A Method of Determining the Extreme Points of a Function Defined on the Integers

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Abstract. The standard method of determining the extreme points of a function $f(x)$ is to set its first derivative equal to zero and solve for x. However, this method requires that the function be continuous (at least piecewise) and differentiable; it won't work for a function defined on the integers. Described herein is a method of determining the extreme points of a function defined on the integers. This method is illustrated by using it to solve two example problems.

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The standard method of determining the extreme points of a function $f(x)$ is to set its first derivative equal to zero and solve for x. However, this method requires that the function be continuous (at least piecewise) and differentiable; it won't work for a function defined on the integers.

For example, consider the binomial function

$$B(n,k,p) = \frac{n!}{k!\,(n-k)!}p^k(1-p)^{n-k},$$

where $B(n,k,p)$ is the probability that for n independent trials, each having a probability p of success, there will be exactly k successful trials.

Suppose we wish to determine the most likely number of successes, say, k_m for fixed n and p. The standard method described above won't work because the factorial function in which k occurs is defined only for integers. Described below is a method that will make it possible to determine k_m.

We will make use of the forward difference operator $\Delta[f(x)]$, where

$$\Delta[f(x)] = f(x+1) - f(x)$$

for any function $f(x)$.

Theorem. Let $I(n)$ be a function defined for integers and let $f(x)$ be an extension of $I(n)$ to the real numbers. Suppose that

(1) $\Delta[f(x_0)] = 0$,

(2) $\Delta[f(x)] > 0$ for $x < x_0$, and

(3) $\Delta[f(x)] < 0$ for $x > x_0$.

Then if x_0 is an integer, both $I(x_0)$ and $I(x_0 + 1)$ are the maxima of $I(n)$. Otherwise, the maximum of $I(n)$ occurs at the integer between x_0 and $x_0 + 1$.

Proof.

Case 1: x_0 is an integer.

It follows from the definition of the forward difference operator that for $n < x_0$,

$$I(x_0) = I(n) + \sum_{i=n}^{x_0-1} \Delta[f(i)].$$

From Condition (2), it follows that all of the terms in the sum are positive, so $I(x_0) > I(n)$ for $n < x_0$

For $n > x_0 + 1$,

$$I(n) = I(x_0 + 1) + \sum_{i=x_0+1}^{n-1} \Delta[f(i)].$$

From Condition (3), it follows that all of the terms in the sum are negative, so $I(x_0 + 1) > I(n)$ for $n > x_0 + 1$. Since from Condition (1), $I(x_0) = I(x_0 + 1)$, it follows that both $I(x_0)$ and $I(x_0 + 1)$ are the maxima of $I(n)$.

Case 2: x_0 is not an integer.

Let n_m be the integer between x_0 and $x_0 + 1$. For $n < n_m$,

$$I(n_m) = I(n) + \sum_{i=n}^{n_m-1} \Delta[f(i)].$$

Since $n_m - 1 < x_0$, it follows from Condition (2) that all of the terms in the sum are positive, so $I(n_m) > I(n)$ for $n < n_m$.

For $n > n_m$,

$$I(n) = I(n_m) + \sum_{i=n_m}^{n-1} \Delta[f(i)].$$

Since $n_m > x_0$, it follows from Condition (3) that all of the terms in the sum are negative, so $I(n_m) > I(n)$ for $n > n_m$. Therefore, $I(n_m)$ is the maximum of $I(n)$. □

A similar theorem can be proved for use in finding the location of the minimum value(s) of a function defined for integer values. This is left as an exercise for the interested reader.

Now let's apply the above theorem to finding the location of the maximum value(s) of $B(n,k,p)$ for fixed n and p. In order to satisfy the hypothesis of the theorem, we need to extend $B(n,k,p)$ to be defined for real values of k; let $B(n,x,p)$ be such an extension. Since k appears in the factorial function, we need to likewise extend that function to the real numbers; call that extension $x!$. As we shall see, the solution does not depend on the extension we choose; it is sufficient that $x! > 0$ for all $x \geq 0$. Since $n! > 0$ for all $n \geq 0$, the piecewise linear function constructed by connecting the points of $n!$ defined at the integers will do the trick.

Applying the forward difference operator to the binomial function results in

$$\begin{aligned}\Delta_x[B(n,x,p)] &= B(n,x+1,p) - B(n,x,p)\\ &= \frac{n!}{(x+1)!\,[n-(x+1)]!}p^{x+1}(1-p)^{n-(x+1)} - \frac{n!}{x!\,(n-x)!}p^x(1-p)^{n-x}\\ &= \frac{n!\,p^x(1-p)^{n-x-1}}{(x+1)!\,(n-x)!}[(n-x)p-(x+1)(1-p)]\\ &= \frac{n!\,p^x(1-p)^{n-x-1}}{(x+1)!\,(n-x)!}[np+p-1-x].\end{aligned}$$

For $p = 0$, the most likely and in fact the only possible number of successes is zero. For $p = 1$, the most likely and in fact the only possible number of successes is n. Otherwise, $0 < p < 1$, in which case all of the factors in the above fraction are positive, in which case setting $\Delta_x[B(n,x_0,p)] = 0$ requires the quantity in brackets to be zero. Solving for x_0 results in

$$x_0 = np + p - 1.$$

Since the quantity in brackets is a decreasing function of x, it follows that $\Delta_x[B(n,x,p)] > 0$ for $x < x_0$ and $\Delta_x[B(n,x,p)] < 0$ for $x > x_0$. Thus, according to the theorem above, if x_0 is an integer, then the maxima of $B(n,x,p)$ occur at $x = np + p - 1$ and $x = np + p$; otherwise, the maximum occurs at the integer between those two values. In the special case of $p = ½$, the maxima occur at $½(n \pm 1)$ for n odd and at $½n$ for n even, as expected.

Now let's apply this method to solving a practical problem. Suppose that a restaurant is having a weekly drawing for free lunches. Those who wish to enter the drawing place their business cards in a glass jar. Ten winning entries are to be drawn from the jar. You estimate that there are 200 entries already in the jar and it's near closing time on Friday, the last day to enter for the week, so there isn't likely to be more than a very few if any additional entries. How many business cards should you submit in order to maximize your chance of winning, given that if more than one of your cards are drawn, you will be disqualified?

I find it advantageous to represent the known quantities by variables rather than numbers. Doing so reduces the amount of computation required as well as the likelihood of error and enables the problem to be solved in greater generality. Once the general problem has been solved, the numbers can be substituted for the variables and the answer can then be computed for the problem at hand. The only exception is for problems that cannot be solved analytically and must, therefore, be solved numerically. However, even in this case, the numerical substitution should be postponed as long as possible.

Let n be the number of entries initially in the jar (200 in this case); let k be the number of winning entries drawn (10 in this case); let x be the number of entries that you submit; and let $P(x)$ be the probability of winning if you submit x entries. I leave it as an exercise for the interested reader to verify that

$$P(x) = \frac{kx\prod_{i=0}^{k-2}(n-i)}{\prod_{i=0}^{k-1}(x+n-i)}.$$

Unlike in the previous example, x does not appear in the argument of any function whose values (i.e., the values of the argument) are restricted to integers, so one could treat x as a continuous variable, set the derivative of $P(x)$ equal to zero and solve for x. While the differentiation process is fairly straightforward, the end result is a polynomial equation in x of degree k, which cannot, in general, be solved analytically for $k > 4$.

Let's try solving this problem using the method described above.

$$\begin{aligned}\Delta[P(x)] &= \frac{k(x+1)\prod_{i=0}^{k-2}(n-i)}{\prod_{i=0}^{k-1}[(x+1)+n-i)]} - \frac{kx\prod_{i=0}^{k-2}(n-i)}{\prod_{i=0}^{k-1}(x+n-i)} \\ &= \frac{k\prod_{i=0}^{k-2}(n-i)}{\prod_{i=0}^{k}(x+n-i+1)}[(x+1)(x+n-k+1)-x(x+n+1)] \\ &= \frac{k\prod_{i=0}^{k-2}(n-i)}{\prod_{i=0}^{k}(x+n-i+1)}[(1-k)x+n-k+1].\end{aligned}$$

Since all of the factors in the above fraction are positive, setting $\Delta[P(x_0)] = 0$ requires the quantity in brackets to be zero. Solving for x_0 results in

$$x_0 = \frac{n-k+1}{k-1}.$$

For $k = 1$, x_0 becomes infinite, which reflects the fact that if a single entry is drawn, you maximize your chance of winning by submitting as many entries as possible, since there is no possibility that you will be disqualified for submitting multiple entries.

For $k > 1$, the present case, the quantity in brackets is a decreasing function of x, in which case it follows that $\Delta[P(x)] > 0$ for $x < x_0$ and $\Delta[P(x)] < 0$ for $x > x_0$, so the hypothesis of the above theorem is satisfied.

Substituting the values for n and k in the above equation results in

$$x_0 = 21.2,$$

so you should submit 22 entries in order to maximize your chance of winning – provided the manager doesn't see you stuffing the jar!

The associated probability of winning is given by

$$P(22) = \frac{(10)(22)\prod_{i=0}^{10-2}(200-i)}{\prod_{i=0}^{10-1}(22+200-i)} = 0.39644\,.$$

As a consistency check,

$$P(21) = 0.39627 < P(22) \text{ and}$$

$$P(23) = 0.39587 < P(22),$$

as required.

References

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