A nice rational estimator of $\{\sqrt{n}\}$

Juan Moreno Borrallo

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email: juan.morenoborrallo@gmail.com

Abstract

In this paper it is proposed a nice rational estimator of the fractional part of the square root of any positive integer n.

1 Main result

Theorem. *Let it be n some positive integer number, $\lfloor\sqrt{n}\rfloor$ the integer part of the square root of n, and $\{\sqrt{n}\}$ the fractional part of the square root of n. Then, we can affirm that*

$$n \geq \left(\lfloor\sqrt{n}\rfloor + \frac{n-\lfloor\sqrt{n}\rfloor^2}{2\lfloor\sqrt{n}\rfloor+1}\right)^2 > n - \frac{1}{4}$$

Corollary.

$$\{\sqrt{n}\} \geq \frac{n-\lfloor\sqrt{n}\rfloor^2}{2\lfloor\sqrt{n}\rfloor+1} > \{\sqrt{n}\} - \left(\sqrt{n} - \sqrt{n-\frac{1}{4}}\right)$$

2 Proof of the main result

2.1 Proof of the Theorem

Expanding, we find that

$$\left(\lfloor\sqrt{n}\rfloor + \frac{n-\lfloor\sqrt{n}\rfloor^2}{2\lfloor\sqrt{n}\rfloor+1}\right)^2 =$$

$$= \lfloor\sqrt{n}\rfloor^2 + \frac{\left(n-\lfloor\sqrt{n}\rfloor^2\right)^2}{\left(2\lfloor\sqrt{n}\rfloor+1\right)^2} + 2\lfloor\sqrt{n}\rfloor\left(\frac{n-\lfloor\sqrt{n}\rfloor^2}{2\lfloor\sqrt{n}\rfloor+1}\right)$$

Using the identity

$$\frac{a}{a+1}+\frac{1}{a+1}=1$$

And asigning the value $a=2\lfloor\sqrt{n}\rfloor$, we get that

$$2\lfloor\sqrt{n}\rfloor\left(\frac{n-\lfloor\sqrt{n}\rfloor^2}{2\lfloor\sqrt{n}\rfloor+1}\right)=\left(n-\lfloor\sqrt{n}\rfloor^2\right)-\frac{n-\lfloor\sqrt{n}\rfloor^2}{2\lfloor\sqrt{n}\rfloor+1}$$

Substituting, we find that

$$\left(\lfloor\sqrt{n}\rfloor+\frac{n-\lfloor\sqrt{n}\rfloor^2}{2\lfloor\sqrt{n}\rfloor+1}\right)^2=$$

$$=\lfloor\sqrt{n}\rfloor^2+\frac{\left(n-\lfloor\sqrt{n}\rfloor^2\right)^2}{\left(2\lfloor\sqrt{n}\rfloor+1\right)^2}+\left(n-\lfloor\sqrt{n}\rfloor^2\right)-\frac{n-\lfloor\sqrt{n}\rfloor^2}{2\lfloor\sqrt{n}\rfloor+1}$$

As

$$\lfloor\sqrt{n}\rfloor^2+\left(n-\lfloor\sqrt{n}\rfloor^2\right)=n$$

Substituting, we get that

$$\left(\lfloor\sqrt{n}\rfloor+\frac{n-\lfloor\sqrt{n}\rfloor^2}{2\lfloor\sqrt{n}\rfloor+1}\right)^2=$$

$$=n+\frac{\left(n-\lfloor\sqrt{n}\rfloor^2\right)^2}{\left(2\lfloor\sqrt{n}\rfloor+1\right)^2}-\frac{n-\lfloor\sqrt{n}\rfloor^2}{2\lfloor\sqrt{n}\rfloor+1}$$

Expanding, we obtain that

$$\frac{\left(n-\lfloor\sqrt{n}\rfloor^2\right)^2}{\left(2\lfloor\sqrt{n}\rfloor+1\right)^2}-\frac{n-\lfloor\sqrt{n}\rfloor^2}{2\lfloor\sqrt{n}\rfloor+1}=$$

$$=\frac{\left(n-\lfloor\sqrt{n}\rfloor^2\right)^2-\left(n-\lfloor\sqrt{n}\rfloor^2\right)\left(2\lfloor\sqrt{n}\rfloor+1\right)}{\left(2\lfloor\sqrt{n}\rfloor+1\right)^2}=$$

$$=\frac{\left(n-\lfloor\sqrt{n}\rfloor^2\right)\left(\left(n-\lfloor\sqrt{n}\rfloor^2\right)-\left(2\lfloor\sqrt{n}\rfloor+1\right)\right)}{\left(2\lfloor\sqrt{n}\rfloor+1\right)^2}$$

The maximum of $n-\lfloor\sqrt{n}\rfloor^2$ can be found at $2\lfloor\sqrt{n}\rfloor$, as by definition of the integer part $\lfloor\sqrt{n}\rfloor$,

$$n<\left(\lfloor\sqrt{n}\rfloor+1\right)^2=\lfloor\sqrt{n}\rfloor^2+2\lfloor\sqrt{n}\rfloor+1$$

Subsequently, the value of the expression $\left(n-\lfloor\sqrt{n}\rfloor^2\right)-(2\lfloor\sqrt{n}\rfloor+1)$ is always less than 0.

Besides, the expression $\left(n-\lfloor\sqrt{n}\rfloor^2\right)\left(\left(n-\lfloor\sqrt{n}\rfloor^2\right)-(2\lfloor\sqrt{n}\rfloor+1)\right)$ is maximized at the value $n-\lfloor\sqrt{n}\rfloor^2=\frac{2\lfloor\sqrt{n}\rfloor+1}{2}$. As this value can not exist, being $n-\lfloor\sqrt{n}\rfloor^2$ some positive integer and $\frac{2\lfloor\sqrt{n}\rfloor+1}{2}$ not being some positive integer, we get that

$$0\geq\frac{\left(n-\lfloor\sqrt{n}\rfloor^2\right)\left(\left(n-\lfloor\sqrt{n}\rfloor^2\right)-(2\lfloor\sqrt{n}\rfloor+1)\right)}{\left(2\lfloor\sqrt{n}\rfloor+1\right)^2}>\frac{\left(\frac{2\lfloor\sqrt{n}\rfloor+1}{2}\right)\left(\left(\frac{2\lfloor\sqrt{n}\rfloor+1}{2}\right)-(2\lfloor\sqrt{n}\rfloor+1)\right)}{\left(2\lfloor\sqrt{n}\rfloor+1\right)^2}$$

Expanding the right side of the inequation, we get that

$$\frac{\left(\frac{2\lfloor\sqrt{n}\rfloor+1}{2}\right)\left(\left(\frac{2\lfloor\sqrt{n}\rfloor+1}{2}\right)-(2\lfloor\sqrt{n}\rfloor+1)\right)}{\left(2\lfloor\sqrt{n}\rfloor+1\right)^2}=$$

$$=\frac{\left(\frac{2\lfloor\sqrt{n}\rfloor+1}{2}\right)\left(-\left(\frac{2\lfloor\sqrt{n}\rfloor+1}{2}\right)\right)}{\left(2\lfloor\sqrt{n}\rfloor+1\right)^2}=$$

$$=\frac{-\left(\frac{2\lfloor\sqrt{n}\rfloor+1}{2}\right)^2}{\left(2\lfloor\sqrt{n}\rfloor+1\right)^2}=-\frac{1}{4}$$

Subsequently, substituting, we find that

$$0\geq\frac{\left(n-\lfloor\sqrt{n}\rfloor^2\right)\left(\left(n-\lfloor\sqrt{n}\rfloor^2\right)-(2\lfloor\sqrt{n}\rfloor+1)\right)}{\left(2\lfloor\sqrt{n}\rfloor+1\right)^2}>-\frac{1}{4}$$

And therefore, we get that

$$n\geq\left(\lfloor\sqrt{n}\rfloor+\frac{n-\lfloor\sqrt{n}\rfloor^2}{2\lfloor\sqrt{n}\rfloor+1}\right)^2>n-\frac{1}{4}$$

As we wanted to prove.

2.2 Proof of the Corollary

By definition,

$$\lfloor\sqrt{n}\rfloor+\{\sqrt{n}\}=\sqrt{n}$$

By the Theorem proved,

$$\sqrt{n} \geq \lfloor\sqrt{n}\rfloor + \frac{n - \lfloor\sqrt{n}\rfloor^2}{2\lfloor\sqrt{n}\rfloor + 1} > \sqrt{n - \frac{1}{4}}$$

Therefore, substracting, we get that

$$\lfloor\sqrt{n}\rfloor + \{\sqrt{n}\} - \lfloor\sqrt{n}\rfloor - \frac{n - \lfloor\sqrt{n}\rfloor^2}{2\lfloor\sqrt{n}\rfloor + 1} < \sqrt{n} - \sqrt{n - \frac{1}{4}}$$

$$\{\sqrt{n}\} - \frac{n - \lfloor\sqrt{n}\rfloor^2}{2\lfloor\sqrt{n}\rfloor + 1} < \sqrt{n} - \sqrt{n - \frac{1}{4}}$$

As we wanted to prove.