that published in *Pour la Science* in May 1998, [4].

It shows the meanings of *flight duration, flight altitude, maximum altitude,* its variants and its indecidability.

In 2017, the article : Luc-Olivier Pochon, Alain Favre. La suite de Syracuse, un monde de conjectures. 2017. Hal-01593181, [5] is published, it gathers our current state of information and knowledge about the $3n+1$ problem.

In the Collatz conjecture a number which never land is a number which never reach the number 1 which implies the endless cycle $\{4; 2; 1\}$.

In what follows, in the section II, we will look at the theorical number which never land and grows continually at each step.

The general formula such as uneven steps $(3n+1)$ and even steps $\left(\frac{n}{2}\right)$ are alternated each time will be showed at the section III.

In the paragraph IV, we will presented the properties on expanded powers (we not use $2^{n+1} = 2^n + 2^n$) and in the paragraph V, it will be the properties on agglutinated powers (we use $2^{n+1} = 2^n + 2^n$).
Finally, the conclusion will be in the part VI.

2 Theoretical number which never land and grows at each step

We are interessed by the particular form of n such as :

$$\boldsymbol{n} = \sum_{k=0}^{k} \mathbf{2^k}$$

Let see this particular n number with the examples :

$$\boldsymbol{n_1} = \sum_{k=0}^{3} 2^k$$

$$\boldsymbol{n_2} = \sum_{k=0}^{4} 2^k$$

$$\boldsymbol{n_3} = \sum_{k=0}^{5} 2^k$$

Vocabulary :

$$\boldsymbol{n_1} = 2^3 + 2^2 + 2^1 + 2^0 = 15$$

$\mathbf{2^0}$: represent the term which define the uneven number.

$\mathbf{2^1 + 2^2 + 2^3} + \cdots$: these terms (from 2^1 to 2^n in consecutives powers only) represent the queue.
The head represent the others terms.
Calculate the serie with first element equal n_1 (At each step we apply $\frac{3n_m+1}{2}$)(2)

$$\boldsymbol{n_1} = 2^3 + 2^2 + 2^1 + 2^0 = 15$$
$$\boldsymbol{n_1} = 2^4 + 2^2 + 2^1 + 2^0 = 23$$
$$\boldsymbol{n_1} = 2^5 + 2^1 + 2^0 = 35$$
$$\boldsymbol{n_1} = 2^5 + 2^4 + 2^2 + 2^0 = 53$$

The queue disappears after **3** steps of $\frac{3n_m+1}{2}$.(2)

Calculate the serie with first element equal n_2 (At each step we apply $\frac{3n_m+1}{2}$)(2)

$$\boldsymbol{n_2} = 2^4 + 2^3 + 2^2 + 2^1 + 2^0 = 31$$
$$\boldsymbol{n_2} = 2^5 + 2^3 + 2^2 + 2^1 + 2^0 = 47$$
$$\boldsymbol{n_2} = 2^6 + 2^2 + 2^1 + 2^0 = 71$$
$$\boldsymbol{n_2} = 2^6 + 2^5 + 2^3 + 2^1 + 2^0 = 107$$
$$\boldsymbol{n_2} = 2^7 + 2^5 + 2^0 = 161$$

The queue disappears after **4** steps of $\frac{3n_m+1}{2}$.(2)

Calculate the serie with first element equal n_3 (At each step we apply $\frac{3n_m+1}{2}$)(2)

$$\boldsymbol{n_3} = 2^5 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0 = 63$$
$$\boldsymbol{n_3} = 2^6 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0 = 95$$
$$\boldsymbol{n_3} = 2^7 + 2^3 + 2^2 + 2^1 + 2^0 = 143$$
$$\boldsymbol{n_3} = 2^7 + 2^6 + 2^4 + 2^2 + 2^1 + 2^0 = 215$$
$$\boldsymbol{n_3} = 2^8 + 2^6 + 2^1 + 2^0 = 323$$
$$\boldsymbol{n_3} = 2^8 + 2^7 + 2^6 + 2^5 + 2^2 + 2^0 = 485$$

The queue disappears after **5** steps of $\frac{3n_m+1}{2}$.(2)

The agglutination with powers and the construction of the theoretical number.

When the terms are composed by consecutives powers as permitted by the formula $\sum_{k=0}^{k} 2^k$, there is agglutination (since $2^n + 2^n = 2^{n+1}$)

Example :

$n = 2^3 + 2^2 + 2^1 + 2^0$

According to the series when a number is even :

$$\begin{aligned} 3n + 1 &= (2+1)n + 2^0 \\ &= (2+1)(2^3 + 2^2 + 2^1 + 2^0) + 2^0 \\ &= 2^4 + 2^3 + 2^3 + 2^2 + 2^2 + 2^1 + 2^1 + 2^0 + 2^0 \\ &= 2^4 + 2^4 + 2^3 + 2^2 + 2^1 \text{ (Agglutination)} \\ &= 2^5 + 2^3 + 2^2 + 2^1 \text{ (Agglutination)} \end{aligned}$$

We divide by 2 :

$$= 2^4 + 2^2 + 2^1 + 2^0$$

If $n = \sum_{k=0}^{k} 2^k$

When we apply between 0 et k steps $\frac{3n+1}{2}$, we have :

$$n = \underbrace{\sum_{m=k+p}^{m} 2^m}_{\textbf{Head}} + \underbrace{\sum_{k=0}^{k} 2^k}_{\textbf{Queue}} \text{ (With p set)}$$

The term of the highest power of the queue always agglutinate (2^k of $\sum_{k=0}^{k} 2^k$)

Because we have :

$$(2+1)\sum_{n=0}^{n} 2^n + 2^0$$

$$= (2+1)(2^n + 2^{n-1} + 2^{n-2} + \cdots + 2^2 + 2^1 + 2^0) + 2^0$$

$= 2^{n+1} + 2^n + 2^n + 2^{n-1} + 2^{n-1} + 2^{n-2} + \cdots + 2^3 + 2^2 + 2^2 + 2^1 + 2^1 + 2^0 + 2^0$
$= 2^{n+1} + 2^{n+1} + 2^n + 2^{n-1} + \cdots + 2^4 + 2^3 + 2^2 + 2^1$ (Agglutination)
$= 2^{n+2} + 2^n + 2^{n-1} + \cdots + 2^4 + 2^3 + 2^2 + 2^1$ (Agglutination)
$= \ 2^{n+2} + \sum_{n=1}^{n} 2^n$ (Even)

So we divide by 2.

Then :

$$\frac{2^{n+2} + \sum_{n=1}^{n} 2^n}{2}$$

$$= \frac{2^{n+2} + 2^n + 2^{n-1} + \cdots + 2^4 + 2^3 + 2^2 + 2^1}{2}$$

$$= 2^{n+1} + 2^{n-1} + 2^{n-2} + \cdots + 2^3 + 2^2 + 2^1 + 2^0$$

We notice that the highest power of the queue (2^n) have disappeared.
We delete the highest consecutive power of the queue without disrupt the rest of the queue.
At each compressed operation $\frac{3n+1}{2}(2)$, we delete the highest power of the queue, by providing the head.
So the series $\sum_{n=0}^{n} 2^n$ oscillate n times (step $\frac{3n+1}{2}$)(2) before to have a behaviour not calculated because the series will not to have queue anymore.
A queue composed by consecutive terms endless will oscillate endless.
Moreover, the step $\frac{3n+1}{2}(2)$ is strictly increasing.
Hence the number :
$\sum_{n=0}^{\infty} 2^n$ oscillate between the uneven terms and even terms in a perfect way (symmetrical) and never land. It is also strictly increasing.

3 General formula such as uneven steps and even steps are alternated

We want to write the general formula such as :

$$\frac{Uneven\ \ steps}{Even\ \ steps} = 1$$

If $n \in \mathbb{N}^* = N_0$ (1^{st} term of hypothetical series)

$$N_1 = \frac{3N_0+1}{2}$$

$$N_2 = \frac{3^2N_0+(3+2)}{2^2}$$

$$N_3 = \frac{3^3N_0+(3^2+3\times2+2^2)}{2^3}$$

$$N_4 = \frac{3^4N_0+(3^3+3^2\times2+3\times2^2+2^3)}{2^4}$$

$$N_5 = \frac{3^5N_0+(3^4+3^3\times2+3^2\times2^2+3\times2^3+\ 2^4)}{2^5}$$

The terms was reduced under the expression of the first term.
It comes from the recurring series : $N_m = \frac{3N_{m-1}+1}{2}$

Example
Calculated for N_2:
$N_1 = \frac{3N_0+1}{2}$ and $N_2 = \frac{3N_1+1}{2}$
We replace N_1 by $\frac{3N_0+1}{2}$ in the expression $N_2 = \frac{3N_1+1}{2}$
Hence :

$$N_2 = \frac{3\left(\frac{3N_0+1}{2}\right)+1}{2}$$

We develop :

$$N_2 = \frac{\frac{3^2N_0+3+2}{2}}{2} = \frac{3^2N_0+3+2}{2^2}$$

Due to the differents terms calculated previously :

$$\boldsymbol{N_m} = \frac{\boldsymbol{3^m N_0 + \sum_{k=0}^{m-1} 3^{m-1-k}.2^k}}{\boldsymbol{2^m}}$$

Where in our example,

$\boldsymbol{N_m}$ corresponds to $\boldsymbol{N_2}$

$\boldsymbol{3^m N_0}$ corresponds to $\boldsymbol{3^2 N_0}$

$\boldsymbol{\sum_{k=0}^{m-1} 3^{m-1-k}.2^k}$ corresponds to $\boldsymbol{3+2}$

$\boldsymbol{2^m}$ corresponds to $\boldsymbol{2^2}$

4 General formula on the expanded powers of $2^m + 2^0$

Considerate the uneven number such as : $2^m + 2^0$
We will apply the operation $\frac{3n+1}{2}$ between each step.
We will not agglutinate the powers such as $2^m + 2^m = 2^{m+1}$ to obtain the expanded powers.
Step 0 : $2^m + 2^0$
Step 1 : $2^{m-1} + 2^{m-2} + 2^0$
Step 2 : $2^{m-2} + 2^{m-3} + 2^{m-3} + 2^{m-4} + 2^0$
Step 3 : $2^{m-3} + 2^{m-4} + 2^{m-4} + 2^{m-4} + 2^{m-5} + 2^{m-5} + 2^{m-5} + 2^{m-6} + 2^0$
Step 4 : $2^{m-4} + 2^{m-5} + 2^{m-5} + 2^{m-5} + 2^{m-5} + 2^{m-6} + 2^{m-6} + 2^{m-6} + 2^{m-6} + 2^{m-6} + 2^{m-6} + 2^{m-7} + 2^{m-7} + 2^{m-7} + 2^{m-7} + 2^{m-8} + 2^0$
Step 5 : $2^{m-5} + 2^{m-6} + 2^{m-6} + 2^{m-6} + 2^{m-6} + 2^{m-6} + 2^{m-7} + 2^{m-7} + 2^{m-7} + 2^{m-7} + 2^{m-7} + 2^{m-7} + 2^{m-7} + 2^{m-7} + 2^{m-7} + 2^{m-7} + 2^{m-8} + 2^{m-8} + 2^{m-8} + 2^{m-8} + 2^{m-8} + 2^{m-8} + 2^{m-8} + 2^{m-8} + 2^{m-8} + 2^{m-8} + 2^{m-9} + 2^{m-9} + 2^{m-9} + 2^{m-9} + 2^{m-9} + 2^{m-10} + 2^0$
We notice :
According to the step N, if we gather the number of identical powers such as :
Step N : $\sum_{p=N}^{p} \sum_{q=0}^{q} 2^{n-p} + 2^0$
Example :
At the step 4, we count the occurrences of the differents powers :
Occurrence of $m-4$: 1
Occurrence of $m-5$: 4
Occurrence of $m-6$: 6
Occurrence of $m-7$: 4

Occurrence of $m-8$: 1

We notice the number of occurrences for the powers without agglutinate at the step N corresponds to $N+1$ line of Pascal triangle (Figure 1).

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1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

Figure 1. Pascal's triangle

We have also for the p step :

First power : $n-p$

Last power : $n-2.p$

Number of power : 2^p

5 General formula on the agglutinated powers of 2^m+2^0

Considerate the uneven number such as : 2^m+2^0

We will apply the operation $\frac{3n+1}{2}$ between each step.

We will not agglutinate the powers such as $2^m+\ 2^m=\ 2^{m+1}$ to obtain the agglutinated powers.

Step 0 : 2^m+2^0

Step 1 : $2^{m-1}+2^{m-2}+\ 2^0$

Step 2 : $2^{m-2}+2^{m-3}+2^{m-3}+2^{m-4}+2^0$

$=\ 2^{m-2}+2^{m-2}+2^{m-4}+2^0$

$=\ 2^{m-1}+2^{m-4}+2^0$

Step 3 : $2^{m-3}+2^{m-4}+2^{m-4}+2^{m-4}+2^{m-5}+2^{m-5}+2^{m-5}+2^{m-6}+2^0$

$=2^{m-3}+2^{m-3}+2^{m-4}+2^{m-4}+2^{m-5}+2^{m-6}+2^0$

$=2^{m-2}+2^{m-3}+2^{m-5}+2^{m-6}+2^0$

Step 4 : $2^{m-4}+2^{m-5}+2^{m-5}+2^{m-5}+2^{m-5}+2^{m-6}+2^{m-6}+2^{m-6}+2^{m-6}+2^{m-6}+2^{m-6}+2^{m-7}+2^{m-7}+2^{m-7}+2^{m-7}+2^{m-8}+2^0$

$=2^{m-4}+2^{m-4}+2^{m-4}+2^{m-5}+2^{m-5}+2^{m-5}+2^{m-6}+2^{m-6}+2^{m-8}+2^0$

$=2^{m-3}+2^{m-4}+2^{m-4}+2^{m-5}+2^{m-5}+2^{m-8}+2^0$

$=2^{m-3}+2^{m-3}+2^{m-4}+2^{m-8}+2^0$

$=2^{m-2}+2^{m-4}+2^{m-8}+2^0$

Step 5 : $2^{m-5}+2^{m-6}+2^{m-6}+2^{m-6}+2^{m-6}+2^{m-6}+2^{m-7}+2^{m-7}+2^{m-7}+2^{m-7}+2^{m-7}+2^{m-7}+2^{m-7}+2^{m-7}+2^{m-7}+2^{m-7}+2^{m-8}+2^{m-8}+2^{m-8}+2^{m-8}+2^{m-8}+2^{m-8}+2^{m-8}+2^{m-8}+2^{m-8}+2^{m-8}+2^{m-9}+2^{m-9}+2^{m-9}+2^{m-9}+2^{m-9}+2^{m-10}+2^0$

$=2^{m-5}+2^{m-5}+2^{m-5}+2^{m-6}+2^{m-6}+2^{m-6}+2^{m-6}+2^{m-6}+2^{m-6}+2^{m-7}+2^{m-7}+2^{m-7}+2^{m-7}+2^{m-7}+2^{m-8}+2^{m-8}+2^{m-9}+2^{m-10}+2^0$

$=2^{m-4}+2^{m-5}+2^{m-5}+2^{m-5}+2^{m-5}+2^{m-6}+2^{m-6}+2^{m-7}+2^{m-7}+2^{m-9}+2^{m-10}+2^0$

$=2^{m-4}+2^{m-4}+2^{m-4}+2^{m-5}+2^{m-6}+2^{m-9}+2^{m-10}+2^0$

$$= 2^{m-3} + 2^{m-4} + 2^{m-5} + 2^{m-6} + 2^{m-9} + 2^{m-10} + 2^0$$

Here the agglutinated powers of the finals steps :
Step 1 : $m - 1; m - 2$
Step 2 : $m - 1; m - 4$
Step 3 : $m - 2; m - 3; m - 5; m - 6$
Step 4 : $m - 2; m - 4; m - 8$
Step 5 : $m - 3; m - 4; m - 5; m - 6; m - 9; m - 10$
Represent the writing of agglutinated powers under the binary form :
Step 1 : 1 1
Step 2 : 1 0 0 1
Step 3 : 1 1 0 1 1
Step 4 : 1 0 1 0 0 0 1
Step 5 : 1 1 1 1 0 0 1 1
Let see the representation of binary number under the decimal form :
Step 1 : $3 = 3^1$
Step 2 : $9 = 3^2$
Step 3 : $27 = 3^3$
Step 4 : $81 = 3^4$
Step 5 : $729 = 3^5$
We have :
Agglutinated powers of step $\text{N} = (3^N)_2$ with the least significant bit $= (n - 2.N)$
Example :
Step 3 :

$$(3^3)_2 = 1\ 1\ 0\ 1\ 1$$

2^4	2^3	2^2	2^1	2^0
1	1	0	1	1

Table 1. Mapping 3^3 and agglutinated powers

We take the least significant bit (2^0)

We apply : $n - 2.3 = n - 6$.
We have :

2^4	2^3	2^2	2^1	2^0
1	1	0	1	1
$n - 2$	$n - 3$	$n - 4$	$n - 5$	$n - 6$

Table 2. Mapping 3^3 and agglutinated powers with powers displayed

For the agglutinated powers of step 3 :

$$n - 2; n - 3; n - 5; n - 6$$

Which is the case. At the step 3, the number $2^m + 2^0$ is write :

$$2^{m-2} + 2^{m-3} + 2^{m-5} + 2^{m-6} + 2^0$$

6 Conclusion

Through this paper, we have seen a theoretical number which never land and continually grows under the form $\sum_{n=0}^{\infty} 2^n$,

We have made a relation between the number $2^n + 2^0$ and the Pascal's triangle and the binary forms of (3^m).
Also the general formula such as uneven and even steps are alternated had been showed.

For another paper we will look at the behaviour for powers greater than n with the number $\sum_{n=0}^{n} 2^n$ during the steps $(\frac{3n+1}{2})(2)$.

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