Elementary Geometric Proofs Of Euler's Rotation Theorem

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1 Introduction

Euler's Rotation Theorem, proved by Euler [1] in 1775, is an important theorem in the study of general 3D motion of rigid bodies, as well as an early example of a fixed point theorem in mathematics. It states that given an arbitrary rigid motion of a sphere about its center, there exists a diameter of the sphere (the 'Euler Axis') and axial rotation about it which produces the same net displacement.

For any non-zero net motion of the sphere this implies there exist exactly two fixed points, the endpoints of the diameter, and thus the Euler Axis must be unique.

Euler's original proof [1, §24-28] makes use of spherical 'non-Euclidean' geometry, for example spherical triangles, and is discussed in [2] and [3].

Here two alternative geometric proofs are given, requiring only elementary Euclidean geometry. The proofs also suggest how the Euler Axis can be constructed. An intuitive notion of 'rigid motion' as in mechanics is assumed with various properties regarded as 'self evident', ie a 'whole body' motion which does not alter shape, nor involve reflection. For example in Lemmas 1–3 such an underlying notion and its properties is being assumed. However in Appendices A and B it is shown how these assumed properties can be derived from the more formal mathematical definition of 'rigid motion' as an orientation preserving isometry of Euclidean Space[1].

The following terms are used :

- a 'motion' of a sphere means a general arbitrary rigid motion in 3D about its center
- an 'axial rotation' is a special case of a 'motion' which is a rotation about a fixed axis (diameter) of the sphere
- two motions are 'equivalent' if they produce the same net displacement
- a 'zero motion' is one with no net displacement
- a 'fixed point' is a point whose final position equals its initial position
- the 'axis' of a circle on the sphere (great or non-great) is the sphere diameter perpendicular to the circle plane
- the 'poles' of a circle on the sphere are the end points of its axis
- the 'great circle of a diameter' is the great circle perpendicular to it
- the antipode of a point on the sphere is the diametrically opposite point
- any 'circle' will be assumed to have a non-zero radius

The proofs both start off as in Euler's proof, by considering the image C_2 of a great circle C_1 under the motion. In proof (1) this is used to construct a non-great circle which must be mapped onto itself due to a certain orientation property that is preserved - the axis of that circle is then the Euler Axis. In proof (2) it is shown the final displacement of the great circle can be achieved via a composition of two 180° axial rotations which then gives the Euler Axis as the

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[1]There are different formal definitions of 'rigid motion' found in the literature — at the time of writing [8] is compatible with the definition given here, whilst other sources define it as distance preserving only, without the orientation requirement — [8] uses the term 'rigid transformation' for the latter.

normal to the plane containing these two axes.

The proofs use only elementary Euclidean geometry and don't require any algebra or matrix theory. It is thus hoped they may be of some interest in the early stages of study in geometry and mechanics.

The lemmas cover the simple special cases and define the notion of 'orientation' used in Proof (1).

Euler's comment on his theorem [1, §28], praises the value of the geometric approach :

> 'On this account this excellent property, the truth of which is so easily shown geometrically will be most hidden by rules of analytical formulas; and owing to this we can anticipate the rules themselves to be the most important advancement for the whole science of mechanics.'

This principle can also operate in reverse however, with the algebraic approach sometimes giving a much simpler solution to a geometric problem — for example in finding the centroid of a tetrahedron, as discussed in [4, p12-13], by conjecturing the generalization of the 2-dimensional formula $\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$ to $\frac{1}{4}(\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d})$.

2 Lemmas

Lemma 1. A motion of a sphere about its center O which leaves a point P on the sphere fixed is equivalent to an axial rotation about OP. Hence the antipode Q of P is fixed also, and if the motion is non-zero P and Q are the only fixed points.

Proof. There are no possible final positions of the sphere in which P is fixed, other than axial rotations about OP from the original position, since with P fixed the situation of the sphere is constrained from every other possible motion. The antipode Q is the opposite end of this axis and hence is fixed also. For a non-zero motion the angle of axial rotation cannot be a multiple of 360°, thus ALL points other than P and Q must be moved.

QED

Lemma 2. Given a motion M_1 of a sphere S about its center O, then a second motion M_2 which places a circle C of S (great or non-great) identically to M_1 is equal to M_1.

Proof. No other possible final position of S than that of M_1 can have C placed completely 'correctly' because the sphere is completely constrained by this criteria — for, once the final positions of all the points of a circle on a fixed center sphere have been determined the final positions of all the other points of the sphere have been determined also. Thus M_2 must equal M_1.

QED

Lemma 3. A motion of a sphere about its center O which overlays a circle C (great or non-great) onto itself in some manner is equivalent to an axial rotation.

Proof.

(i) If C is non-great then as in Lemma 1 the sphere is constrained so no net displacement other than a rotation about the circle's axis is possible.

(ii) If C is a great circle then it must either be :

- (a) overlayed the 'same way up', in which case the same argument as (i) applies, or
- (b) overlayed but 'flipped over'. Consider an arbitrary point P on C and its image P' under the motion (P' may equal P), as in the 'plan view' of Figure 1.

A 180° rotation ϕ about axis D of symmetry of P and P' places C the 'right way up' and puts P onto P'. This must then place all the other points of C in the correct position, and so by Lemma 2, ϕ is equivalent to the original motion.

QED

Lemma 4. Any motion of a sphere about its center O in which a diameter is flipped is equivalent to a 180° axial rotation.

Proof. This causes the great circle of the diameter to be flipped over onto itself, and thus by Lemma 3 case (ii)(b), the result follows.

QED

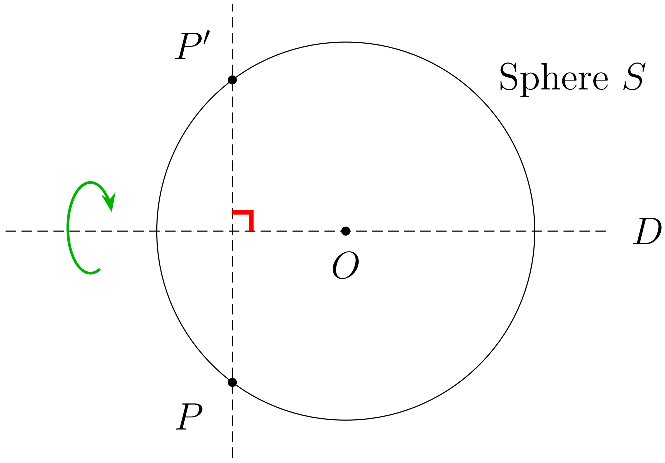

Figure 1: Flipping a great circle 180°

Lemma 5. Given two non-diametrical points A and B on a sphere S of radius R, then if d is the straight-line distance AB, the circles on the sphere which contain A, B are :

(i) a unique minimal radius circle of radius $r = \frac{1}{2}d$,

(ii) a unique maximal radius great circle of radius $r = R$,

(iii) for every intermediate radius $r \in (\frac{1}{2}d, R)$, exactly 2 circles of radius r.

Proof. In the 'Hoopla Construction' in Figure 2, A and B are viewed at D, with A in front of B. The set of circles on S containing A, B corresponds to the set of planes through the axis AB, as they cut S, such as Γ and Δ. $\theta = 0°$ gives case (i), $\theta = 90°$ gives case (ii), and $\theta \in (0, 90°)$ gives case (iii), with $r = \sqrt{R^2 - l^2 cos^2\theta}$ (an increasing function of θ).

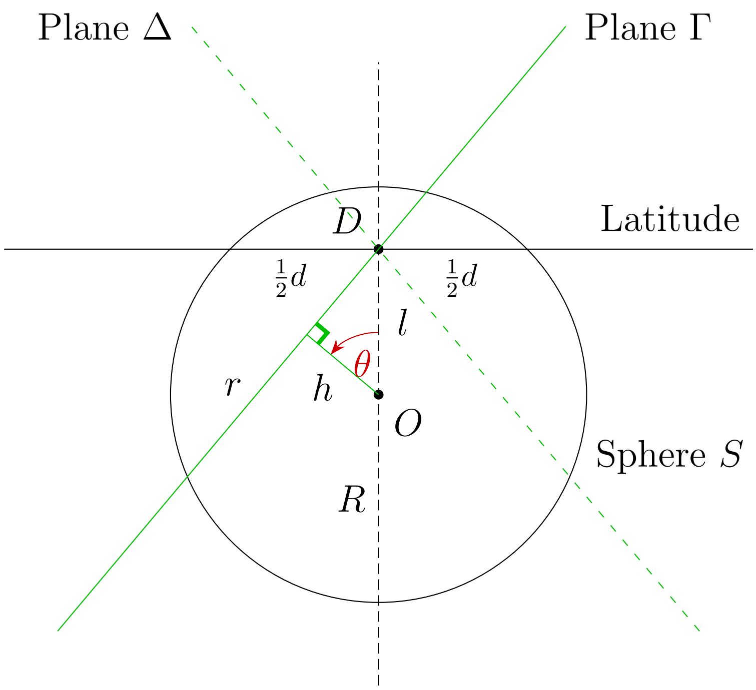

Figure 2: 'Hoopla' construction

QED

Definition. Given two non-diametrical points A and B on a circle C, 'orientation of A, B on C' is either CW or ACW according to the sense of the minor arc from A to B.

□

With this definition :

(i) 'orientation of A, B on C' flips when we view from the other side of C,

(ii) the orientation is undefined for diametrical points A, B of C, and

(iii) the 'orientation of B, A on C' is opposite from the 'orientation of A, B on C'.

Lemma 6. Given any non-great circle C on a sphere S and two non-diametrical points A, B of C, the orientation of A, B (as viewed from 'non-O' side of circle C, ie the 'outside' of S) is preserved after any motion of S about O.

Proof. Center O of sphere never crosses or touches the plane of circle C, so circle C is always being viewed from the same side, and so as A, B are fixed onto C, their orientation on C remains the same.

QED

Lemma 7. If two non-diametrical points A, B on a sphere S of radius R lie on two distinct circles of common radius r on the sphere, then A, B (viewed from non-O side) have opposite orientations on these respective circles.

Proof. From Lemma 5, the two distinct circles of same radius implies case (iii), so the circles are non-great.

Thus from the 'Hoopla' diagram of Figure 2, with A, B seen at D with A in front of B, we have $\theta \in (0, 90°)$.

The circle in plane Γ gives A, B with orientation ACW, whilst the other circle in the mirror image plane Δ gives A, B with orientation CW.

QED

Lemma 8. Given two diameters L and M of sphere S, then the motion which is the composition of a 180° rotation about L followed by a 180° rotation about M is equivalent to a single axial rotation.

Proof. The case $L = M$ is trivial as the composition is a zero motion.

Otherwise consider the great circle C defined by the plane containing L and M, and let P and P' be the poles of C, as in Figure 3.

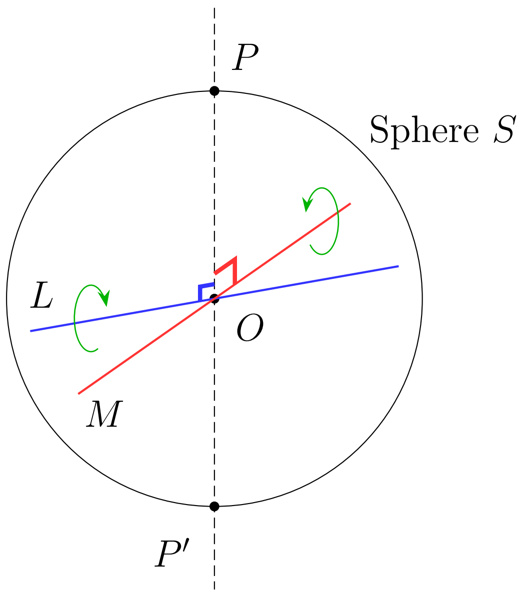

Figure 3: Two 180° axial rotations

The rotation about L flips P and P', as does the rotation about M. Hence the composition leaves P fixed. By Lemma 1 the result then follows, with the axis being the normal to the plane containing L and M.

QED

3 Proofs

3.1 Proof 1

Suppose great circle C_1 is mapped onto great circle C_2. Assume planes C_1 and C_2 do not coincide (otherwise Lemma 3 completes the proof).

Let C_1 and C_2 intersect along a diameter BF (the 'line of nodes'), as shown in Figure 4.

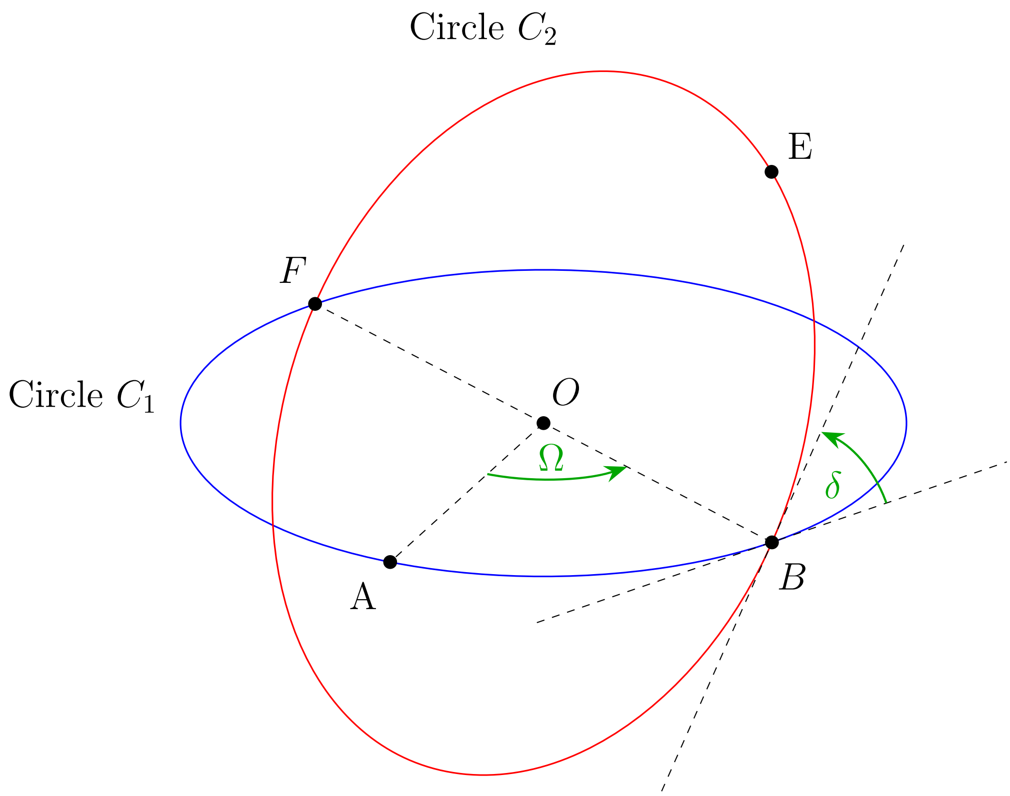

Figure 4: Great circle C_1 and its image C_2

Since B lies on C_2, it must have been mapped from some point A on C_1. Assume $A \neq F$ (otherwise the proof follows from Lemma 4), and $A \neq B$ (otherwise the proof follows from Lemma 1). A is shown on the left of BF in Figure 4 - if it was on the right, we could rotate the diagram around 180° about BF so A is on the left. The dihedral angle $\delta \in (0, 180°)$.

Let $\Omega \in (0, 180°)$ be the angle $\angle A\widehat{O}B$.

B also lies on C_1 so it maps to some point E on C_2. So from the rigid motion of C_1, angle $\angle B\widehat{O}E = \Omega$, and chord $|AB| =$ chord $|BE|$ (on C_1, C_2 respectively). E is shown above C_1 in Figure 4, but the same argument as below applies if it is below.

Also plane $A, O, B =$ plane C_1, and plane $B, O, E =$ plane C_2.

A, E, B cannot be collinear because that would imply E to be in plane C_1, so the plane C_2 defined by B, O, E would then be in plane C_1 - a contradiction.

Thus A, E, B define a unique plane, containing 3 distinct points of sphere S. That plane cannot pass through O since then all of A, E, B, O would lie in the same plane, again implying C_1, C_2 coincident - a contradiction. Let the non-great circle defined by this plane be C, and let its image be D.

We show $C = D$. Firstly note that although C is a smaller radius circle than $C_{1,2}$, chords AB and BE can't be diameters of C because that would imply $A = E$ - a contradiction - so the orientations below are well-defined. Consider the points B, E which lie on C. They must also lie on D, being the image of A, B. But (viewing from non-O side) :

orientation of B, E on C = orientation of A, B on C,

because the non-diametrical equal length chords AB and BE of C subtend the same angle within C, and these chords lie to either side of point B by virtue of $A \neq E$.

And secondly, considering the rigid motion taking C to D :

orientation of B, E on D = orientation of A, B on C,

by Lemma 6.

So B, E have the same orientation on circles C, D. But by Lemma 7, as C, D have common radius, this means $C = D$, from which the proof now follows from Lemma 3 case (i), the Euler Axis being the axis of circle C.

QED

3.2 Proof 2

View C_1 and C_2 as shown in Figure 5. Cases $\theta = 0°$ and $\theta = 90°$ follow from Lemma 3 case (ii), so assume $\theta \in (0, 90°)$.

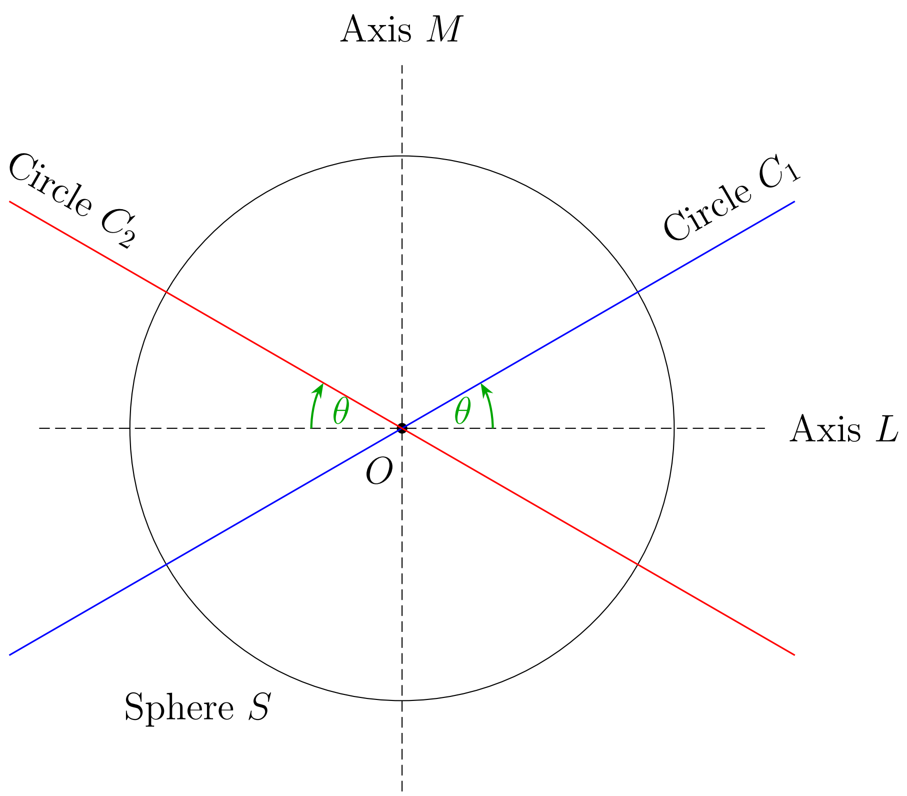

Figure 5: Great Circle C_1 and its image circle C_2 viewed along line of intersection of their planes

C_1 can be made to overlay C_2 by 180° rotation about L or about M.

The first of these places upper side of C_1 onto lower side of C_2, while the second places the upper side of C_1 onto the upper side of C_2.

Choose whichever of these results in C_1 being overlayed onto C_2 the 'wrong way up'. Then by Lemma 3 case (ii) (b) a 180° rotation about some axis within C_2 places C_1 exactly in the 'correct' position of C_2.

Thus we have achieved the correct final position for C_1 by a succession of two 180° axial rotations, and thus by Lemma 8 and Lemma 2 the proof follows.

QED

Appendix A

Formal Proofs of Rigid Motion Properties

Formally a 'rigid motion' ϕ is an isometry (ie distance preserving bijection) from ES (the set of points of Euclidean Space) to ES which preserves orientation, where 'preserves orientation' means given any tetrahedron $ABCD$ the sense of ABC (CW or ACW) as viewed from the D-side is preserved. Clearly the intuitive notion of rigid motion would have these properties. Certain isometries, such as a reflection in a plane, do not preserve orientation.

Certain properties of general isometries of ES are assumed below — proofs of these properties are given in Appendix B. In particular, the manner in which an isometry maps a circle and a circular segment is used. The proofs below of rigid motion properties use only the formal definition of 'rigid motion' as a mapping from ES to ES, and are given wherever intuitive properties of 'rigid motion' were assumed previously — the remaining parts of the original proof remain unchanged.

A.1 Proof of Lemma 1

By distance preservation the antipode Q of fixed point P is still at distance $2R$ from P after the motion ϕ, therefore it can only have one possible location which is Q itself. Thus both P and Q are fixed. In Figure A.1 consider a longitude half-circle H wrt axis PQ. By the properties of isometries H is mapped bijectively to a radius $2R$ semi-circle with end points P and Q, which must then be a longitude half-circle, H' say.

Taking positive angles of rotation about PQ as ACW when looking down axis PQ from P-side, let the dihedral angle which rotates half-plane H into half-plane H' be $\theta \in [0, 360°)$. After the mapping, a typical point A on H remains at the same distance from P, Q respectively, and is thus on the intersection of the same two spheres centered on P, Q respectively, ie on the same latitude L at A'. Thus the rotation θ about PQ takes H pointwise to H'.

Take an arbitrary second longitude half-circle K at angle $\alpha \in (0, 360°)$ from H, and let its image be the longitude half-circle K'. Because isometries preserve dihedral angles between half-planes, half-plane K' is inclined at angle α to H', which gives it one of two possible positions wrt H' (or just one if $\alpha = 180°$), ie at $+\alpha$ or $-\alpha$. Assuming $\alpha \neq 180°$, a position of $-\alpha$ would in all cases result in orientation of tetrahedron $ABPO$ being reversed, where $B \in K$ lies on the latitude L — thus the position must always be $+\alpha$.

But now dihedral angle from $K \to K' = -\alpha$ (for $K \to H$) $+ \theta$ (for $H \to H'$) $+ \alpha$ (for $H' \to K'$) $= \theta$, and since every point on K must map to the same latitude on K' it follows that the entire final displacement can be achieved by the axial rotation θ about PQ.

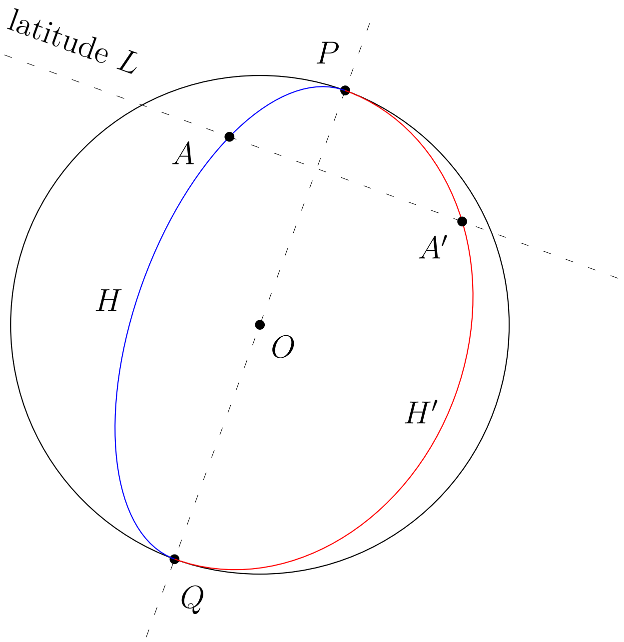

Figure A.1: Mapping longitude lines when ϕ has a fixed point P

A.2 Proof of Lemma 2

Firstly consider C a non-great circle on S, with nearest pole P and furthest pole Q. Its image under an isometry must be a circle in S of the same radius. By hypothesis, any point of C is then mapped identically in ES by ϕ_1 and ϕ_2, onto coincident circles C' and C'' respectively. By distance preservation, image of P must lie on perpendicular axis of image of C and be at same distance from C as before. Thus we have the situation in Figure A.2, with $P'(= P'')$ the image of P, and $Q'(= Q'')$ the image of Q, under ϕ_1 and ϕ_2, and $O' = O''$.

Take any point A on a longitude half-circle H wrt PQ. If A is one of P, Q or is on C then both motions map it identically. Otherwise let H meet C at B. Under isometry ϕ_1, H maps to some longitude half-circle H' wrt $P'Q'$, and images A', B' are located on it. Similarly with ϕ_2. But since by hypothesis $B' = B''$ in ES, H' and H'' then coincide. So A', A'' lie on the same longitude half-circle in ES. But they also must lie on the same latitude circle because the distance of A from P and Q remains unchanged by either mapping, and thus $A' = A''$.

Only distance preservation was needed in the above case. When C is a great circle, a similar diagram as Figure A.2 still applies, except distance preservation alone does not imply $P' = P''$ and $Q' = Q''$, for the pole positions could be reversed. But because C', C'' are point for point identical images of C, a reversal of the poles would imply that any tetrahedron $ABPO$ formed from two non-diametrical points A, B of C would gain opposite orientations under ϕ_1 and ϕ_2 — a contradiction. Then the same argument as for non-great C applies.

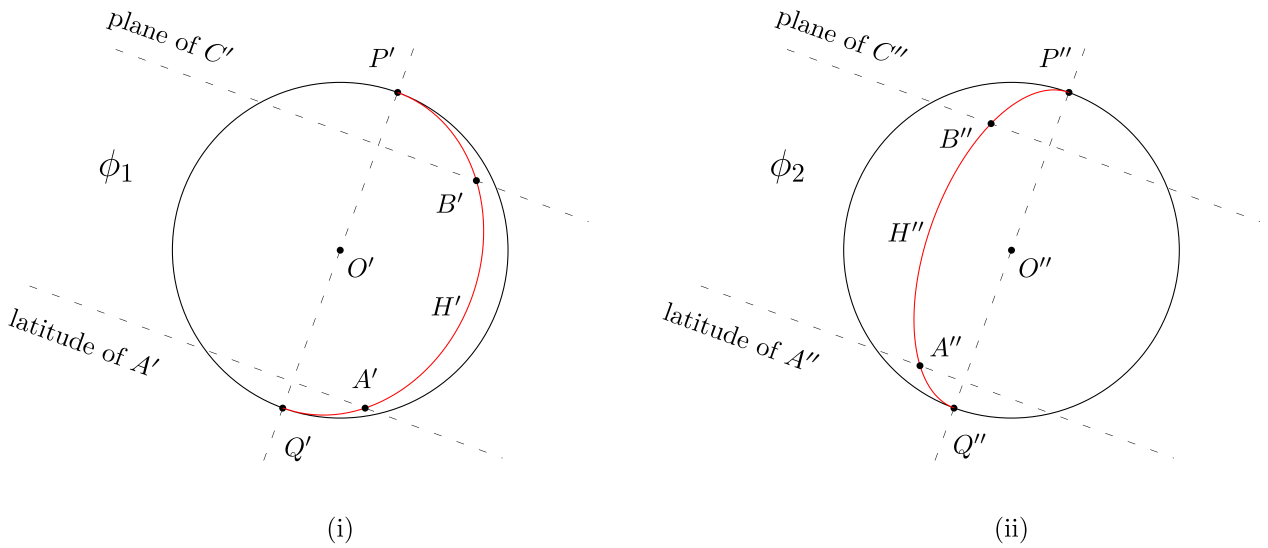

Figure A.2: A circle C mapped identically by ϕ_1 and ϕ_2

A.3 Proof of Lemma 3

(i) For ϕ overlaying non-great C onto itself, nearest pole of C must be fixed (cf §A.2) so proof follows from Lemma 1.

(ii) For ϕ overlaying great circle C onto itself :

  (a) C mapped the 'right way up' (see note after Theorem 4, Appendix B below) means the original position of C and its overlayed image are separated by a rotation about their common axis PQ. Preservation of distance to C implies either $P' = P$ or $P' = Q$. But the latter cannot happen because due to the aforementioned separation by a rotation, a change would then occur in the orientation of a tetrahedron $ABPO$ where A, B are any non-diametrical points of C, so proof now follows from Lemma 1.

  (b) C mapped the 'wrong way up' means flipping over as in the original proof makes the sense of C correct (since CW sense appears ACW when viewed from the opposite side and v.v.), and matching a single point P of C then matches all the other points, as in Figure 1 — proof then follows from Lemma 2.

A.4 Proof of Lemma 6

Let the center of C be E. Considering tetrahedron $ABEO$, the sense of triangle ABE as viewed from the non-O-side is preserved under the mapping ϕ since it is preserved by ϕ from the O-side. But the earlier definition of orientation of A, B on C (from non-O-side) equals this sense, and so is unchanged.

Appendix B

Properties of Isometries

The properties of an isometry ϕ of Euclidean Space as required in Appendix A are derived here. An intuitive approach to Euclidean Geometry is adopted as in [5, Chap 2 & p45], but all the needed properties of ϕ are derived assuming only the conditions in the definition below — nothing else is assumed about ϕ. (This contrasts with the intuitive approach where a 'rigid motion' is a mapping assumed to possess a range of properties that we regard as 'self-evident').

For convenience the following notation is used — ES denotes either the overall concept of Euclidean Space, or if refered to as a set (eg as in '$x \in$ ES', or 'f is a function on ES') the set of points of Euclidean Space. ES is a metric space with the natural distance function. EV denotes the set of Euclidean Vectors in ES, where each Euclidean Vector comprises a length and a direction — EV is in 1-1 correspondence with the set of equivalence classes of directed line segments in ES wrt the relation 'of same length and parallel' (ie the 'equipollence' relation [6]). EV is a vector space with the natural operations of addition and scalar multiplication, and has a norm which is the vector length $|x|$. $|x, y|$ denotes the distance between $x, y \in$ ES, whilst $|x-y|$ denotes the distance between vectors $x, y \in$ EV. $[x, y]$ denotes the finite line segment from x to y in ES, $]x, y[$ the corresponding open line segment. $L(x, y)$ is the infinite straight line in ES containing the distinct points x, y.

Choosing an origin point O in ES brings ES into 1-1 correspondence with EV, with point P mapping to position vector $\overrightarrow{OP}$ — thus with any choice of such O a function on (or to) ES becomes a function on (or to) EV and vice-versa. We can pass freely between ES and EV with O understood — which interpretation is being used at any point will be determined from the context, for example if an arithmetic operation such as $x + y$ appears, x and y are considered in EV, if $|x, y|$ appears, x and y are considered in ES. Also in passing from ES to EV we have $|x, y| = |x - y|$, where on the lhs $|.|$ is the ES distance function, and on the rhs the EV norm.

Definition. An isometry of ES is a bijection $\phi :$ ES $\to$ ES which preserves distance, ie $|\phi(x), \phi(y)| = |x, y| \ \forall x, y \in$ ES.
□

Theorem 1. An isometry ϕ of ES maps three independent (ie non-collinear) points to three independent points.

Proof. Three independent points x, y, z form a triangle. From Figure B.1(i) we have : $|x, y| < |x, z| + |z, y|$, $|x, z| < |x, y| + |y, z|$, $|y, z| < |y, x| + |x, z|$, and hence $|\phi(x), \phi(y)| < |\phi(x), \phi(z)| + |\phi(z), \phi(y)|$, $|\phi(x), \phi(z)| < |\phi(x), \phi(y)| + |\phi(y), \phi(z)|$, $|\phi(y), \phi(z)| < |\phi(y), \phi(x)| + |\phi(x), \phi(z)|$. These preclude respectively the three cases of Figure B.1(ii). But since $\phi(x), \phi(y), \phi(z)$ are distinct these are the only possible cases with these three dependent (ie collinear), hence
QED

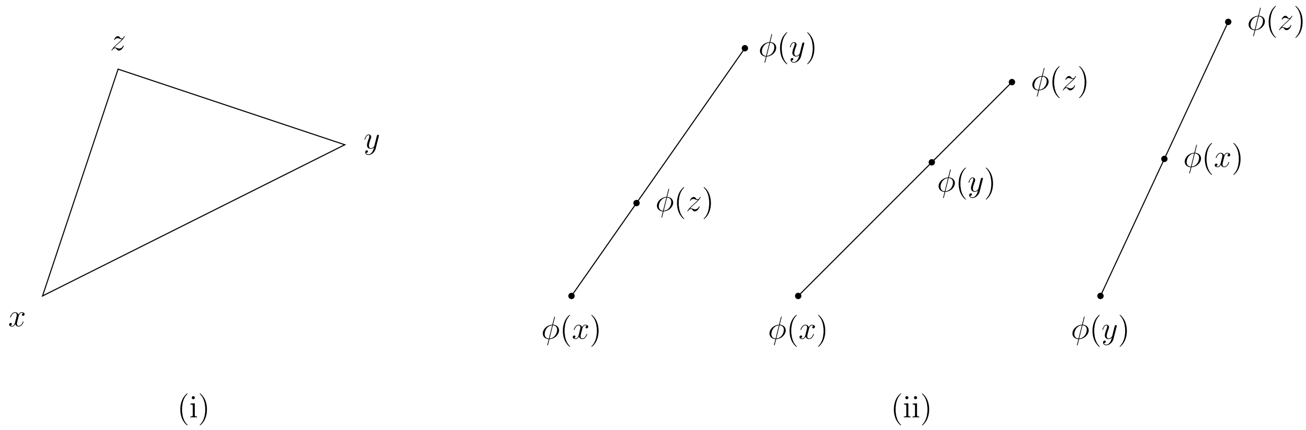

Figure B.1: Excluding cases on mapping three independent points

Theorem 2. Given an isometry ϕ of ES, and $x \neq y$ in ES :

(i) $\phi([x, y]) = [\phi(x), \phi(y)]$

(ii) $\phi(]x, y[) =]\phi(x), \phi(y)[$

(iii) $\phi(L(x, y)) = L(\phi(x), \phi(y))$

(iv) $\forall t \in \mathbb{R}$, $\phi(x + t(y - x)) = \phi(x) + t(\phi(y) - \phi(x))$

(v) ϕ maps an infinite (or half-infinite) straight line to an infinite (resp. half-infinite) straight line, with every point mapped to the geometrically corresponding point in the image line.

Proof.

(i) Consider case $x \neq y$.
$\subseteq$. Let $z \in]x, y[$. Then $\phi(z) \neq \phi(x)$ and $\phi(z) \neq \phi(y)$. But if $\phi(z) \notin]\phi(x), \phi(y)[$ then from Figure B.2, $|\phi(x), \phi(y)| < |\phi(x), \phi(z)| + |\phi(z), \phi(y)|$, and thus $|x, y| < |x, z| + |z, y|$, contradicting $z \in]x, y[$.
$\supseteq$. It suffices to show $\forall\, t \in [0, 1]$ we have $\phi(x + t(y - x)) = \phi(x) + t(\phi(y) - \phi(x))$. Consider such $z = x + t(y - x)$. Then from $\subseteq$ we know $\phi(z) \in [\phi(x), \phi(y)] \Rightarrow \exists\, t_2 \in [0, 1]$ such that $\phi(z) = \phi(x) + t_2(\phi(y) - \phi(x)) \Rightarrow |\phi(z) - \phi(x)| = t_2|\phi(y) - \phi(x)|$, ie $|z - x| = t_2|y - x|$. But $|z - x| = t|y - x|$, $\therefore$ as $x \neq y$, $t_2 = t$. Hence etc.

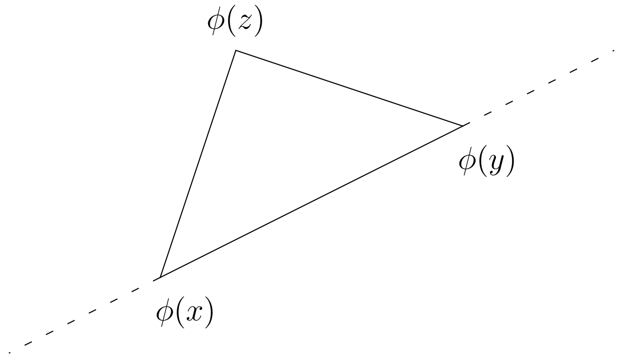

Figure B.2: Case of $\phi(z) \notin]\phi(x), \phi(y)[$

(ii) Follows from (i) and ϕ being bijective.

(iii) $\subseteq$. Let $z \in L(x, y)$. Case $z \in [x, y]$ follows from (i). Otherwise consider z to the x-side of $[x, y]$ as in Figure B.3. Then $[\phi(x), \phi(y)] = \phi([x, y]) \subseteq \phi([z, y]) = [\phi(z), \phi(y)] \Rightarrow L(\phi(x), \phi(y)) = L(\phi(z), \phi(y)) \ni \phi(z)$. A similar argument applies when z is to y-side of $[x, y]$.
$\supseteq$. As ϕ surjective, a point in $L(\phi(x), \phi(y))$ has form $\phi(z)$ for some $z \in$ ES. Then $\phi(x), \phi(y), \phi(z)$ are dependent $\Rightarrow$ by Theorem 1, x, y, z dependent $\Rightarrow z \in L(x, y) \Rightarrow \phi(z) \in \phi(L(x, y))$.

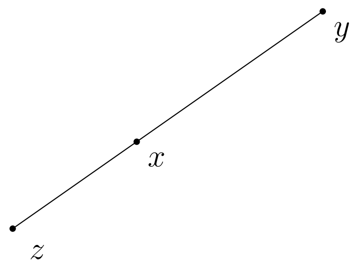

Figure B.3: Case of z to the x-side of $[x, y]$

(iv) Let $z = x + t(y - x)$, $(t \in \mathbb{R})$. The case $t \in [0, 1]$ was proved in (i) above. By (iii), $\phi(z) \in L(\phi(x), \phi(y))$, $\therefore$ $\phi(z) = \phi(x) + t_2(\phi(y) - \phi(x))$ for some $t_2 \in \mathbb{R}$. But then $|z - x| = |t_2| \cdot |y - x|$. But $|z - x| = |t| \cdot |y - x| \Rightarrow$ as $x \neq y, |t_2| = |t|$, $\therefore$ $t_2 = \pm t$. Consider case $t < 0$ as shown in Figure B.4(i), with z to the x-side of $[x, y]$. Then $|z - x| + |x - y| = |z - y| \Rightarrow |\phi(z) - \phi(x)| + |\phi(x) - \phi(y)| = |\phi(z) - \phi(y)| \Rightarrow \phi(z)$ can only lie to $\phi(x)$ side of $[\phi(x), \phi(y)] \Rightarrow t_2 < 0$, and hence $t_2 = t$, as required. A similar argument applies in the case of $t > 1$ in Figure B.4(ii).

(v) From (iii) and (iv) above, when we progress along an infinite line $L[x, y]$, by varying t, the image point covers the same distance (since $|\phi(y) - \phi(x)| = |y - x|$), and only changes direction when the original direction changes. With a half-infinite line this applies also, except we are restricted to $t \geq 0$, x being the end point of the original line, $\phi(x)$ being the end point of the image line.

QED

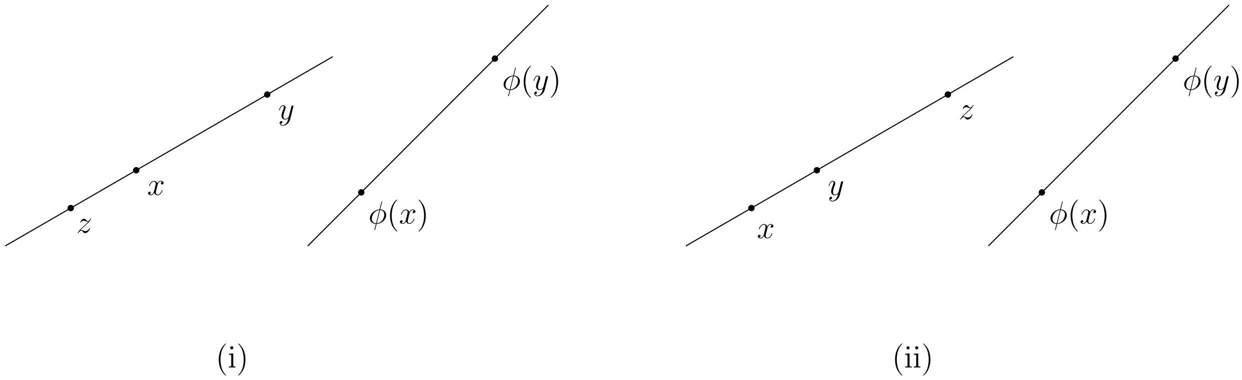

Figure B.4: Cases $t < 0$ and $t > 1$

Theorem 3. An isometry ϕ of ES preserves angles between line segments.

Proof. By Theorem 1, a triangle is mapped to a triangle, and the side lengths are preserved, thus it is congruent to the original triangle but re-oriented in some manner in ES. Taking the triangle $\triangle ABC$ formed by the two line segments $\overrightarrow{AB}$ and $\overrightarrow{AC}$, and its image $\triangle A'B'C'$, the angles $\angle B\widehat{A}C$ and $\angle B'\widehat{A'}C'$ between respective line segments are thus equal.

QED

Theorem 4. An isometry ϕ of ES :

(i) maps a circle onto a circle of the same radius, such that if the original circle is swept out from a point P then the image circle is swept out by the corresponding angle from the image point P', ie all points are mapped to geometrically corresponding points on the image circle

(ii) maps a circular segment onto a circular segment of the same radius and angle, such that if the original segment is swept out from a point P then the image segment is swept out by the corresponding angle from the image point P', ie all points are mapped to geometrically corresponding points on the image segment.

Proof.

(i) Firstly we show ϕ maps a plane *into* a plane. Consider two line segments $\overrightarrow{AB}$ and $\overrightarrow{AC}$ separated by some angle θ, and a line segment $\overrightarrow{AD}$ lying in the plane between them, separated from $\overrightarrow{AB}$ by angle β and separated from $\overrightarrow{AC}$ by angle γ (so $\theta = \beta + \gamma$). Then by Theorem 3, these angular separations are all preserved, so that $\overrightarrow{A'D'}$ lies in the intersection of two cones with apex A', one with axis $\overrightarrow{A'B'}$ and angle β, and the other with axis $\overrightarrow{A'C'}$ and angle γ. But because $\theta = \beta + \gamma$ these cones intersect tangentially in the plane of A', B', C', and therefore $\overrightarrow{A'D'}$ which is in this intersection lies in this plane. Any plane is a union of a number of such sectors ABC with a common center A, which are overlapping and thus the entire image is coplanar.

By distance preservation the image must then be a circle of the same radius. Choose any point A on the circle whose diameter does not contain P, and consider the circle and its image as shown in Figure B.5 (i) and (ii) (note the angle α must be preserved, and we can always view the image circle as shown in (ii)). Let B be any point on the circle. The distance of B to A is preserved, so the only possible images of B are B' and B'' shown in (ii) (these degenerate to a single point when B is on diam A). But the distance BP must be preserved also — and B' and B'' are at different distances from P' since P' is not on the perpendicular bisector of $B'B''$, by choice of A. Thus image of B can only be B', the geometrically corresponding point to B.

(ii) This follows similarly to (i) by choosing A on the segment and considering the complete circle.

QED

Note: Theorem 4 gives rise to the notion of a circle (or circular segment) being overlayed onto itself in ES by ϕ, either the 'right way up' or the 'wrong way up' — the former meaning the angles are swept out in the same sense as we view the two overlayed circles, the latter meaning they are in opposite sense.

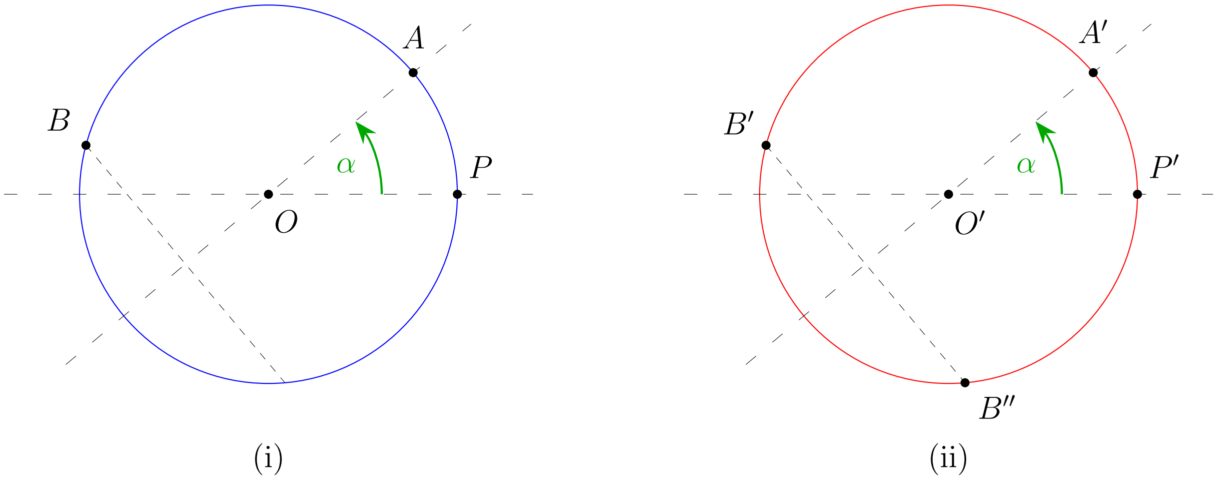

Figure B.5: Mapping a circle

Theorem 5. An isometry ϕ of ES :

(i) maps an infinite (or half-infinite) plane onto an infinite (resp. half-infinite) plane, such that all the points are mapped to geometrically corresponding points in the image

(ii) maps parallel lines to parallel lines

(iii) preserves the dihedral angle between two planes (or two adjoining half-planes)

(iv) maps four independent (ie non-coplanar) points to four independent points.

Proof.

(i) In proof of Theorem 4 (i) we proved the image must lie in a single plane. For an infinite plane take any point O of it and consider the plane as a union of circles centered on O which are then mapped to concentric circles about O' in the image plane. These circles can all be traced out by a single ray L starting at O, and then the image ray L' will trace out the image circles in unison. No two image circles could be traced out in opposite sense during this because that would produce points lying along L that were not mapped to collinear points in the image. Thus the image is a union of circles which geometrically correspond to the original plane. The half-infinite plane is then the corresponding union of semi-circles, the boundary mapping to the boundary.

(ii) By (i) and Theorem 2, the image of the lines are straight lines lying in a common plane, and as ϕ is a bijection they never meet, thus they are parallel.

(iii) The dihedral angle must be preserved because perpendiculars to the line of intersection remain so in the image, and the angle between them is preserved. If the planes are parallel then by (i) and the bijectivity of ϕ they are planes in the image which never meet and hence parallel.

(iv) Take three of the points — by (i) and Theorem 1 these map to three independent points in a common plane. If the fourth point is also in that plane then since ϕ^{-1} is also an isometry, (i) then implies the original four points are coplanar — a contradiction. Thus

QED

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