RECURRING PAIRS OF CONSECUTIVE ENTRIES IN THE NUMBER-OF-DIVISORS FUNCTION

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Abstract. The Number-of-Divisors Function $\tau(n)$ is the number of divisors of a positive integer n, including 1 and n itself. Searching for pairs of the format $(\tau(n), \tau(n+1))$, some pairs appear (very) often, some never and some — like (1,2), (4,9), or (10,3) — exactly once. The manuscript provides proofs for 46 pairs to appear exactly once and lists 12 pairs that conjecturally appear only once. It documents a snapshot of a community effort to verify sequence A161460 of the Online Encyclopedia of Integer Sequences that started ten years ago.

1. Scope

The number-theoretic function $\tau(n)$ counts the divisors of n, including 1 and n itself:

$$\tau(n) = 1, 2, 2, 3, 2, 4, 2, 4, 3, 4, 2, 6, 2, 4, 4, 5, 2, 6, 2, 6, 4, 4, 2, 8, 3, 4, 4, \ldots (n \geq 1). \tag{1}$$

$\tau(n)$ is tabulated in the Online Encyclopedia of Integer Sequences [6, A000005] and a reverse lookup arranged in [6, A073915]. It is multiplicative. If $n = \prod_i p_i^{e_i}$ is the unique prime factorization of n, then

$$\tau(\prod_i p_i^{e_i}) = \prod_i (e_i + 1). \tag{2}$$

In the list (1), the pair (6,2) appears at least 2 times — once at $\tau(12) = 6, \tau(13) = 2$ and again at $\tau(18) = 6, \tau(19) = 2$, see Appendix C.2. The pair (4,4) appears at least three times (see Appendix C.1). The pair (3,3), indeed any pair of odd primes, appears never.

Proof. Formula (2) shows that only the squared primes have $\tau(p^2) = 3$, and because the squared primes have a mutual distance larger than one, a 3 cannot be immediately followed by a 3. □

The manuscript deals with the question: which pairs appear only once in the infinite sequence of $\tau(n)$.

[6, A161460] searches for pairs of numbers $n \neq m$ such that the numbers of divisors $\tau(.)$ and the numbers of divisors of the next integer match:

$$\tau(n) = \tau(m), \quad \text{and} \quad \tau(n+1) = \tau(m+1). \tag{3}$$

If the two equations can be solved only by setting $n = m$, the number $n = m$ is in the sequence.

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2. Strategies and Recipes

2.1. **Modular Arithmetic.** We use the elementary relation (2) that $\tau(n)$ can be computed from the product of the incremented exponents in the prime factorization of n [13]. The prime signatures of n with a given τ are obtained by inspection of a table of all factorizations of τ as counted in [6, A001055] and made explicit in [6, A162247]. Once an n is fixed, $n = p^{\cdots}q^{\cdots}r^{\cdots}\cdots$ and $n+1 = s^{\cdots}t^{\cdots}\cdots$ are known with some known prime sets $\{p, q, r, \ldots\}$ and $\{s, t, \ldots\}$ plus their exponents and define a pair of $\tau's$ in the τ-sequence. The proofs center around diophantine equations of the type

$$p^{\cdots}q^{\cdots}r^{\cdots}\cdots = s^{\cdots}t^{\cdots}\cdots - 1. \tag{4}$$

We ask for a given number of distinct prime factors on the left hand side (LHS) and a given number of distinct prime factors on the right hand side (RHS), and a given set of exponents, whether more than one solution exists.

Theorem 1. *All primes of both sides of (4) are distinct.*

Proof. If one prime would appear on both sides, reducing both sides modulo that prime would lead to a contradiction with residues of 0 and −1. □

A first generic constraint is derived from the −1 on the RHS of Eq. (4), which requires that either the prime product $p^{\cdots}q^{\cdots}\cdots$ on the RHS or the prime product: $s^{\cdots}t^{\cdots}\cdots$ on the LHS is even, but not both:

Theorem 2. *(Parity Argument) Exactly one of the terms $p^{\cdots}q^{\cdots}r^{\cdots}\cdots$ or $s^{\cdots}t^{\cdots}$ is even, so the prime 2 appears exactly once in the set of primes of both sides.*

Usually the primes p, $q\ldots$ on the LHS and the primes s, $t\ldots$ on the RHS are in the residue classes $\equiv \{1, 2\} \pmod 3$. In cases where the prime exponents of $s^{\cdots}t^{\cdots}\cdots$ are all even, that product is a perfect square and usually the entire term is $\equiv 1 \pmod 3$ such that the RHS is $\equiv 0 \pmod 3$, whereas the LHS is $\equiv \{1, 2\}$ (mod 3). To salvage that mismatch of congruences modulo 3 we need a prime in the $\equiv 0 \pmod 3$ class, which is just the 3 itself:

Theorem 3. *(Mod-3 Criterion) If $s^{\cdots}t^{\cdots}$ on the right hand side in (4) is a perfect square, one of the primes in the set $\{p, q, r, \ldots, s, t, \ldots\}$ must be 3.*

2.2. **Notations.** The text uses the $\sim$ or $\nsim$ symbol to indicate that two expressions are (or are not) in the same prime signature class.

Definition 1. *(omega) $\omega(n)$ is the number of distinct prime divisors of n, the cardinality of the set of i with positive e_i that contribute to (2).*

Definition 2. *(Big-omega) $\Omega(n)$ is the number of prime divisors of n, counted with multiplicity.*

$$\Omega(\prod_i p_i^{e_i}) = \sum_i e_i. \tag{5}$$

Definition 3. *(gcd) (x, y) is the greatest common divisor of x and y.*

3. Proven Unique

3.1. **1.** $n = 1$, $\tau(n) = 1$, $\tau(n+1) = 2$. This pair is simple since 1 is the only number with $\tau(1) = 1$, all the others having at least two divisors (namely 1 and themselves).

3.2. **2.** $n = 2$, $\tau(n) = 2$, $\tau(n+1) = 2$. Since prime numbers are the only numbers with $\tau = 2$, and since this here is the case with two consecutive numbers having $\tau = 2$, and since 2 and 3 are the only consecutive prime numbers, this case is also settled easily.

3.3. **3.** $n = 3$, $\tau(n) = 2$, $\tau(n+1) = 3$. This requires a prime number $n = p$ followed by a square of a prime number $n+1 = q^2$. Therefore $p + 1 = q^2$, equivalent to $p = q^2 - 1 = (q+1)(q-1)$. Since we have a prime on the left hand side of this equation and a composite on the right hand side unless $q = 2$, we must have $q = 2$, therefore $n + 1 = 4$. So there is only the pair $(n, n+1) = (3, 4)$ in this category.

3.4. **4.** $n = 4$, $\tau(n) = 3$, $\tau(n+1) = 2$. The pair of τ is in the opposite order of the one discussed above for $n = 3$. Here we need a square of a prime $n = p^2$ followed by a prime $n + 1 = q$, which implies $p^2 + 1 = q$. With the exception of $p = 2$, the squares of primes on the left hand side are odd, so the left hand side is even, and the right hand side is odd. Therefore $p = 2$ and $n = 4$ are a unique solution to this pair of τ.

3.5. **8.** $n = 8 = 2^3$, $\tau(n) = 4$ [6, A030513] , $\tau(n+1) = 3$ [6, A001248] . This requires a product of two distinct primes $n = pq$ or a prime cubed $n = p^3$ followed by a squared prime $n + 1 = r^2$.

- $pq + 1 = r^2$ has two sub-formats by the parity argument:
  - $2q + 1 = r^2$ with $q \neq 2$ is equivalent to $2q = (r+1)(r-1)$ equivalent to $r - 1 = 2$ and $r + 1 = q$, which does not have a solution over primes r.
  - $pq + 1 = 2^2$ has no solution over pairs (p, q) of distinct primes.
- The second sub-case of $p^3 + 1 = r^2$ is also settled by the parity argument: If $p = 2$ we have the solution $n = 8$, and if $r = 2$ there is no solution.

This concludes the track of all sub-cases.

3.6. **15.** $n = 15 = 3 \cdot 5$, $\tau(n) = 4$ [6, A030513] , $\tau(n+1) = 5$ [6, A030514] . Demonstrated in the OEIS comment by the original author of the sequence.

3.7. **16.** $n = 16 = 2^4$, $\tau(n) = 5$, $\tau(n+1) = 2$. This requires a fourth power of a prime $n = p^4$ followed by a prime $n + 1 = q$. By the parity argument either $p = 2$ or $q = 2$ are needed to solve this equation. Since $q = 2$ does not lead to a solution, the case $n = 2^4$ remains the only one.

3.8. **24.** $n = 24 = 2^3 \cdot 3$, $\tau(n) = 8$ [6, A030626] , $\tau(n+1) = 3$. $\tau = 8$ requires $n = p^7$ or $n = pq^3$ or $n = pqr$ with distinct primes p, q and r; and $\tau = 3$ requires $n + 1 = s^2$ with prime s. By the parity argument one of the primes involved equals 2, and this prime is not s because $n = 2^2 - 1$ is no solution.

- $2^7 + 1 = s^2$ has no solution in integers s.
- $2q^3 + 1 = s^2$ implies $2q^3 = (s+1)(s-1)$. On the RHS s is an odd prime such that $(s+1)(s-1)$ is a product of two even numbers and a multiple of 4. This does not match the prime signature of the LHS, so there are on solutions.
- $p \cdot 2^3 + 1 = s^2$, $p \geq 3$ means $8p = (s+1)(s-1)$. The sub-cases of factorizations are $1 = s - 1$, $8p = s + 1$ (no solution in primes), or $2 = s - 1$, $4p = s + 1$ (no solution in primes), or $p = s - 1$, $8 = s + 1$ (no solution in primes),

or $2p = s-1$, $4 = s+1$ (no solution in primes), or $2p = s+1$, $4 = s-1$ (solution $s = 5$, $n = 24$ as known).

- $pq^3 + 1 = 2^2$ does not provide solutions with distinct primes p, q.
- $pqr + 1 = 2^2$ does not provide solutions with distinct primes p, q.
- $2qr + 1 = s^2$ implies $2qr = (s+1)(s-1)$. On the RHS s is an odd prime, so $(s+1)(s-1)$ is a product of two even numbers, therefore a multiple of 4, which does not match the prime signature of $2qr$: no solution in this branch.

3.9. **35.** $n = 35 = 5 \cdot 7$, $\tau(n) = 4$ [6, A030513] , $\tau(n+1) = 9$ [6, A030627] . seqfan list [12]: This requires $n = p^3$ or $n = pq$, $n+1 = r^8$ or $n+1 = r^2s^2$. There are two cases:

- n even: Then $n = 2^3$ or $n = 2q$ with odd q, and $n+1 = r^8$ or $n+1 = (rs)^2$ with odd r, s. But odd squares are $\equiv 1 \pmod 8$, hence $n \equiv 0 \pmod 8$, which excludes $n = 2q$, so $n = 2^3$. But 9 is neither r^8 nor $(rs)^2$. Hence n cannot be even.
- n odd: Then $n+1 = 2^8$ or $n+1 = 4s^2$ and $n = p^3$ or $n = pq$ (with p, q and s odd). $255 = 3 \cdot 5 \cdot 17$, hence $n+1 = 2^8$ can be excluded. Then $n = 4s^2 - 1 = (2s+1)(2s-1)$.
  - If $n = p^3$, this implies
    - either $2s - 1 = 1$ which yields a contradiction $s = 1$,
    - or $2s - 1 = p$, $2s + 1 = p^2$. Therefore $p^2 = p + 2$ which enforces $p \in \{2, -1\}$, a contradiction.
  - Remains the case $n = pq = (2s+1)(2s-1)$, assuming $p > q$, for example. Then $p = 2s+1$ and $q = 2s-1$ are twin primes, hence $2s$ is a multiple of 6, s a multiple of 3. Therefore $s = 3$, leading to $n = 35$.

3.10. **48.** $n = 48 = 2^4 \cdot 3$, $\tau(n) = 10$ [6, A030628] , $\tau(n+1) = 3$ [6, A001248] . $\tau = 10$ requires $n = p^9$ or $n = pq^4$; $\tau = 3$ requires $n+1 = r^2$. seqfan list [12]: There are two cases:

- n odd: then $n+1 = 2^2$, but $\tau(3) = 2$ does not match.
- n even: $n = 2^9$ is discarded by direct evaluation. Then $n = 2p^4$ or $n = 2^4p$ (p odd), and $n+1 = r^2$ (r odd). Since $r^2 = 1 \pmod 8$ by (9), only $n = 16p$ remains. Then $16p = (r+1)(r-1)$ implies that $r-1$ and $r+1$ are adjacent even divisors of $16p$ with a greatest common divisor of 2. The factorizations of that form are $16p = 2 \times 8p$ and $16p = 2p \times 8$. The first form requires $r+1 = 2$ or $r-1 = 2$ which both do not generate solutions by direct inspection. We are left with $16p = 8 \times 2p$. Then $r-1 = 8$ does not yield a prime r, but $r+1 = 8$ (with $p = 2$) and $n = 48$ as was to be shown.

3.11. **63.** $n = 63 = 3^2 \cdot 7$, $\tau(n) = 6$ [6, A030515] , $\tau(n+1) = 7$ [6, A030516] . n is of the form p^5 or pq^2. $n+1$ is of the form r^6. seqfan list [12] (All prime variables denote odd primes):

- n even. Then $n = 2^5$ or $2q^2$ or 2^2p. $\tau(n+1) = 7$, hence $n+1$ must be an odd square r^6 and $n = 1 \pmod 8$ by Theorem 4. This matches only $n = 2^5$ because $2q^2 \equiv 2 \pmod 8$ and $4q^2 \equiv 4 \pmod 8$, which is no solution: $2^5 + 1 \nsim r^6$.
- n odd. Then $n+1$ is even and must be 2^6. Hence $n = 63$.

3.12. **64.** $n = 64 = 2^6$, $\tau(n) = 7$ [6, A030516] , $\tau(n+1) = 4$ [6, A030513] . $\tau = 7$ requires $n = p^6$. $\tau = 4$ requires $n+1 = q^3$ or qr. seqfan list [12] (All prime variables denote odd primes):

- n odd: $n = p^6$ and $n+1 = 2^3$ (but $2^3 - 1 \nsim p^6$) or $n+1 = 2q$. But $q = \frac{n+1}{2} = \frac{p^6+1}{2} = \frac{p^2+1}{2} \cdot (p^4 - p^2 + 1)$ is a nontrivial factorization of q — contradiction..
- n even: Then $n = 2^6$ as was to be shown.

3.13. **80.** $n = 80 = 2^4 \cdot 5$, $\tau(n) = 10$ [6, A030628] , $n+1 = 81 = 3^4$, $\tau(n+1) = 5$ [6, A030514] . $\tau = 10$ requires $n = p^9$ or pq^4. $\tau = 5$ requires $n+1 = r^4$. seqfan list [12] (All prime variables denote odd primes):

- n odd. Then $n+1 = 2^4$, but $\tau(15) \neq 10$.
- n even. Then $n = 2^9$ [but $\tau(2^9+1) \nsim r^4$] or $n = 2p^4$ or $n = 2^4 p$. Note that $n+1 = r^4$ is an odd square, is $\equiv 1 \pmod 8$, hence $n = 0 \pmod 8$, which leaves only $n = 16p$. But $r^4 - 1 = (r-1)(r+1)(r^2+1)$. All three factors are even, hence exactly one is divisible by 4. Also, exactly one is divisible by p. If $p \mid r^2+1$, we conclude $r^2+1 \leq 4$ – contradiction. Hence $r-1 = 2$, $r+1 = 4$, $r^2+1 = 2p$, i.e., $n = 80$.

3.14. **99.** $n = 99 = 3^2 \cdot 11$, $\tau(n) = 6$ [6, A030515] , $n+1 = 100 = 2^2 \cdot 5^2$, $\tau(n+1) = 9$ [6, A030627] . $\tau = 6$ requires $n = p^5$ or pq^2. $\tau = 9$ requires $n+1 = s^8$ or s^2t^2. seqfan list [12] (All prime variables denote odd primes):

- n even. $n = 2^5$ [but $\tau(33) \neq 9$] or $n = 2p^2$ or $n = 4p$ and once again $n+1$ is an odd square, hence n must be $\equiv 0 \pmod 8$ – contradiction.
- n odd. $n = p^5$ or $n = pq^2$. $n+1 = 2^8$ [but $\tau(255) \neq 6$] or $n+1 = 4r^2$. Then $pq^2 = (2r+1)(2r-1)$, a product of two adjacent odd coprime numbers. At most one of $2r+1$, $2r-1$ is divisible by q, hence
  - either $p = 2r-1$, $q^2 = 2r+1$. Since r is prime, $r = \pm 1 \pmod 6$, hence $2r - 1 = 1$ or $3 \pmod 6$. Since p is prime, only $p = 2r-1 \equiv 1 \pmod 6$ remains. But then $q^2 = 3 \pmod 6$ – contradiction.
  - or $p = 2r+1$, $q^2 = 2r-1$. Then $p = q^2+2$. If $q = 3$, we obtain $p = 11$ and $n = 99$. Otherwise, this implies $p \equiv 0 \pmod 3$ and $q^2 = 1$ – contradiction Hence we must have $n = 99$.

3.15. **288.** $n = 288 = 2^5 \cdot 3^2$, $\tau(n) = 18$ [6, A030636] , $n+1 = 289 = 17^2$, $\tau(n+1) = 3$ [6, A001248] . [9]:

The forms of numbers having 18 divisors are $n = p^{17}$, pq^8, p^2q^5, pq^2r^2. The forms of numbers having 3 divisors is $n+1 = s^2$.

- Suppose n is odd. Then $n+1$ is even, which implies that $n+1 = 2^2$. But $n = 3$ is not possible because $\tau(3) \neq 18$. So n cannot be odd.
- Suppose n is even. Then one of the prime factors of n must be 2. So the forms of even n are 2^{17}, $2q^8$, $256q$, $4q^5$, $32p^2$, $2q^2r^2$, or $4pq^2$. Also $n = s^2 - 1 = (s-1)(s+1)$. Because n is even, s must be odd. Hence $2^3 | s^2 - 1$ (Theorem 4). This eliminates the forms $2q^8$, $4q^5$, $2q^2r^2$, and $4pq^2$. The remaining forms are 2^{17}, 2^8p, and 2^5p^2. It cannot be 2^{17} because $\tau(2^{17}+1) \neq 3$.

  Writing $s = 2y+1$ for some y, $n = s^2 - 1 = 4y(y+1)$.
  - Setting $4y(y+1) = 2^8 p$, we see that either y or $y+1$ must be prime. If $y+1$ is prime, then we obtain $4y = 2^8$, which means $y = 2^6$. But

then $y+1$ is not prime – a contradiction. If y is prime, then we obtain $4(y+1) = 2^8$, which means $y = 63$ – also not prime. So the form cannot be $2^8 p$.

– Finally, we have the case $4y(y+1) = 2^5 p^2$. Equivalently, $y(y+1) = 8p^2$. It is well known that y and $y+1$ are coprime for all y. Hence, either y or $y+1$ must have the factor 8. If $y = 8$, then we obtain the known solution $n = 288$. If $y = 8k$ for some odd $k > 1$, then $y(y+1) = 8k(8k+1)$, which means that $k(8k+1)$ must the square of a prime. However, this is not possible because it has a factor of k and the other factor is not k. Similarly, if $y+1 = 8k$ for some $k > 1$, then $y(y+1) = 8k(8k-1)$, which means that $k(8k-1)$ is the square of a prime for $k > 1$, which is not possible.

Hence, $n = 288$ is the only solution.

3.16. **528.** $n = 528 = 2^4 \cdot 3 \cdot 11$, $\tau(n) = 20$ [6, A030638] , $n+1 = 529 = 23^2$, $\tau(n+1) = 3$ [6, A001248] . seqfan list [12]: $n+1$ is the square of a prime and surely $\neq 4$, hence n is even. n is one of 2^{19} [but then $\tau(n+1) \neq 3$], $2^9 p$, $2^4 p^3$, $2^4 pq$, $2^3 p^4$, $2p^9$, $2p^4 q$. Since $n+1$ is an odd prime square r^2, $n = 0 \pmod 8$ by Eq. (9). This leaves cases where n supports at least the factor 2^3: $n = 2^9 p$, $2^4 p^3$, $2^4 pq$, $2^3 p^4$. In $n = (r-1)(r+1)$, both factors are even with a greatest common divisor of 2.

- If $n = 2^9 p$, these factors cannot be 2 and $2^8 p$ but must be 2^8 and $2p$, which implies $r = 257$ (255 is composite) and hence $p = (r+1)/2 = 129$, composite – contradiction.
- If $n = 2^4 p^3$, $r \pm 1$ must be 8 and $2p^3$, which is impossible.
- If $n = 2^3 p^4$, $r \pm 1$ must be 4 and $2p^4$, which is impossible.
- If $n = 2^4 pq$, $r \pm 1$ must be 8 and $2pq$ or, without loss of generality, $8p$ and $2q$. The first case is quickly ruled out by inspection. In the second case, $r \neq 3$ (as $n \neq 8$), hence one of $r \pm 1$ is a multiple of 3 [since in the three consecutive numbers $r-1$, r, $r+1$ one must be a multiple of 3, and this cannot be the prime $r > 3$], i.e., $p = 3$ or $q = 3$. (Note that the factor of 3 cannot come from 8 or 2, and p and q must remain prime.) If $q = 3$, then $r = 6 \pm 1$, $n = 24$ or $n = 48$, but then $\tau(n) \neq 20$. If $p = 3$, then $r = 24 \pm 1$, hence $r = 23$ and $n = 528$ as was to be shown.

3.17. **575.** $n = 575 = 5^2 \cdot 23$, $\tau(n) = 6$ [6, A030515] , $n+1 = 576 = 2^6 \cdot 3^2$, $\tau(n+1) = 21$ [6, A137484] . seqfan list [12]: Then $n = p^5$ or $p^2 q$ and $n+1 = r^{20}$ or $r^6 s^2$.

- If $n+1 = r^{20}$, then $n = r^{20} - 1 = (r-1)(r+1)(r^2+1)(r^4 - r^3 + r^2 - r - 1)(r^4 + r^3 + r^2 + r + 1)(r^8 - r^6 + r^4 - r^2 + 1)$. The case $r = 2$ is eliminated explicitly: $\tau(2^{20} - 1) \neq 6$. Hence n can be written as product of 6 factors ≥ 2, contradicting the prime signature(s) of n.
- Hence $n+1 = r^6 s^2$. If $n+1$ is odd, then as usual $n = 0 \pmod 8$, hence not of the form $p^2 q$ but $n = 2^5$—but $\tau(2^5 + 1) \neq 21$. Therefore, n is odd and can be written as $n = (r^3 s - 1)(r^3 s + 1)$, i.e., as product of two adjacent and hence coprime odd numbers. This eliminates the case $n = p^5$ ($r^3 s \pm 1$ would be of the form 1 and p^5) and enforces $|p^2 - q| = 2$. $q = 3$ is impossible and the case $p = 3$ (with $q = 7$ or $q = 11$) can be eliminated explicitly: $\tau(3^2 \cdot 7 + 1) \neq 21$, $\tau(3^2 \cdot 11 + 1) \neq 21$. Equations (9) and (25) mean $p^2 \equiv 1 \pmod{24}$, so $q \equiv -1 \pmod{24}$ and $q = p^2 - 2$. But then $p^2 = r^3 s + 1$

and $r^3 s$ must be $0 \pmod 8$, hence $r = 2$. Thus we arrive at the conditions $p^2 = 8s+1$, $q = 8s-1$. Exactly one of s, $8s+1$, $8s-1$ must be $0 \pmod 3$, hence one of p, q, s is 3. The only valid possibility is $s = 3$, $p = 5$, $q = 23$, leading to $n = 575$.

3.18. **624.** $n = 624 = 2^4 \cdot 3 \cdot 13$, $\tau(n) = 20$ [6, A030638] , $n + 1 = 625 = 5^4$, $\tau(n+1) = 5$ [6, A030514] [9]: The forms of numbers having 20 divisors are $n = p^{19}$, pq^9, p^3q^4, pqr^4 The forms of numbers having 5 divisors is $n + 1 = s^4$.

- Suppose n is odd. Then $n + 1$ is even, which implies that $n + 1 = 2^4$. But $n = 15$ is not possible because $\tau(15) \neq 20$. So n cannot be odd.
- Suppose n is even. Then one of the prime factors of n must be 2. So the forms of even n are 2^{19}, $2p^9$, $512p$, $8p^4$, $16p^3$, $2pq^4$, or $16pq$. Also $n = s^4 - 1 = (s-1)(s+1)(s^2+1)$. Because n is even, s must be odd. Hence 16 divides $s^4 - 1$ by (10). This eliminates the forms $2p^9$, $8p^4$, and $2pq^4$. The remaining forms are 2^{19}, $512p$, $16p^3$, and $16pq$. It cannot be 2^{19} because $\tau(2^{19} + 1) \neq 5$.

  Prime s is either 3 or a number of the form $6k - 1$ or $6k + 1$. If $s = 3$, then $n = 80$ and $\tau(80) = 10$, which contradicts the requirement. For s of the form $6k + 1$, we obtain $s^4 - 1 = 24k(3k + 1)(18k^2 + 6k + 1)$. For s of the form $6k - 1$, we obtain $s^4 - 1 = 24k(3k - 1)(18k^2 - 6k + 1)$. For $k > 1$, the three factors k, $3k + 1$, and $18k^2 + 6k + 1$ are coprime and greater than 1, as are the three factors k, $3k - 1$, and $18k^2 - 6k + 1$. When $s > 3$, then 2 and 3 divide $s^4 - 1$.
  - Case $512p$. Suppose $s^4 - 1 = 512p$. As noted above, 3 divides $s^4 - 1$. Hence, we can assume $p = 3$. This p produces $n = 512 \cdot 3 = 1536$. But $\tau(n + 1) = 4$, a contradiction.
  - Case $16p^3$. Suppose $s^4 - 1 = 16p^3$. As noted above, 3 divides $s^4 - 1$. Hence, we can assume $p = 3$. This produces $n = 16 \cdot 27$. But $\tau(n+1) = 2$, a contradiction.
  - Case $16pq$. Suppose $s^4 - 1 = 16pq$. For prime s of the form $6k - 1$, we have $s^4 - 1 = 24k(3k - 1)(18k^2 - 6k + 1)$. Letting $k = 1$, we obtain the known solution $n = 624$. As noted above, 3 divides $s^4 - 1$. Hence, we can assume $q = 3$. So we can assume $s^4 - 1 = 48p$, or equivalently, $k(3k - 1)(18k^2 - 6k + 1) = 2p$. For $k > 1$, the left hand side is the product of at least 3 distinct primes because the three factors are all relatively prime to each other. The right hand side has only 2 distinct prime factors. Hence, there is no solution for $k > 1$. The same argument applies to primes s of the form $6k + 1$.

Hence, $n = 624$ is the only solution.

3.19. **728.** $n = 728 = 2^3 \cdot 7 \cdot 13$, $\tau(n) = 16$ [6, A030634] , $n + 1 = 729 = 3^6$, $\tau(n+1) = 7$ [6, A030516] . $\tau = 16$ requires that n is of the form p^{15} or pq^7 or p^3q^3 or pqr^3 or $pqrs$ with p, q, r and s prime. $\tau = 7$ requires that $n + 1 = t^6$ with t prime. Sub-cases:

- $1 + p^{15} = t^6$. The parity argument requires that either p or t is even. $p = 2$ is no solution since $1 + 2^{15} \not\approx t^6$ and $t = 2$ is no solution since $2^6 - 1 \not\approx p^{15}$. So this case does not create solutions.
- $1 + pq^7 = t^6$. Again comparing parities on both sides, exactly one of the primes must be 2.

- $1+2q^6 = t^6$ is equivalent to $2q^6 = t^6-1$. Eq. (11) requires that t^6-1 has a factor 2^3 which is not supported by $2q^6$. This contradiction means this sub-case does not provide solutions.
- $1+2^7p = t^6$. By the parity argument, t is an odd prime; the odd primes $3, 5$ and 7 are excluded by direct evaluation. Then Theorem 6 says that t^6-1 has at least 3 distinct prime factors, which is not supported by the other side, 2^7p, so there are no solutions from this sub-case.
- $1+pq^7 = 2^6$. Has no solutions since p and q are distinct primes > 2.

- $1+p^3r^3 = t^6$. For primes $t = 2, 3, 5, 7$ the corresponding t^6-1 do not have the $(pq)^3$ signature. For primes $t > 7$, Theorem 6 requires the t^6-1 has at least 3 distinct prime factors, but p^3q^3 does not support his: no solutions.
- $1+pqr^3 = t^6$. By the parity argument, exactly one of the primes must be 2.
  - $1+2qr^3 = t^6$, i.e. $2qr^3 = t^6-1$. Here (11) means that t^6-1 has at least a factor 2^3, which is not supported by the format $2qr^3$. This mismatch means there are no solutions from this branch.
  - $1+2^3pq = t^6$. Insertion of $t = 2k+1$, $k \ge 1$, yields $2^3pq = 4k(k+1)(4k^2+6k+3)(4k^2+2k+1)$, therefore $pq = T(k)(4k^2+6k+3)(4k^2+2k+1)$, where $T(k)$ is the k-th triangular number. $k = 1$ gives the known solution with $p = 7$, $q = 13$. For $k > 1$, $T(k)$ may be even, with an odd parity of the pq on the left hand side and an even parity on the right hand side (no solutions). If $T(k)$ is odd, we have also two coprime odd factors $4k^2+6k+3$ and $4k^2+2k+1$, both distinct from $T(k)$, so Ω applied to the right hand side must be ≥ 3. So this prevents further solutions because $\Omega(pq) = 2$ on the left hand side.
  - $1+pqr^3 = 2^6$ does not yield solutions because $2^6-1 \nsim pqr^3$.
- $1+pqrs = t^6$. By the parity argument, one of the primes must be 2.
  - $1+2qrs = t^6$. As above t^6-1 for odd primes t contains a factor 2^3, but only 2^1 appears in $2qrs$. This mismatch means there are no solutions from this branch.
  - $1+pqrs = 2^6$ does not contribute solutions because $2^6-1 \nsim pqrs$.

3.20. **960.** $n = 960 = 2^6 \cdot 3 \cdot 5$, $\tau(n) = 28$ [6, A137491] , $n+1 = 961 = 31^2$, $\tau(n+1) = 3$ [6, A001248] [9]:

The forms of numbers having 28 divisors are $n = p^{27}$, pq^{13}, p^3q^6, pqr^6.

The forms of numbers having 3 divisors is $n+1 = s^2$.

- Suppose n is odd. Then $n+1$ is even, which implies that $n+1 = 2^2$. But $n = 3$ is not possible because $\tau(3) \ne 28$. So n cannot be odd.
- Suppose n is even. Then one of the prime factors of n must be 2. So the forms of even n are 2^{27}, $2q^{13}$, $2^{13}q$, 2^3q^6, 2^6p^3, $2qr^6$, or 2^6pq. We also know that $n+1 = s^2$ for some s. Hence $n = s^2-1 = (s-1)(s+1)$. Because n is even, s must be odd. Hence 8 divides s^2-1. This eliminates the forms $2q^{13}$ and $2qr^6$. The remaining forms are 2^{27}, $2^{13}q$, 2^3q^6, 2^6p^3, and 2^6pq. It cannot be 2^{27} because $\tau(2^{27}+1) \ne 3$.

  Writing $s = 2y+1$ for some y, we obtain $n = 4y(y+1)$.
  - Setting $4y(y+1) = 2^{13}q$, we see that either y or $y+1$ must be prime q. If $y+1$ is prime, then we obtain $4y = 2^{13}$, which means $y = 2^{11} = 2048$. But then $y+1$ is not prime—a contradiction. If y is prime, then we

obtain $4(y+1) = 2^{13}$, which means $y = 2047$—also not prime. So the form cannot be $2^{13}q$.

– The next case is $4y(y+1) = 2^3q^6$. Equivalently, $y(y+1) = 2q^6$. Either y or $y+1$ must have the factor 2. If $y = 2k$ for some odd $k > 1$, then $y(y+1) = 2k(2k+1)$, which means that $k(2k+1)$ must the sixth power of a prime. However, this is not possible because it has a factor of k and for the other factor to be a fifth power, we would need $k^5 = 2k+1$, which has no integer solution. Similarly, if $y+1 = 2k$ for some odd k, then $y(y+1) = 2k(2k-1)$, which means that $k(2k-1)$ is the sixth power of a prime, which is only possible if $k^5 = 2k-1$, which has the solution $k = 1$. So we obtain $y = 1$, which produces $n = 8$, but $\tau(8) \neq 28$.

– The next case is $4y(y+1) = 2^6p^3$. Equivalently, $y(y+1) = 16p^3$. Either y or $y+1$ must have the factor 16. If $y = 16k$ for some odd k, then $y(y+1) = 16k(16k+1)$, which means that $k(16k+1)$ must the cube of a prime. However, this is not possible because it has a factor of k and for the other factor to be a square, we would need $k^2 = 16k+1$, which has no integer solution. Similarly, if $y+1 = 16k$ for some odd k, then $y(y+1) = 16k(16k-1)$, which means that $k(16k-1)$ is the cube of a prime, which is only possible if $k^2 = 16k-1$, which has no integer solution.

– The final case is $(s-1)(s+1) = 2^6pq$. Recall that s must be prime. There are three possibilities (considering that $s-1$ and $1+s$ have a greatest common divisor of 2): $s \pm 1$ are 2 and 2^5pq, $s \pm 1$ are 2^5 and $2pq$, or $s \pm 1$ are $2p$ and 2^5q. The first possibility 2×2^5pq forces n to be 1 or 3. The prime $s = 3$ produces $n = 8$, which is clearly not a solution. The second possibility $2^5 \times 2pq$ forces s to be 31 or 33; the prime $s = 31$ produces the known solution $n = 960$. For the third possibility $2p \times 2^5q$, note that if prime $s > 3$, then 3 divides either $s-1$ or $s+1$ by Eq. (25). Hence p or q must be 3. This forces $s \pm 1$ to be $2 \cdot 3$ or $2^5 \cdot 3$. The primes $s = 5$, 7 and 97 produce $n = 24$, 48, and 9408, none of which have the required form 2^6pq.

Hence, $n = 960$ is the only solution.

3.21. **1023.** $n = 1023 = 3 \cdot 11 \cdot 31$, $\tau(n) = 8$ [6, A030626] , $n+1 = 1024 = 2^{10}$, $\tau(n+1) = 11$ [6, A030629] . seqfan list [10]: If n has 8 divisors, then $n = p^7$, pq^3, or pqr for distinct primes p, q, r. If $n+1$ has 11 divisors, then $n+1 = s^{10}$ is the only possible form with s prime.

- Suppose n is even. Since $n = p^7$, pq^3, or pqr, one of the primes is 2. So n must be either 2^7, $2q^3$, 2^3p, or $2qr$ with q and r odd primes. The case $n = 2^7$ is not possible because then $n+1 \nsim s^{10}$. $n+1 = s^{10}$ means s must be odd and $n = s^{10}-1 = (s-1)(s+1)(s^4+s^3+s^2+s+1)(s^4-s^3+s^2-s+1)$. 2^3 divides $s^{10}-1$ by Eq. (13), implying that the forms $2q^3$ and $2qr$ are not possible. So the remaining case is $n = 2^3p$. The factorization of $s^{10}-1$ shows that after dividing by 8, we are always left with a composite number. Hence the form 2^3p is also not possible.
- Suppose n is odd. Then $n+1$ is even, which implies s^{10} is even, which implies $s = 2$. This gives us the only solution, $n = 1023$.

3.22. **1024.** $n = 1024 = 2^{10}$, $\tau(n) = 11$ [6, A030629] , $n + 1 = 1025 = 5^2 \cdot 41$, $\tau(n+1) = 6$ [6, A030515] . seqfan list [10]: If n has 11 divisors, then $n = s^{10}$ is the only possible form with s prime. If $n + 1$ has 6 divisors, then $n + 1 = p^5$ or pq^2 for distinct primes p and q.

Suppose n is odd. Then n+1 is even, implying that $n+1$ is either 2^5, $2q^2$, or $4p$. Also $n + 1 = s^{10} + 1 = (s^2 + 1)(s^8 - s^6 + s^4 - s^2 + 1)$. Because s is odd, the $s^2 + 1$ factor equals $2f$ for some odd f, and $s^8 - s^6 + s^4 - s^2 + 1$ is anyway odd. Hence, the form of $n + 1$ is not 2^5 or $4p$. It cannot be $2q^2$ because that would require $(s^2 + 1)/2 = s^8 - s^6 + s^4 - s^2 + 1$, an equation whose integer roots are $s = -1$ and 1.

Suppose n is even. This implies s^{10} is even, which implies $s = 2$. This gives us the only solution, $n = 1024$.

3.23. **1088.** $n = 1088 = 2^6 \cdot 17$, $\tau(n) = 14$ [6, A030632] , $n + 1 = 1089 = 3^2 \cdot 11^2$, $\tau(n + 1) = 9$ [6, A030627] . $\tau(n) = 14$ requires the prime signature $n = p^{13}$ or $n = pq^6$; $\tau(n + 1) = 9$ requires $n + 1 = r^8$ or $n + 1 = r^2s^2$. The case $n = r^8 - 1 = (r^4+1)(r^2+1)(r^2-1)$ is ruled out because a fourth power minus one is not supported by the p^{13} or pq^6, see Appendix B. Therefore $n = (rs)^2 - 1 = (rs + 1)(rs - 1)$.

- Suppose n is odd. Then p^{13} or pq^6 are odd and rs is even, which means $r = 2$ by our name convention. So $n = 4s^2 - 1$, where s is an odd prime. We can rule out $s = 3$ because $n = (2 \cdot 3)^2 - 1 = 5 \cdot 7$ does not match $\tau(n) = 14$. By the Mod-3 criterion, p^{13} or pq^6 must have a prime factor 3. The case of $n = 3^{13}$ is ruled out because $\tau(3^{13} + 1) \neq 9$. So $n = 3q^6$ or $n = 3^6p$.
  - $n = 3q^6 = 4s^2-1$ and the branch $s = 3k+1$ means $q^6 = (2k+1)(6k+1)$. Since $6k+1$ and $2k+1$ are coprime, this requires $2k+1 = 1 \wedge 6k+1 = q^6$, therefore $k = 0$, which is no solution.
  - $n = 3q^6 = 4s^2-1$ and the branch $s = 3k+2$ means $q^6 = (2k+1)(6k+5)$. Since $6k+5$ and $2k+1$ are coprime, this requires $2k+1 = 1 \wedge 6k+5 = q^6$, therefore $k = 0$, which is no solution.
  - $n = 3^6p = 4s^2 - 1$ and the branch $s = 3k+1$ means $3^5p = (2k+1)(6k+1)$, $(k > 4)$, Since $6k + 1$ and $2k + 1$ are coprime, either $3^5 = 2k + 1$ (but $k = 121$ implies $s = 364$, composite) or $3^5 = 6k + 1$ (but this k is not integer).
  - $n = 3^6p = 4s^2 - 1$ and the branch $s = 3k + 2$ means $3^5p = (2k + 1)(6k + 5)$. Again $6k + 5$ and $2k + 1$ are coprime so either $3^5 = 2k + 1$ (but $k = 121$ implies $s = 365$, composite) or $3^5 = 6k + 5$ (but k is not integer).
- Suppose n is even. Then p^{13} or pq^6 is even and rs is odd. The case $n = 2^{13}$ is excluded since $\tau(2^{13} + 1) \neq 9$ does no match the requirement.
  - The case $2q^6 = (rs)^2 - 1 = (rs + 1)(rs - 1)$ is excluded, because Eq. (9) requires a minimum factor 2^3 which is not supported by the form $2q^6$.
  - The case $2^6q = (rs + 1)(rs - 1)$ remains. $rs + 1$ and $rs - 1$ are two consecutive even numbers with a greatest common divisor of 2.
    * The split $2 = rs-1$ and $2^5q = rs+1$ is excluded because $rs \geq 15$, the smallest product of two odd primes.
    * The split $2^5 = rs - 1$ and $2q = rs+1$ requires $rs = 33$, so $r = 3$, $s = 11$, $q = 17$, which furnishes the known solution.

3.24. **1295.** $n = 1295 = 5 \cdot 7 \cdot 37$, $\tau(n) = 8$ [6, A030626] , $n+1 = 1296 = 2^4 \cdot 3^4$, $\tau(n+1) = 25$ [6, A137488] . $\tau = 8$ requires the prime signature $n = p^7$ or pq^3 or pqr; $\tau = 25$ requires $n+1 = s^{24}$ or s^4t^4. In both cases $n+1$ is a fourth power which rules out the signatures $n = p^7$ or $n = pq^3$, see Appendix B. The format $n = pqr$ and $n = (st)^4 - 1$ requires that $p = st - 1$, $q = st + 1$ and $r = (st)^2 + 1$; the format $n = pqr$ and $n = s^{24} - 1$ requires $p = s^6 - 1$, $q = s^6 + 1$ and $r = s^{12} + 1$. [We tacitly follow the convention that in unique factorizations with the same exponent, like in pqr, the smaller prime is the one named earlier in the alphabet; here $p < q < r$.] In the first case

- Assume st is odd, then $(st)^4$ is odd and n must be even such that the smallest of p, q and r must be even, i.e., $p = 2$, the only even prime, which cannot occur because $p = st - 1 \geq 5$ since $st \geq 6$, the smallest product of distinct primes.
- So st must be even, actually $s = 2$ if we let $s < t$, which requires $n = 16t^4 - 1$. The Pisano Period for $16t^4 - 1$ modulo 5 has period length 5: [4,0,0,0,0], which means for all primes t of the form $5k+1$, $5k+2$, $5k+3$ and $5k+4$ we have $16t^4 - 1 \equiv 0 \pmod 5$, which requires that $5 \mid pqr$, so at least one of p, q or r must be a multiple of 5, and since these are distinct primes, one of them must be 5.
  - Assume 5 is the largest, $r = (st)^2 + 1 = 5$: this cannot happen because $t \neq 1$.
  - Assume 5 is the middle factor, $q = st + 1 = 5$: this cannot happen because $s \neq t$.
  - So 5 is the smallest factor, $5 = st - 1$ and therefore $t = 3$. Having thus fixed $s = 2$ and $t = 3$, $n = 16 \cdot 3^4 - 1 = 1295$ is uniquely determined and no further solutions exist.

In the second case

- Assume s^6 is odd, then $(s^6)^4$ is odd and n must be even such that the smallest of p, q and r must be even, i.e., $p = 2$, the only even prime, which cannot occur because $p = s^6 - 1 \geq 63$ since $s \geq 2$.
- So s^6 must be even, actually $s = 2$. This case is ruled out by direct inspection, because $\tau(2^{24} - 1) \neq 8$.

3.25. **2303.** $n = 2303 = 7^2 \cdot 47$, $\tau(n) = 6$ [6, A030515] , $n+1 = 2304 = 2^8 \cdot 3^2$, $\tau(n+1) = 27$ [6, A137490] [9]:

The forms of numbers having 6 divisors are $n = p^5$ and $n = pq^2$. The forms of numbers having 27 divisors are $n+1 = r^{26}$, r^2s^8, and $r^2s^2t^2$.

- Suppose n is even. This forces n to have one of the forms 2^5, $2p^2$, or $4p$. It is clearly not 2^5 because $\tau(2^5+1) \neq 27$. So we must equate $2p^2$ and $4p$ to each of the forms for $n+1$ with odd r, s, t. There are no solutions for even n:
  - Case $2p^2 = r^{26} - 1$. 2^3 divides the RHS by Eq. (20), but not the LHS.
  - Case $4p = r^{26} - 1$. 2^3 divides the RHS by Eq. (20), but not the LHS.
  - Case $2p^2 = r^2s^8 - 1$. 2^3 divides the RHS by Eq. (9), but not the LHS.
  - Case $4p = r^2s^8 - 1$. 2^3 divides the RHS by Eq. (9), but not the LHS.
  - Case $2p^2 = r^2s^2t^2 - 1$. 2^3 divides the RHS by Eq. (9), but not the LHS.
  - Case $4p = r^2s^2t^2 - 1$. 2^3 divides the RHS by Eq. (9), but not the LHS.

- Suppose n is odd. Then $n+1$ is even, forcing one of its prime factors to be 2. Hence, the forms of $n+1$ are 2^{26}, 2^2r^8, 2^8r^2, and $2^2r^2s^2$. Clearly 2^{26} is not a solution because $\tau(2^{26}-1) \neq 6$. There are six cases to consider:
  - Case $p^5 = 4r^8 - 1$. This equation of the form $a^2 - 1 = p^5$ has no solutions, see Appendix B.1.
  - Case $p^5 = 2^8r^2 - 1$. This equation of the form $a^2 - 1 = p^5$ has no solutions, see Appendix B.1.
  - Case $p^5 = 2^2r^2s^2 - 1$. This equation of the form $a^2 - 1 = p^5$ has no solutions, see Appendix B.1.

  For the next three cases we use that fact that an odd prime is either 3 or a number of the form $6k-1$ or $6k+1$.
  - Case $pq^2 = 2^2r^8 - 1 = (2r^4+1)(2r^4-1)$. This is a product of two adjacent odd (therfore coprime) numbers, and by the Mod-3 criterion one of p, q or r must be 3. $r = 3$ is ruled out because $2^2 \cdot 3^8 - 1 \nsim pq^2$, so p or q is 3. Since $3 \nmid 2r^4 - 1$ we need $3 \mid 2r^4 + 1$, but this requires $r \equiv 0 \pmod 3$ — contradiction because r must be a prime distinct from p and q.
  - Case $pq^2 = 2^8r^2 - 1 = (2^4r+1)(2^4r-1)$. If $r = 3$, we obtain the known solution, $n = 256 \cdot 9 - 1 = 2303$. Otherwise the Mod-3 criterion requires $p = 3$ or $q = 3$. Since $2^4r \pm 1$ are adjacent odd coprime numbers, the splits are $2^4r - 1 = 1 \wedge 2^4r + 1 = pq^2$ (obviously no solution) or $2^4r + 1 = p \wedge 2^4r - 1 = q^2$ (solution not for $q = 3$ nor for $p = 3$) or $2^4r - 1 = p \wedge 2^4r + 1 = q^2$ (solution not for $q = 3$ nor for $p = 3$).
  - Case $pq^2 = 4r^2s^2 - 1 = (2rs+1)(2rs-1)$. The Mod-3 criterion requires one of p, q, r to be 3.
    * If $r = 3$ and $s = 6k+1$, then $4r^2s^2 - 1 = (5+36k)(7+36k)$, product of two coprime factors. But $5+36k$ is not a square because a square is never $\equiv 5 \pmod 6$. Similarly, $7+36k$ is not a square because a square is $\not\equiv 3 \pmod 4$. The case $r = 3$ and $s = 6k-1$ is similar.
    * If $rs = 3k+1$, then $(2rs)^2 - 1 = pq^2 = 3(2k+1)(6k+1)$, which is impossible for $p = 3$ because $(2k+1)(6k+1) \nsim q^2$ is a product of two coprime odd numbers, and gives no solution for $q = 3$ because $3p = (2k+1)(6k+1)$ yields $k = 1$ and the invalid $rs = 4$.
    * If $rs = 3k+2$, then $(2rs)^2 - 1 = pq^2 = 3(2k+1)(6k+5)$, which is impossible for $p = 3$ because $(2k+1)(6k+5) \nsim q^2$ is a product of two coprime odd numbers, and gives no solution for $q = 3$ because $3p = (2k+1)(6k+5)$ yields $k = 1$ and the invalid $rs = 5$.

Hence, $n = 2303$ is the only solution.

3.26. **2400.** $n = 2400 = 2^5 \cdot 3 \cdot 5^2$, $\tau(n) = 36$ [6, A175746] , $n+1 = 2401 = 7^4$, $\tau(n+1) = 5$ [6, A030514] . n is of the form p^{35} or pq^{17} or p^2q^{11} or p^3q^8 or p^5q^5 or $p^2q^2r^3$ or pq^2r^5 or pqr^8 or pqr^2s^2, and $n+1 = t^4$, where p, q, r, s are distinct primes, and t is prime. We consider how the factorizations of $t^4 - 1$ may match these products of prime powers [11]: For primes $t < 5$ we swiftly observe by direct computation that these do not generate solutions. All larger primes t generate

$t^4 - 1 \equiv 0 \pmod{240}$, Theorem 5. As $240 = 2^4 \times 3 \times 5$, the prime signatures p^{35}, pq^{17}, p^2q^{11}, p^3q^8 and p^5q^5 are already excluded for n, because they do not support the minimum 3 prime factors of $t^4 - 1$. Furthermore $p^2q^2r^3$ and pqr^2s^2 are excluded because they do not support the fourth power in $2^4 \times 3 \times 5$ in their greatest exponent. This leaves two prime signatures of n:

- pq^2r^5: here $r = 2$ because the other two factors do not support the 4th power in $2^4 \cdot 3 \cdot 5$. p and q must be 3 and 5 or vice versa, because there is no fourth prime factor left for flexibility, and $p = 3 \wedge q = 5$ is the known case; $p = 5 \wedge q = 3$ produces $n = 2^5 \cdot 5 \cdot 3^2$, but $\tau(2^5 \cdot 5 \cdot 3^2 + 1) \neq 5$.
- pqr^8: here $r = 2$ because the other two factors do not support the 4th power in $2^4 \cdot 3 \cdot 5$. p and q must be 3 and 5 or vice versa, because there is no fourth prime factor left for flexibility. This demands $n = 3 \cdot 5 \cdot 2^8$ but $\tau(3 \cdot 5 \cdot 2^8 + 1) \neq 5$.

3.27. **4095.** $n = 4095 = 3^2 \cdot 5 \cdot 7 \cdot 13$, $\tau(n) = 24$ [6, A137487] , $n + 1 = 4096 = 2^{12}$, $\tau(n + 1) = 13$ [6, A030631] . seqfan list [7]: If $n + 1$ has 13 divisors, then p^{12} is the only possible form, with p prime. However, $p^{12} - 1 = (p - 1)(p + 1)(p^2 - p + 1)(p^2 + 1)(p^2 + p + 1)(p^4 - p^2 + 1)$ which is, for $p > 2$, of the form $2a \cdot 2b \cdot c \cdot 2d \cdot e \cdot f$ with a, b, c, d, e, f odd numbers. A number of this form cannot have less than 28 divisors.

So $p = 2$ is the only possibility.

3.28. **4096.** $n = 4096 = 2^{12}$, $\tau(n) = 13$ [6, A030631] , $n + 1 = 4097 = 17 \cdot 241$, $\tau(n + 1) = 4$ [6, A030513] . seqfan list [7]: $\tau = 13$ requires $n = p^{12}$ with p prime. $\tau = 4$ requires $n + 1 = q^3$ or qr for distinct primes q and r.

$n + 1 = p^{12} + 1 = (p^4 + 1)(p^8 - p^4 + 1)$. For odd $p > 2$, this is the product of the even number $p^4 + 1 \geq 82$ and the odd number $p^8 - p^4 + 1 \geq 3^8 - 3^4 + 1$, which can't be of the form q^3 or qr with even q.

So the only solution is $p = 2$.

3.29. **5328.** $n = 5328 = 2^4 \cdot 3^2 \cdot 37$, $\tau(n) = 30$ [6, A137493] , $n + 1 = 5329 = 73^2$, $\tau(n + 1) = 3$ [6, A001248] . The requirements are $n = p^{29}$ or pq^{14} or p^2q^9 or p^4q^5 or pq^2r^4, and $n + 1 = s^2$. Cases where n is odd do not exist, because this requires s to be even, and $s = 2$, the only even prime, generates $n = 3$ which matches none of these prime signatures. The sub-cases to be examined for even n, odd s, are

- $n = 2^{29} = s^2 - 1$. This yields $\tau(n + 1) \neq 3$, contradicting the requirement.
- $n = 2q^{14} = s^2 - 1$. Theorem 4 requires that the LHS is divisible by 2^3, which it does not support.
- $n = 2^{14}p = s^2 - 1$. The Mod-3 criterion requires either $p = 3$ (but $2^{14} \cdot 3 + 1 \not\sim s^2$) or $s = 3$ (but $3^2 - 1 \not\sim 2^{14}p$).
- $n = 2^2q^9 = s^2 - 1$. Theorem 4 requires that the LHS is divisible by 2^3, which it does not support.
- $n = 2^9p^2 = s^2 - 1$. The Mod-3 criterion requires either $p = 3$ (but $2^9 \cdot 3^2 + 1 \not\sim s^2$) or $s = 3$ (but $3^2 - 1 \not\sim 2^9p^2$).
- $n = 2^4q^5 = s^2 - 1 = (s + 1)(s - 1)$. With the Mod-3 criterion either $q = 3$ or $s = 3$ is needed, but $2^4 \cdot 3^5 + 1 \not\sim s^2$ and $3^2 - 1 \not\sim 2^4q^5$.
- $n = 2^5p^4 = s^2 - 1$. With the Mod-3 criterion either $p = 3$ or $s = 3$ is needed, but $2^5 \cdot 3^4 + 1 \not\sim s^2$ and $3^2 - 1 \not\sim 2^5p^4$.

- $n = 2q^2r^4 = s^2 - 1$. No solutions because the RHS has a prime factor 2^3 according to Eq. (9), which is not supported by the LHS.
- $n = 2^2pr^4 = s^2 - 1$. No solutions because the RHS has a prime factor 2^3 according to Eq. (9), which is not supported by the LHS.
- $n = 2^4pq^2 = s^2 - 1$, the branch of the known solution. We need $s = 2^4l + 1$ or $2^4l + 7$ or $2^4l + 9$ or $2^4l + 15$ to ensure the RHS is a multiple of 2^4. In the two cases $2^4l + 1$ and $2^4l + 15$, pq^2 would be even and cannot generate solutions. The remaining cases are
  - $s = 2^4l + 7$ with $pq^2 = (2l + 1)(8l + 3)$ and coprime $8l + 3$ and $2l + 1$, with 2 possible associations:
    - $p = 2l + 1$, $q^2 = 8l + 3$ is excluded because for all odd primes q $q^2 \equiv 1 \pmod 8$ and never $\equiv 3 \pmod 8$ (Theorem 4).
    - $p = 8l+3$ and $q^2 = 2l+1$ demands $p = 4q^2-1 = (2q+1)(2q-1)$, but that form requires that p is composite (since $2q - 1 = 1$ is excluded).
  - $s = 2^4l + 9$ with $pq^2 = (2l + 1)(8l + 5)$ and coprime $8l + 5$ and $2l + 1$, with 2 possible associations:
    - $p = 2l + 1$, $q^2 = 8l + 5$ is excluded because for all odd primes q $q^2 \equiv 1 \pmod 8$ and never $\equiv 5 \pmod 8$.
    - $p = 8l + 5$ and $q^2 = 2l + 1$ is solved by $l = 4$ to generate the known solution $s = 73$, $p = 37$ and $q = 3$. Now suppose $q > 3$, then $q^2 \equiv 1 \pmod 3$ by Eq. 25, and because $s = 8q^2 + 1$ also $s \equiv 0 \pmod 3$. Therefore $3 \mid s$ which contradicts that s is a prime. So no further solutions exist.

3.30. **6399.** $n = 6399 = 3^4 \cdot 79$, $\tau(n) = 10$ [6, A030628] , $n + 1 = 6400 = 2^8 \cdot 5^2$, $\tau(n + 1) = 27$ [6, A137490] . $\tau = 10$ requires $n = p^9$ or pq^4, and $\tau = 27$ requires $n + 1 = r^{26}$ or r^2s^8 or $r^2s^2t^2$. The 6 combinations of these prime signatures are

- $p^9 = r^{26} - 1$. The parity argument requires $p = 2$ or $r = 2$, but $2^9 + 1 \nsim r^{26}$ nor $2^{26} - 1 \nsim p^9$ give solutions.
- $p^9 = r^2s^8 - 1$. The format $p^9 = a^2 - 1$ does not have solutions (Appendix B.1)
- $p^9 = r^2s^2t^2 - 1$. The format $p^9 = a^2 - 1$ does not have solutions (Appendix B.1)
- $pq^4 = r^{26} - 1$. The case $r = 3$ is not a solution because $3^{26} - 1 \nsim pq^4$. So considering $r \equiv \{1, 5\} \pmod 6$ we find that $6 \mid r^{26} - 1$ such that p and q must be 2 and 3, but $2 \cdot 3^4 \nsim r^{26} - 1$ and $3 \cdot 2^4 \nsim r^{26} - 1$.
- $pq^4 = r^2s^8 - 1$. In terms of the general solutions in Appendix B.2.4, we require $a = rs^4$, and find only $a = 5 \cdot 2^4 = 80 = 79 + 1 = 3^4 - 1$ (smallest case of the first bullet) equivalent to known solution.
- $pq^4 = r^2s^2t^2 - 1$. This is a special case of the solutions considered in Appendix B.2.4. Considering the various possible forms of the variable a listed there, no solutions exist where a is a product of three distinct primes.

3.31. **6723.** $n = 6723 = 3^4 \cdot 83$, $\tau(n) = 10$ [6, A030628] , $n + 1 = 6724 = 2^2 \cdot 41^2$, $\tau(n+1) = 9$ [6, A030627] . $\tau = 10$ requires $n = p^9$ or pq^4; $\tau = 9$ requires $n+1 = r^8$ or r^2s^2. The combinations are

- $p^9 = r^8 - 1$. As discussed in Appendix B.1 this has no solutions.
- $p^9 = r^2s^2 - 1$. As discussed in Appendix B.1 this has no solutions.

- $pq^4 = r^8 - 1$ is discussed in Appendix B.2.4: solutions with $a = r^4$ do not exist.
- $pq^4 = r^2s^2 - 1$ is discussed in Appendix B.2.4, searching solutions where $a = rs$, the product of two distinct primes. Only $a = 2 \cdot 41$ matches this format and yields the known solution.

3.32. **9408.** $n = 9408 = 2^6 \cdot 3 \cdot 7^2$, $\tau(n) = 42$ [6, A175750] , $n+1 = 9409 = 97^2$, $\tau(n+1) = 3$ [6, A001248] . $\tau = 42$ requires $n = p^{41}$ or pq^{20} or p^2q^{13} or p^5q^6 or pq^2r^6. $\tau = 3$ requires $n+1 = s^2$. We discard the case $s = 2$ right away because $n = 3$ does not match any of the prime signatures. The space of solutions with odd s is

- $p^{41} = s^2 - 1$. There are no solutions, see Appendix B.1.
- $pq^{20} = s^2 - 1$. For odd s, Eq. (9) demands that $q = 2$, so $2^{20}p = s^2 - 1$. The Mod-3 criterion calls for either $p = 3$ (which does not work since $3 \cdot 2^{20} + 1 \nsim s^2$) or $s = 3$ (which does not work since $3^2 - 1 \nsim 22^{20}p$).
- $p^2q^{13} = s^2 - 1$. Eq. (9) means that $q = 2$ to support the factor 2^3 on the left hand side. Furthermore case $2^{13}p^2 = s^2 - 1$ requires $p = 3$ or by $s = 3$ by the Mod-3 criterion, but by direct inspection $3^2 - 1 \nsim 2^{13}p^2$ and $2^{13} \cdot 3^2 + 1 \nsim s^2$: no solutions exit here.
- $p^5q^6 = s^2 - 1$. Guided by the parity argument we have either $2^5q^6 = s^2 - 1$ (and either $q = 3$ or $s = 3$ by the Mod-3 criterion), or $2^6p^5 = s^2 - 1$ (and either $p = 3$ or $s = 3$ by the Mod-3 criterion). These 4 ways are all excluded by direction inspection: $2^5 \cdot 3^6 + 1 \nsim s^2$, $3^2 - 1 \nsim 2^5q^6$, $2^6 \cdot 3^5 + 1 \nsim s^2$, and $3^2 - 1 \nsim 2^6p^5$.
- $pq^2r^6 = s^2 - 1$, the branch with the known solution. Since s is odd, Eq. (9) requires $r = 2$ to provide at least a factor 2^3 on the LHS, so $2^6pq^2 = s^2 - 1$. By the Mod-3 criterion either $p = 3$ or $q = 3$ or $s = 3$ (the latter obviously discarded because $3^2 - 1 \nsim 2^6pq^2$).
  - $2^6 \cdot 3 \cdot q^2 = s^2 - 1$ requires $s \equiv \{1, 31, 65, 95, 97, 127, 161, 191\} \pmod{2^6 \cdot 3}$,
    and insertion of these s yields the 8 cases $q^2 = 2k(96+1)$ with $k \ge 1$, $q^2 = (6k+1)(32k+5)$ with $k \ge 0$, $q^2 = 2(3k+1)(32k+11)$ with $k \ge 0$, $q^2 = (2k+1)(96k+47)$ with $k \ge 0$, $q^2 = (2k+1)(96k+49)$ with $k \ge 0$, $q^2 = 2(3k+2)(32k+21)$ with $k \ge 0$, or $q^2 = 2(k+1)(96k+95)$ with $k \ge 0$. The second and fourth in this list generate the primes 5 and 47 on the right hand side if $k = 0$ (not wanted, we seek q^2), but the fifth generates uniquely the known solution $q^2 = 49$ and no other.
  - Alternatively $2^6 \cdot 3^2 p = s^2 - 1$ requires $s \equiv \{1, 127, 161, 287, 289, 315, 449, 575\} \pmod{2^6 \cdot 3^2}$,
    and insertion of these s yields the 8 cases $p = 2k(288k+1)$ with $k \ge 1$, $p = 2(9k+2)(32k+7)$ with $k \ge 0$, $p = (18k+5)(32k+9)$ with $k \ge 0$, $p = (2k+1)(288k+143)$ with $k \ge 0$, $p = (2k+1)(288k+145)$ with $k \ge 0$, $p = (18k+13)(32k+23)$ with $k \ge 0$, $p = 2(9k+7)(32k+25)$ with $k \ge 0$, or $p = 2(k+1)(288k+287)$ with $k \ge 0$. These forms are incompatible with the requirement that p is prime, even if $k = 0$.

3.33. **9999.** $n = 9999 = 3^2 \cdot 11 \cdot 101$, $\tau(n) = 12$ [6, A030630] , $n+1 = 10000 = 2^4 \cdot 5^4$, $\tau(n+1) = 25$ [6, A137488] . $\tau = 12$ requires $n = p^{11}$, or pq^5 or pqr^2. $\tau = 25$

requires $n+1=s^{24}$ or $n+1=s^4t^4$. The 6 combinations of these prime signatures are

- $p^{11}=s^{24}-1$. Solutions are ruled out with the aid of Appendix B.1.
- $p^{11}=(st)^4-1$. Solutions are ruled out with the aid of Appendix B.1.
- $pq^5=s^{24}-1$. The parity argument requires either $2q^5=s^{24}-1$ or $2^5p=s^{24}-1$ or $pq^5=2^{24}-1$. $2q^5$ does not match because Eq. (19) requires at least a factor 2^5 on the LHS. The case $pq^5=2^{24}-1$ is ruled out by direct evaluation. Finally the case $2^5p=s^{24}-1$ needs either $p=3$ or $s=3$ by the Mod-3 Argument, but $2^5\cdot 3+1 \nsim s^{24}$, and $3^{24}-1\nsim 2^5p$.
- $pq^5=(st)^4-1$. The parity argument requires either $2q^5=(st)^4-1$ or $2^5p=(st)^4-1$ or $pq^5=2^4t^4-1$.
  - $2q^5=(st)^4-1$ lacks the minimum power 2^4 on the LHS required by Eq. (10).
  - $2^5p=(st)^4-1=(s^2t^2+1)(s^2t^2-1)$ has a product of two even terms on the RHS which have a greatest common divisor of 2. [This might be taken as a result of running the Euclidean Algorithm on the two factors.] So the only possible matching splits on the LHS are $2p=(st)^2-1\wedge 2^4=(st)^2+1$ or $2=(st)^2-1\wedge 2^4p=(st)^2+1$. The first case is impossible because $2^4-1\nsim (st)^2$ and the second case because $2+1\nsim (st)^2$.
  - $pq^5=(2t)^4-1=(4t^2+1)(4t^2-1)$ has two adjacent odd factors on the RHS with a greatest common divisor of 1. [This might be taken as a result of running the Euclidean Algorithm on the two factors.] The only associated splits of the LHS are $p=(2t)^2-1\wedge q^5=(2t)^2+1$ or $p=(2t)^2+1\wedge q^5=(2t)^2-1$ or $1=(2t)^2-1\wedge pq^5=(2t)^2+1$ [rejected because $2\nsim(2t)^2$]. The Mod-3 criterion requires that exactly one of p, q or t equals 3. In the first two cases $p\neq 3$ because that violates $p\pm 1=(2t)^2$ for odd prime t. In the first two cases $q\neq 3$ because then $q^5\pm 1$ is not a square. In the first two cases $t\neq 3$ because then $(2t)^2\pm 1$ is not of the form q^5.
- $pqr^2=s^{24}-1$. The parity argument requires $2qr^2=s^{24}-1$ or $2^2pq=s^{24}-1$ or $pqr^2=2^{24}-1$. Eq. (19) requires for odd s at least a factor 2^5 on the other side; this dismisses the first two cases. Finally $2^{24}-1\nsim pqr^2$ does not provide a solution.
- $pqr^2=(st)^4-1$. The parity argument requires $2qr^2=(st)^4-1$ or $2^2pq=(st)^4-1$ or $pqr^2=2^4t^4-1$. The first two expressions miss the minimum factor 2^4 on the left hand side required by Eq. 10, so $pqr^2=2^4t^4-1=(4t^2+1)(2t+1)(2t-1)$ remains. The Mod-3 criterion requires the presence of a factor 3, one of
  - $3qr^2=(2t)^4-1=(4t^2+1)(2t+1)(2t-1)$ with 3 pairwise coprime factors on the RHS. Sorting by magnitude, the LHS must be factorized such that $3=2t-1$ because q and r are larger than 3. But this means $t=2$ which is invalid because already $s=2$ in use.
  - $9pq=(2t)^4-1$. The only t where $(2t)^4-1$ is divisible by 9 are the residue classes $t=9k+4$ ($k\geq 1$) and $t=9k+5$ ($k\geq 0$), giving $pq=(2k+1)(18k+7)(324k^2+288k+65)$ with $k\geq 1$ or $pq=(2k+1)(18k+11)(324k^2+360k+101)$ with $k\geq 0$. The first case has 3 distinct factors and does not produce solutions, but the second

case has 2 distinct factors if $k = 0$, $pq = 11 \cdot 101$, which is the unique known solution.
– $pqr^2 = (2 \cdot 3)^4 - 1$. This fails because $6^4 - 1 \nsim pqr^2$.

3.34. **14640.** $n = 14640 = 2^4 \cdot 3 \cdot 5 \cdot 61$, $\tau(n) = 40$ [6, A175749] , $n+1 = 14641 = 11^4$, $\tau(n+1) = 5$ [6, A030514] . $\tau = 40$ requires $n = p^{39}$ or $p^{19}q$ or p^9q^3 or p^7q^4 or p^9qr or p^4q^3r or p^4qrs. $\tau = 5$ requires $n + 1 = t^4$.

- $p^{39} = t^4 - 1$. Impossible according to B.1 because the exponent 39 is too large.
- $p^{19}q = t^4 - 1$. For the primes $t = 2$, 3 and 5 the equation is invalid by direct inspection, because $2^4 - 1 \sim pq$, $3^4 - 1 \sim p^4q$ and $5^4 - 1 \sim p^4qr$. For primes $t > 5$ the LHS fails to have at least 3 distinct prime divisors of Theorem 5.
- $p^9q^3 = t^4 - 1$. For the primes $t = 2$, 3 and 5 the equation is invalid by direct inspection. For primes $t > 5$ the LHS fails to have at least 3 distinct prime divisors of Theorem 5.
- $p^7q^4 = t^4 - 1$. For the primes $t = 2$, 3 and 5 the equation is invalid by direct inspection. For primes $t > 5$ the LHS fails to have at least 3 distinct prime divisors of Theorem 5.
- $p^9qr = t^4 - 1$. For the primes $t = 2$, 3 and 5 the equation is invalid by direct inspection. For primes $t > 5$ the LHS must meet Theorem 5, which requires $p = 2$ and $qr = 3 \cdot 5$. But $2^9 \cdot 3 \cdot 5 + 1 \nsim t^4$.
- $p^4q^3 = t^4 - 1$. For the primes $t = 2$, 3 and 5 the equation is invalid by direct inspection. For primes $t > 5$ the LHS fails to have at least 3 distinct prime divisors of Theorem 5.
- $p^4qrs = t^4 - 1$. For the primes $t = 2$, 3 and 5 the equation is invalid by direct inspection. For primes $t > 5$ the LHS must meet Theorem 5, which requires $p = 2$ and $qr = 3 \cdot 5$ with a free slot for the prime $s \geq 7$. Since $2^4 \cdot 3 \cdot 5 \mid t^4 - 1$, the Pisano period of the sequence $t^4 - 1$ modulo $2^4 \cdot 3 \cdot 5$ requires $t \equiv \{1, 7, 11, 13, 17, 19, 23, \ldots 227, 229, 233, 239\} \pmod{240}$, which leads to polynomial expressions for s with at least 3 factors. Only in the residue class $t = 240 + 11k$ we get an expression, $s = (24k + 1)(20k + 1)(28800k^4 + 2640k + 61)$, where, at $k = 0$, $s = 61$ is prime. So this is the only solution.

3.35. **15624.** $n = 15624 = 2^3 \cdot 3^2 \cdot 7 \cdot 31$, $\tau(n) = 48$ [6, A175754] , $n+1 = 15625 = 5^6$, $\tau(n + 1) = 7$ [6, A030516] . $\tau = 48$ requires $n = p^{48}$, $p^{23}q$, $p^{15}q^2$, $p^{11}q^3$, p^7q^5, $p^3q^3r^2$, p^7q^2r, $p^{11}qr$, p^5q^3r, p^5qrs, p^3q^2rs or p^2qrst. $\tau = 7$ requires $n + 1 = u^6$. First we check the prime signatures of $u^6 - 1$ for the primes $u = 2, 3, 5, 7$ against those 12 supported by n: $2^6 - 1 = 3^2 \cdot 7$, $3^6 - 1 = 2^3 \cdot 7 \cdot 13$, $5^6 - 1 = 2^3 \cdot 3^2 \cdot 7 \cdot 31$, $7^6 - 1 = 2^4 \cdot 3^2 \cdot 19 \cdot 43$. Only $5^6 - 1 \sim p^3q^2rs$, the known solution, fits in here. The prime signatures of $n \geq 11^6 - 1$ that have less than three distinct primes or not at least a prime cubed and another prime squared are discarded with the aid of Theorem 6. The remaining cases to be checked with $u > 7$ for further solutions are

- $p^3q^3r^2 = u^6 - 1 = (u^2 + u + 1)(u^2 - u + 1)(u + 1)(u - 1)$. The 4 factors on the RHS are well ordered: $u^2 + u + 1 > u^2 - u + 1 > u + 1 > u - 1$. Because $u+1$ and $u-1$ are adjacent even numbers, their greatest common divisor is $(u+1, u-1) = 2$. The first step of the Euclidean Algorithm shows that the greatest common divisor (u^2+u+1, u^2-u+1) is $(u^2-u+1, 2u)$. Because u is odd and u^2-u+1 is odd, this is the same as (u^2-u+1, u), which is 1. Similarly $(u^2+u+1, u\pm 1)=1$; $(u^2-u+1, u\pm 1)=1$. We know from Theorem 6 that $p=2$ which leaves either $q=3\wedge r=7$ or $q=7\wedge r=3$. But $2^3\cdot 3^3\cdot 7^2+1 \nsim u^6$, and $2^3\cdot 7^3\cdot 3^2+1\nsim u^6$: there are no solutions here.

- $p^7q^2r=u^6-1$. As above, $p=2$ is enforced by the theorem, with either $q=3\wedge r=7$ or $q=7\wedge r=3$. By direct inspection $2^7\cdot 3^2\cdot 7+1\nsim u^6$ and $2^7\cdot 7^2\cdot 3+1\nsim u^6$.
- $p^5q^3r=u^6-1$. As above, $p=2$ or $q=2$ with $r=7$ is enforced by the theorem.
  (1) $p=2$, $q=3$, $r=7$ does not match u^6-1 by direct inspection.
  (2) $q=2$, $p=3$, $r=7$ does not match u^6-1 by direct inspection.
- $p^3q^2rs=u^6-1$. As above, there is a product of 4 polynomials in u on the RHS, pairwise coprime except a common divisor $(u+1,u-1)=2$. $p=2$, $q=3$, $r=7$ are enforced by the theorem. The factors q^2, r and s must be matched in some permutation by the polynomial factors u^2+u+1, u^2-u+1 and $(u^2-1)/8$. $(u+1)(u-1)/8$ cannot contribute with more than one odd prime to the triplet $q^2\times r\times s$, because at least one odd factor is contributed by each of $u^2\pm u+1$, so at least one of $(u+1)/2$, $(u+1)/4$, $(u-1)/2$ and $(u-1)/4$ must be 1. This means $u=1$ or $u=3$ or $u=3$ or $u=5$. $u=1$ or $u=3$ are discarded by direct evaluation of u^6-1, and $u=5$ with $5^6-1=2^3\cdot 3^2\cdot 7\cdot 31$ is the known solution.

3.36. **28223.** $n=28223=13^2\cdot 167$, $\tau(n)=6$ [6, A030515] , $n+1=28224=2^6\cdot 3^2\cdot 7^2$, $\tau(n+1)=63$. $\tau=6$ requires $n=p^5$ or p^2q. $\tau=63$ requires $n+1=s^{62}$ or $s^{20}t^2$ or s^8t^6 or $s^6t^2u^2$.

- $p^5=s^{62}-1$. Rejected in Appendix B.1.
- $p^5=s^{20}t^2-1$. Rejected in Appendix B.1.
- $p^5=s^8t^6-1$. Rejected in Appendix B.1.
- $p^5=s^6t^2u^2-1$. Rejected in Appendix B.1.
- $p^2q=s^{62}-1$. For odd s, Eq. (23) requires the presence of a factor 2^3 in p^2q, which is not supported. So the alternative is $s=2$, but $2^{62}-1\nsim p^2q$ does not fit the prime signature.
- $p^2q=s^{20}t^2-1$. Appendix B.2.2 mandates that $s=2$ or $t=2$, so the RHS is a product $(s^{10}t+1)(s^{10}t-1)$ of two adjacent odd coprime numbers. Therefore one of $s^{10}t\pm 1$ must be p^2, the other q.
  - $p^2=s^{10}t+1\wedge q=s^{10}t-1$. The Mod-3 Argument requires that one of the four primes is 3. $q=3$ or $p=3$ are immediately discarded because they do not fit $s^{10}t\pm 1$. This leaves $s=2$ with $t=3$ or $s=3$ with $t=2$, but in both cases $s^{10}t+1$ is not a perfect square p^2.
  - $p^2=s^{10}t-1\wedge q=s^{10}t+1$. The Mod-3 Argument requires that one of the four primes is 3. $q=3$ or $p=3$ are immediately discarded because they do not fit $s^{10}t\pm 1$. This leaves $s=2$ with $t=3$ or $s=3$ with $t=2$, but in both cases $s^{10}t-1$ is not a perfect square p^2.
- $p^2q=s^8t^6-1$. Appendix B.2.2 mandates that $s=2$ or $t=2$, so the RHS is a product $(s^4t^3+1)(s^4t^3-1)$ of two adjacent coprime odd numbers. Therefore one of $s^4t^3\pm 1$ must equal p^2, the other q.

  - $p^2 = s^4t^3+1 \wedge q = s^4t^3-1$. The Mod-3 Argument requires that one of the four primes is 3. $q = 3$ or $p = 3$ are immediately discarded because they do not fit $s^4t^3 \pm 1$. This leaves $s = 2$ with $t = 3$ or $s = 3$ with $t = 2$, but in both cases $s^4t^3 + 1$ is not a perfect square p^2.
  - $p^2 = s^4t^3-1 \wedge q = s^4t^3+1$. The Mod-3 Argument requires that one of the four primes is 3. $q = 3$ or $p = 3$ are immediately discarded because they do not fit $s^4t^3 \pm 1$. This leaves $s = 2$ with $t = 3$ or $s = 3$ with $t = 2$, but in both cases $s^4t^3 - 1$ is not a perfect square p^2.
- $p^2q = s^6t^2u^2-1$. Branch of the known solution. Appendix B.2.2 mandates that $s = 2$ or $t = 2$, so the RHS is a product $(s^3tu + 1)(s^3tu - 1)$ of two adjacent coprime odd numbers. Therefore one of $s^3tu \pm 1$ must equal p^2, the other q.
  - $p^2 = s^3tu + 1 \wedge q = s^3tu - 1$. The Mod-3 Argument requires that one of the four odd primes is 3. $q = 3$ or $p = 3$ are immediately discarded because they do not fit $s^3tu \pm 1$. This leaves $s = 2$ with $t = 3$ or $s = 3$ with $t = 2$.
    * $p^2 = 2^3{\cdot}3u+1 \wedge q = 2^3{\cdot}3u-1$. Here $p^2-1 = (p+1)(p-1) = 2^3{\cdot}3u$ has 2 adjacent even factors on the LHS, such that the factors can be split as
      · either $p + 1 = 2 \cdot 3u \wedge p - 1 = 2^2$, ($p = 5$ and $u = 1$, not prime)
      · or $p+1 = 2\cdot 3 \wedge p-1 = 2^2\cdot u$, ($p = 5$ and $u = 1$, not prime)
      · or $p + 1 = 2u \wedge p - 1 = 2^2 \cdot 3$ ($p = 13$, $u = 7$, $t = 3$, $s = 2$, $p^2 = 169$, $q = 167$, the known solution.)
      · or $p + 1 = 2 \wedge p - 1 = 2^2 \cdot 3u$, ($p = 1$, not prime)
      · or $p + 1 = 2^2 \cdot 3u \wedge p - 1 = 2$, ($p = 3$ but no u noninteger.)
      · or $p + 1 = 2^2 \cdot 3 \wedge p - 1 = 2 \cdot u$, ($p = 11$, $u = 5$, but $q = 119$ composite.)
      · or $p + 1 = 2^2u \wedge p - 1 = 2 \cdot 3$, ($p = 7$, but requires $u = 2$ already used by s).
      · or $p + 1 = 2^2 \wedge p - 1 = 2 \cdot 3u$, ($p = 3$, but u noninteger.)
    * $p^2 = 3^3\cdot 2u+1 \wedge q = 3^3\cdot 2u-1$. Here $p^2-1 = 3^3\cdot 2u$ violates the requirement of Theorem 9 of supplying a factor 2^3 on the other side; no solutions.
  - $p^2 = s^3tu - 1 \wedge q = s^3tu + 1$. The Mod-3 criterion requires that one of the 4 odd primes is 3. $q = 3$ or $p = 3$ are immediately discarded because they do not fit $s^3tu \pm 1$. This leaves $s = 2$ with $t = 3$ or $s = 3$ with $t = 2$. In both cases $s^3tu \equiv 0 \pmod 3$ which leaves $p^2 \equiv -1 \pmod 3$ and violates Eq. (25).

3.37. **36863.** $n = 36863 = 2^2{\cdot}9209$, $\tau(n) = 4$ [6, A030513] , $n+1 = 36864 = 2^{12}\cdot 3^2$, $\tau(n + 1) = 39$ [6, A175748] . Proof by Jon Schoenfield [11]: $\tau(n) = 4$ so $n = p^3$ or pq, and $\tau(n + 1) = 39$, so $n + 1 = s^{38}$ or s^2t^{12}.

- $p^3 = s^{38} - 1$ is ruled out via App. B.1.
- $pq = s^{38} - 1$ has the sub-cases $2q = s^{38} - 1$ and $pq = 2^{28} - 1$ (which has 3 distinct prime factors and does not work). The Mod-3 Argument requires either $q = 3$ or $s = 3$ which are discarded by direct evaluation.
- So $n+1$ is the square of a composite $c = st^6$, so $n = c^2 - 1 = (c-1)(c+1)$, which cannot be p^3 (see App. B.1), so $n = pq = (c - 1)(c + 1)$. c cannot

be odd, because then Eq. (9) requires a factor 2^3 which is not supported by the form pq. So c is even, and $c-1$ and $c+1$ are twin primes. They are not 3 and 5 (since $c = 4 \nsim st^6$), so they are of the form $6j-1$ and $6j+1$, so $n+1 = c^2 = 2^2 \cdot 3^2 \cdot j^2$ is divisible by both 2 and 3. Since $c = st^6$, $n+1 = s^2t^{12}$ and j must be 2^5 or 3^5. So $n+1 = 2^{12} \cdot 3^2$ or $2^2 \cdot 3^{12}$; the latter won't work (it would give $\tau(n) = 8$), so the only solution is $n = 2^{12} \cdot 3^2 - 1 = 36863$.

3.38. **38415.** $n = 38415 = 3 \cdot 5 \cdot 13 \cdot 197$, $\tau(n) = 16$ [6, A030634] , $n+1 = 38416 = 2^4 \cdot 7^4$, $\tau(n+1) = 25$ [6, A137488] . $\tau = 16$ requires that n is of the form p^{15} or pq^7 or pqr^3 or p^3q^3 or $pqrs$ with p, q, r and s prime. $\tau = 25$ requires that $n+1 = t^{24}$ or t^4u^4.

- $p^{15} = t^{24} - 1$ and $p^{15} = (tu)^4 - 1$ are discarded via Appendix B.1.
- $pq^7 = t^{24} - 1$. For primes $t \le 13$ the format of $t^{24} - 1$ never matches pq^7. For primes $t > 13$, the LHS does not support the minimum 5 distinct primes required by Theorem 10.
- $pq^7 = (tu)^4 - 1$.
  - If tu is odd, Eq. (9) requires $q = 2$ to match the LHS. The Mod-3 argument requires $3 \cdot 2^7 = (tu)^4 - 1$ (but $3 \cdot 2^7 + 1 \nsim t^4u^4$) or $2^7p = (3u)^4 - 1$. In $2^7p = (9u^2+1)(9u^2-1)$ the two adjacent even numbers on the RHS have a greatest common divisor of 2, so one of
    - $2^6 = 9u^2 + 1 \wedge 2p = 9u^2 - 1$. Impossible since $2^6 - 1 \nsim 9u^2$.
    - $2^6p = 9u^2 + 1 \wedge 2 = 9u^2 - 1$. Impossible since $2 + 1 \nsim 9u^2$.
    - $2 = 9u^2 + 1 \wedge 2^6p = 9u^2 - 1$. Impossible since $2 - 1 \nsim 9u^2$.
    - $2p = 9u^2 + 1 \wedge 2^6 = 9u^2 - 1$. Impossible since $2^6 + 1 \nsim 9u^2$.
  - If tu is even, the RHS is the product $(2^2u^2+1)(2^2u^2-1)$ of two adjacent coprime odd numbers, so one of
    - $p = (2u)^2 + 1 \wedge q^7 = (2u)^2 - 1$. The Mod-3 argument requires that either $p = 3$ (fails since $3 - 1 \nsim (2u)^2$ with noninteger u) or $u = 3$ [fails since $(2 \cdot 3)^2 - 1 \nsim q^7$] or $q = 3$ (fails since $2^7 + 1 \nsim 4u^2$).
    - $q^7 = (2u)^2 + 1 \wedge p = (2u)^2 - 1$. The Mod-3 argument requires that either $q = 3$ (fails since $3^7 - 1 \nsim 4u^2$) or $u = 3$ [fails since $(2 \cdot 3)^2 + 1 \nsim q^7$] or $p = 3$ (fails since $2 + 1 \nsim 4u^2$).
- $pqr^3 = t^{24} - 1$. For primes $t \le 13$ the format of $t^{24} - 1$ never matches pqr^3. For primes $t > 13$, the LHS does not support the minimum 5 distinct primes required by Theorem 10.
- $pqr^3 = (tu)^4 - 1 = (t^2u^2+1)(tu+1)(tu-1)$.
  - If $tu = 2u$ is even, the RHS is the product of 3 odd coprime distinct factors. So the factorizations may have three splittings, organized by association r^3 with the largest to smallest of the factors:
    - $r^3 = 4u^2 + 1 \wedge p = 2u + 1 \wedge q = 2u - 1$. The Mod-3 argument requires that either $r = 3$ (impossible because $3^3 - 1 \nsim 4u^2$), or $u = 3$ (impossible since $4 \cdot 3^2 + 1 \nsim r^3$), or $p = 3$ (impossible because $u = 1$, nonprime) or $q = 3$ (impossible because then $u = 2$ already taken by t).
    - $r^3 = 2u + 1 \wedge p = 4u^2 + 1 \wedge q = 2u - 1$. The Mod-3 argument requires that either $r = 3$ (impossible because then $u = 13$ but $2 \cdot 13 - 1 \nsim q$), or $u = 3$ (impossible because $2 \cdot 3 + 1 \nsim r^3$),

or $p = 3$ (impossible because $3 - 1 \nsim 4u^2$) or $q = 3$ (impossible because then $u = 2$ already taken by t).
    * $r^3 = 2u - 1 \wedge p = 4u^2 + 1 \wedge q = 2u + 1$ The Mod-3 argument requires that either $r = 3$ (impossible because $3^3 + 1 \nsim 2u$), or $u = 3$ (impossible because $2 \times 3 - 1 \nsim r^3$), or $p = 3$ (impossible because $3 - 1 \nsim 4u^2$) or $q = 3$ (impossible because $3 - 1 \nsim 2ut$).
  - If tu is odd, $2^4 \mid (tu)^4 - 1$ by (10), but pqr^3 does not support a fourth power: no solutions.
- $p^3q^3 = t^{24} - 1$. For primes $t \le 13$ the format of $t^{24} - 1$ never matches p^3q^3. For primes $t > 13$, the LHS does not have at least the 5 distinct primes required by Theorem 10.
- $p^3q^3 = (tu)^4 - 1$.
  - If $tu = 2u$ is even, the RHS is the product of three distinct coprime odd factors. Since the LHS offers only two coprime factors, the smallest of the three must be $2u - 1 = 1$, but then $u = 1$ (not prime).
  - If tu is odd, $2^4 \mid (tu)^4 - 1$ by (10), but the LHS does not support a fourth power: no solutions.
- $pqrs = t^{24} - 1$. For primes $t \le 13$ the format of $t^{24} - 1$ never matches $pqrs$. For primes $t > 13$, the LHS does not have at least the 5 distinct primes required by Theorem 10.
- $pqrs = (tu)^4 - 1$. The branch of the known solution.
  - If $tu = 2u$ is even, the RHS is the product of three distinct coprime odd factors, $pqrs = (4u^2+1)(2u+1)(2u-1)$. For all odd primes $u > 5$, $3 \cdot 5 \mid (2u)^4 - 1$, because $2^4 \equiv 1 \pmod 3$ and $u^4 \equiv 1 \pmod 3$ and $2^4 \equiv 1 \pmod 5$ and $u^4 \equiv 1 \pmod 5$. We check that $(2 \cdot 5)^4 - 1 \nsim pqrs$. So $3 \cdot 5 \cdot rs = (4u^2 + 1)(2u + 1)(2u - 1)$. Distributing the four primes of the LHS over the three factors of the RHS (ignoring the order of r and s which appear with the same exponent) requires that rs is split [because otherwise $rs > 5 > 3$ is the largest factor, which leads leads to $2u + 1 = 5$, $2u - 1 = 3$ and $u = 2$ is impossible because already taken by $t = 2$].
    * $\ldots r = 4u^2 + 1 \wedge \ldots s = 2u + 1 \wedge \ldots = 2u - 1$. Here $\ldots$ are the 3 spots allocated for 3 and 5. Because $u \ge 7$, $2u - 1 \ge 13$, the 3 and 5 must appear at $2u - 1$, so $3 \cdot 5 = 2u - 1$, but $u = 8$ is not prime.
    * $\ldots r = 4u^2 + 1 \wedge \ldots = 2u + 1 \wedge \ldots s = 2u - 1$. Here $\ldots$ are the dispersed 3 and 5. To put an isolated 3 or 5 at $2u + 1$ requires too small u, so we need both there, $3 \cdot 5 = 2u + 1$, $u = 7$, so $s = 13$ and $r = 197$, the known solution.
    * $\ldots = 4u^2 + 1 \wedge \ldots r = 2u + 1 \wedge \ldots s = 2u - 1$. Here $\ldots$ are the dispersed 3 and 5. Since $u \ge 7$, $4u^2 + 1 \ge 197$, we cannot fill the LHS spot of $4u^2 + 1$ because at most the factor $3 \cdot 5 = 15$ is available. So no solutions here.
  - If tu is odd, $2^4 \mid (tu)^4 - 1$ by (10), but the LHS does not support a fourth power: no solutions.

3.39. **46655.** $n = 46655 = 5 \cdot 7 \cdot 31 \cdot 43$, $\tau(n) = 16$ [6, A030634] , $n + 1 = 46656 = 2^6 \cdot 3^6$, $\tau(n + 1) = 49$ [6, A175755] . $\tau = 16$ requires that n is one of the 5 forms p^{15} or pq^7 or p^3q^3 or pqr^3 or $pqrs$ with p, q, r and s prime. $\tau = 49$ requires that

$n+1 = s^{48}$ or s^6t^6 with s,t prime. There are 10 combinations of these prime signatures:

- $p^{15} = s^{48}-1$. There are no solutions according to App. B.1.
- $p^3q^3 = s^{48}-1$ or $pqr^3 = s^{48}-1$ or $pqrs = s^{48}-1$: If s is odd, Eq. (22) requires that the LHS does support the factor 2^6, which they do not. So $s=2$, but $2^{48}-1$ has 9 distinct prime factors and does not match any of the 3 prime signatures of the LHS. $\nexists$
- $pq^7 = s^{48}-1$. The parity argument requires either $s=2$ (but $2^{48}-1 \not\sim pq^7$) or $p=2$ or $q=2$. So s is odd and with Eq. (22) $q=2$, so $2^7p = s^{48}-1$. Then the Mod-3 criterion requires either $p=3$ or $s=3$, but $2^7\cdot 3+1 \not\sim s^{48}$ and $3^{48}-1 \not\sim 2^7p$. $\nexists$
- $p^{15} = s^6t^6-1 = (s^3t^3+1)(s^3t^3-1)$. If st is even, the right hand side is the product of two adjacent coprime odd numbers, but the other side can only be factored that way as $1\cdot p^{15}$, but that would require $s^3t^3-1=1$, impossible. If st is odd, the parity argument requires $p=2$, but $2^{15}+1 \not\sim (st)^6$. $\nexists$
- $pq^7 = s^6t^6-1 = (s^3t^3+1)(s^3t^3-1)$.
  - If st is even, the RHS is a product of two adjacent odd coprime numbers, and the Mod-3 criterion requires one of the odd primes to be 3:
    * $3q^7 = (2^3t^3+1)(2^3t^3-1)$. Sub-cases
      · $3q^7 = (2t)^3+1 \wedge 1 = (2t)^3-1$. Fails because $1+1 \not\sim (2t)^3$.
      · $q^7 = (2t)^3+1 \wedge 3 = (2t)^3-1$. Fails because $3+1 \not\sim (2t)^3$.
      · $3 = (2t)^3+1 \wedge q^7 = (2t)^3-1$. Fails because $3-1 \not\sim (2t)^3$.
      · $1 = (2t)^3+1 \wedge 3q^7 = (2t)^3-1$. Fails because $3q^7 > 1$.
    * $3^7p = (2^3t^3+1)(2^3t^3-1)$. Sub-cases
      · $3^7p = (2t)^3+1 \wedge 1 = (2t)^3-1$. Fails because $1+1 \not\sim (2t)^3$.
      · $p = (2t)^3+1 \wedge 3^7 = (2t)^3-1$. Fails because $3^7+1 \not\sim (2t)^3$.
      · $3^7 = (2t)^3+1 \wedge p = (2t)^3-1$. Fails because $3^7-1 \not\sim (2t)^3$.
      · $1 = (2t)^3+1 \wedge 3^7p = (2t)^3-1$. Fails because $3^7p > 1$.
    * $pq^7 = (2^3\cdot 3^3+1)(2^3\cdot 3^3-1)$. Fails because $(6^3+1)(6^3-1) \not\sim pq^7$.

    $\nexists$
  - If st is odd, Eq. (11) requires $q=2$, so $2^7p = (s^3t^3+1)(s^3t^3-1)$. The RHS is the product of two adjacent even numbers which have a greatest common divisor of 2. The two factors of 2^7p can be dispersed as:
    * $2p = (st)^3+1 \wedge 2^6 = (st)^3-1$. Fails because $2^6+1 \not\sim (st)^3$.
    * $2^6p = (st)^3+1 \wedge 2 = (st)^3-1$. Fails because $2+1 \not\sim (st)^3$.
    * $2 = (st)^3+1 \wedge 2^6p = (st)^3-1$. Fails because $2-1 \not\sim (st)^3$.
    * $2^6 = (st)^3+1 \wedge 2p = (st)^3-1$. Fails because $2^6-1 \not\sim (st)^3$.

    $\nexists$
- $p^3q^3 = s^6t^6-1$.
  - If st is even, $(st)^3\pm 1$ is a pair of coprime odd factors, with two possible splittings:
    * $p^3q^3 = (2t)^3+1 \wedge 1 = (2t)^3-1$. Impossible since $1+1 \not\sim (2t)^3$
    * $p^3 = (2t)^3+1 \wedge q^3 = (2t)^3-1$. Subtraction of both equations gives $p^3-q^3 = 2 = (p-q)(p^2+pq+q^2)$. For odd primes,

$p-q$ is even and p^2+pq+q^2 is odd, so $p-q=2$, but clearly $p^2+pq+q^2>1$, impossible.

- If st is odd the parity argument gives $2^3q^3 = (s^3t^3+1)(s^3t^3-1)$, product of two adjacent even numbers with greatest common divisor 2. The possible splittings are
  - $2^2q^3 = (st)^3+1 \wedge 2 = (st)^3-1$. Impossible since $2+1 \nsim (st)^3$.
  - $2q^3 = (st)^3+1 \wedge 2^2 = (st)^3-1$. Impossible since $2^2+1 \nsim (st)^3$.
  - $2^2 = (st)^3+1 \wedge 2q^3 = (st)^3-1$. Impossible since $2^2-1 \nsim (st)^3$.
  - $2 = (st)^3+1 \wedge 2^2q^3 = (st)^3-1$. Impossible since $2-1 \nsim (st)^3$.

$\nexists$

- $pqr^3 = s^6t^6-1$
  - If st is even, $(2t)^3 \pm 1$ are two adjacent coprime odd factors on the RHS.
    - $pqr^3 = (2t)^3+1 \wedge 1 = (2t)^3-1$. Impossible since $1+1 \nsim (2t)^3$.
    - $qr^3 = (2t)^3+1 \wedge p = (2t)^3-1$. Impossible since $(2t)^3-1 = (2t-1)(2^2t^2+2t+1)$ is composite.
    - $r^3 = (2t)^3+1 \wedge pq = (2t)^3-1$. Implies $(r-1)(r^2+r+1) = 2^3t^3$. Because r is odd, r^2+r+1 is odd, $r-1$ is even. The greatest common divisor of r^2+r+1 and $r-1$ is 1 or 3 (but $t=3$ is discarded, $(2\cdot 3)^3+1 \nsim r^3$). So $r-1 = 2^3 \wedge r^2+r+1 = t^3$, but $2^3+1 \nsim r$.
    - $pq = (2t)^3+1 \wedge r^3 = (2t)^3-1$. Implies $(r+1)(r^2-r+1) = 2^3t^3$. r^2-r+1 is odd, $r+1$ is even, and the greatest common divisor of r^2-r+1 and $r+1$ is 1 or 3 (but $t=3$ is discarded, $(2\cdot 3)^3-1 \nsim r^3$). So $r+1 = 2^3 \wedge r^2-r+1 = t^3$, so $r=7$, but $7^2-7+1 \nsim t^3$.
    - $q = (2t)^3+1 \wedge pr^3 = (2t)^3-1$. Requires that $q = (2t+1)(2^2t^2-2t+1)$ has distinct factors, contradiction to q being prime.
    - $1 = (2t)^3+1 \wedge pqr^3 = (2t)^3-1$. Impossible ordering because $pqr^3 > 1$.

    $\nexists$
  - If st is odd, Eq. (11) leads to $r=2$, $2^3pq = (s^3t^3+1)(s^3t^3-1)$ where the gcd of $(st)^3 \pm 1$ is 2. Associations:
    - $2^2pq = (st)^3+1 \wedge 2 = (st)^3-1$. Impossible $2+1 \nsim (st)^3$.
    - $2pq = (st)^3+1 \wedge 2^2 = (st)^3-1$. Impossible $2^2+1 \nsim (st)^3$.
    - $2^2 = (st)^3+1 \wedge 2pq = (st)^3-1$. Impossible $2^2-1 \nsim (st)^3$.
    - $2 = (st)^3+1 \wedge 2^2pq = (st)^3-1$. Impossible $2-1 \nsim (st)^3$.
    - $2^2p = (st)^3+1 \wedge 2q = (st)^3-1 = (st-1)(s^2t^2+st+1)$. Since st is odd, $(st)^2+st+1$ is odd. The sub-case where $st-1$ and s^2t^2+st+1 have a common factor of 3 is not supported because $2q$ does not support 3^2, so $2 = st-1$, impossible since $2+1 \nsim st$.
    - $2p = (st)^3+1 \wedge 2^2q = (st)^3-1$ Since st is odd, $(st)^2-st+1$ is odd, a common factor 3 not supported as above, so in $2p = (st+1)(s^2t^2-st+1)$, so $2 = st+1$, impossible $2-1 \nsim st$.

    $\nexists$
- $pqrv = s^6t^6-1$. Branch with the known solution.
  - st is even. The Mod-3 criterion requires one of
    - $3qrv = (2^2t^2+2t+1)(2^2t^2-2t+1)(2t+1)(2t-1)$. There are four distinct well ordered $4t^2+2t+1 > 4t^2-2t+1 > 2t+1 > 2t-1$

factors (where $t \geq 5$), so the smallest must match the 3, but $2t-1=3$ is impossible because the prime $s=2$ is already used.

  * or $pqrv = (2^2\cdot 3^2+2\cdot 3+1)(2^2\cdot 3^2-2\cdot 3+1)(2\cdot 3+1)(2\cdot 3-1)$, which gives the known solution $p=5$, $q=7$, $r=31$, $v=43$.

$\exists$

- If st is odd, Eq. (11) requires a factor 2^3 which is not supported by the form $pqrv$. $\nexists$

3.40. **50624.** $n = 50624 = 2^6\cdot 7\cdot 113$, $\tau(n) = 28$ [6, A137491] , $n+1 = 50625 = 3^4\cdot 5^4$, $\tau(n+1) = 25$ [6, A137488] . $\tau = 28$ requires $n = p^{27}$ or pq^{13} or p^3q^6 or p^6qr. $\tau = 25$ requires $n+1 = s^{24}$ or s^4t^4.

- $p^{27} = s^{24}-1$. There are no solutions according to App. B.1. $\nexists$
- $p^{27} = s^4t^4-1 = (s^2t^2+1)(s^2t^2-1)$. If st is even, the RHS is the product of two adjacent coprime odd numbers, but p^{27} cannot be split that way. If st is odd, $p=2$ with the parity argument, but $2^{27}+1 \not\sim (st)^4$. $\nexists$
- $pq^{13} = s^{24}-1$. We check that for primes $s \leq 13$ the RHS does not have the prime signature of the LHS. For the other primes Theorem 10 requires at least 5 distinct prime factors on the LHS, which it does not support. $\nexists$
- $pq^{13} = s^4t^4-1 = (s^2t^2+1)(s^2t^2-1)$.
  - If st is even, $(2^2t^2+1)(2^2t^2-1)$ is the product of two adjacent coprime odd numbers. The possible associations are
    * $pq^{13} = 2^2t^2+1 \wedge 1 = 2^2t^2-1$. Impossible since $1+1 \not\sim (2t)^2$.
    * $q^{13} = 2^2t^2+1 \wedge p = 2^2t^2-1$. Impossible since $p = (2t+1)(2t-1)$ is composite (unless $2t-1=1$ with t nonprime, rejected).
    * $p = 2^2t^2+1 \wedge q^{13} = 2^2t^2-1$. Impossible since $q^{13} = (2t+1)(2t-1)$ is a product of two distinct coprime factors ($2t-1=1$ rejected).
    * $1 = s^2t^2+1 \wedge pq^{13} = s^2t^2-1$. Impossible ordering because $pq^{13} > 1$.
  - If st is odd, Eq. (10) requires $q=2$ and $2^{13}p = (s^2t^2+1)(s^2t^2-1)$, product of two adjacent even numbers with greatest common divisor 2. The possible associations of factors:
    * $2^{12}p = s^2t^2+1 \wedge 2 = s^2t^2-1$. Impossible because $2+1 \not\sim (st)^2$.
    * $2p = s^2t^2+1 \wedge 2^{12} = s^2t^2-1$. Impossible because $2^{12}+1 \not\sim (st)^2$.
    * $2^{12} = s^2t^2+1 \wedge 2p = s^2t^2-1$. Impossible because $2^{12}-1 \not\sim (st)^2$.
    * $2 = s^2t^2+1 \wedge 2^{12}p = s^2t^2-1$. Impossible ordering.

  $\nexists$
- $p^3q^6 = s^{24}-1$. We check that for primes $s \leq 13$ the RHS does not have the prime signature of the LHS. For the other primes Theorem 10 requires at least 5 distinct prime factors on the LHS, which it does not support. $\nexists$
- $p^3q^6 = s^4t^4-1 = (s^2t^2+1)(s^2t^2-1)$.
  - If st is odd, this is a product of two adjacent even factors with a greatest common divisor of 2. Eq. (10) requires $q=2$, so $2^6p^3 = (s^2t^2+1)(s^2t^2-1)$. The associations are
    * $2^5p^3 = s^2t^2+1 \wedge 2 = s^2t^2-1$. Impossible since $2+1 \not\sim (st)^2$.
    * $2p^3 = s^2t^2+1 \wedge 2^5 = s^2t^2-1$. Impossible since $2^5+1 \not\sim (st)^2$
    * $2^5 = s^2t^2+1 \wedge 2p^3 = s^2t^2-1$. Impossible since $2^5-1 \not\sim (st)^2$.
    * $2 = s^2t^2+1 \wedge 2^5p^3 = s^2t^2-1$. Impossible ordering.
  - If st is even, this is a product of two adjacent coprime odd factors.
    * $p^3q^6 = 2^2t^2+1 \wedge 1 = 2^2t^2-1$. Impossible since $1+1 \not\sim (2t)^2$.

  * $q^6 = 2^2t^2+1 \wedge p^3 = 2^2t^2-1$. Subtraction gives $q^6-p^3 = 2 = (q^2-p)(q^4+q^2p+p^2)$. Impossible because the factor $q^4+q^2p+p^2$ is already ≥ 151 for distinct odd primes $p,q\ge 3$.
  * $p^3 = 2^2t^2+1 \wedge q^6 = 2^2t^2-1$. Subtraction requires $p^3-q^6 = 2 = (p-q^2)(q^4+q^2p+p^2)$, and again the factor $q^4+q^2p+p^2$ is already ≥ 151 for distinct odd primes $p,q\ge 3$.
  * $1 = 2^2t^2+1 \wedge p^3q^6 = 2^2t^2-1$. Impossible ordering.

$\nexists$

- $p^6qr = s^{24}-1$. We check that for primes $s\le 13$ the RHS does not have the prime signature of the LHS. For the other primes Theorem 10 requires at least 5 distinct prime factors on the LHS, which it does not support. $\nexists$
- $p^6qr = s^4t^4-1$. Proof by Jon Schoenfield [11]: Without loss of generality, require that $q<r$ and that $s<t$.
  - Suppose st is even. Then $p^6qr = 2^4t^4-1 = (2t-1)(2t+1)(4t^2+1)$: three pairwise coprime odd factors; $2t-1 < 2t+1 < 4t^2+1$.
    So p^6, q, and r are, in some order, $2t-1$, $2t+1$ and $4t^2+1$.
    $p^6 \equiv 1 \pmod 8$, so p^6 cannot be $2t+1$, because $t\equiv\{1,3\} \pmod 4$ means $2t+1\equiv\{3,7\} \pmod 8$. Also, p^6 cannot be $4t^2+1$, because that would require the existence of two positive squares, $(p^3)^2$ and $(2t)^2$, with a difference of 1.
    So $p^6 = 2t-1$, $q = 2t+1$, and $r = 4t^2+1$.
    * If $p=3$, then $3^6 = 2t-1$ (impossible, $3^6+1 \nsim 2t$).
    * If $p\ne 3$, then $p^6\equiv 1 \pmod 3$, and $q = p^6+2$, so $q\equiv 0 \pmod 3$: $3\mid q$ (and $q\ne 3$ since $t\ne 1$) (also impossible).

    $\nexists$
  - So st is odd, Eq. (10) leads to $p=2$, and $2^6qr = s^4t^4-1 = (st-1)(st+1)(s^2t^2+1)$: three even factors with no prime divisors in common other than 2. That means in all 3 possible partitions of the exponent of 2^6 in 3 parts, $6 = 2+2+2 = 1+2+3 = 1+1+4$, only the partition $2^6 = 2^1\cdot 2^1\cdot 2^4$, is possible, so exactly one of the three factors is a multiple of 2^4. That one is not s^2t^2+1; st is odd, so $(st)^2\equiv 1 \pmod 8$, so $s^2t^2+1\equiv 2 \pmod 8$, i.e., 2 divides s^2t^2+1, but 2^2 does not. Since $2^6 \mid s^4t^4-1$, we have $2^5\mid (st-1)(st+1)$.
    So, of $st-1$ and $st+1$, one is 2^4 times an odd number, and the other is 2 times an odd number. The associations of the product qr as either $2qr = s^2t^2+1 \wedge 2^5 = (st+1)(st-1)$ or $2 = s^2t^2+1 \wedge 2^5qr = (st+1)(st-1)$ fail obviously, so it must be distributed as $2q = s^2t^2+1\wedge 2^5r = (st+1)(st-1)$ (without loss of generality). This leaves 4 choices: $2^4 = st+1\wedge 2r = st-1$ or $2^4r = st+1\wedge 2 = st-1$ or $2 = st+1\wedge 2^4r = st-1$ or $2r = st+1\wedge 2^4 = st-1$. Only the first of these matches the prime signatures, $st = 3\cdot 5$, $r = 7$, $q = 113$, the known solution. $\exists$

3.41. **57121.** $n = 57121 = 239^2$, $\tau(n) = 3$ [6, A001248] , $n+1 = 57122 = 2\cdot 13^4$, $\tau(n+1) = 10$ [6, A030628] . $\tau = 3$ requires $n = p^2$ and $\tau = 10$ requires $n+1 = q^9$ or $n+1 = qr^4$. The sub-case $p^2 = q^9-1$ requires by the parity argument $p=2$ or $q=2$, and by explicit evaluation there are no solutions. The sub-case $p^2 = qr^4-1$ requires again by the parity argument either $p=2$ (no solution obviously) or $q=2$ or $r=2$.

- $p^2 = 2r^4 - 1$ provides the known solution with $p = 239$ and $r = 13$, and this is known to be the only solution (besides $p = 1$, $r = 1$) of this Diophantine equation [4, 1].
- $p^2 = q \cdot 2^4 - 1$. This requires $p^2 \equiv -1 \pmod 8$, but this contradicts the statement $p^2 \equiv 1 \pmod 8$ of Eq. (9): no solution here.

3.42. **59048.** $n = 59048 = 2^3 \cdot 11^2 \cdot 61$, $\tau(n) = 24$ [6, A137487] , $n+1 = 59049 = 3^{10}$, $\tau(n+1) = 11$ [6, A030629] . The candidates for $\tau = 24$ have prime signatures $n = p^{23}$, p^2q^7, pq^2r^3, pqr^5, $pqrs^2$, p^3q^5 or pq^{11}. Proof from Jon Schoenfield [11]: For $n + 1$ to have 11 divisors, we must have $n + 1 = s^{10}$ for some prime s.

- For all primes s other than 2, 3, and 11, $n = s^{10} - 1$ is divisible by $2^3 \cdot 3 \cdot 11$, so the prime signatures with less than 3 distinct primes or highest exponent less than 3 are ruled out and only $n = pq^2r^3$ or pqr^5 remain. The only possibilities are $n = 2^5 \cdot 3 \cdot 11 = 1056$, $n = 2^3 \cdot 3^2 \cdot 11 = 792$, and $n = 2^3 \cdot 3 \cdot 11^2 = 2904$, none of which satisfy $\tau(n + 1) = 11$. $\nexists$
- For primes $s = 2, 3$, and 11, $s^{10} - 1$ can be computed explicitly, and the only prime s such that $s^{10} - 1$ has exactly 24 divisors is $s = 3$, so $n = 3^{10} - 1 = 59048$ is the unique known solution. $\exists$

3.43. **65535.** $n = 65535 = 3 \cdot 5 \cdot 17 \cdot 257$, $\tau(n) = 16$, [6, A030634] $n+1 = 65536 = 2^{16}$, $\tau(n+1) = 17$ [6, A030635] . $\tau = 16$ requires the form $n = pqrs$, pqr^3, pq^7, p^3q^3 or p^{15}. $\tau = 17$ requires $n + 1 = s^{16}$. Theorem 9 requires that $n = s^{16} - 1$ is divisible by $2^6 \cdot 3 \cdot 5$ for sufficiently large odd primes s, but none of the 5 prime signatures of n have the minimum three distinct prime factors and at least one exponent ≥ 6. The small primes $s = 2, 3$, and 5 are checked explicitly and only $s = 2$, the known solution, remains.

3.44. **65536.** $n = 65536 = 2^{16}$, $\tau(n) = 17$, $\tau(n + 1) = 2$. $\tau = 17$ requires $n = p^{16}$ and $\tau = 2$ requires $n + 1 = q$. This implies $p^{16} + 1 = q$ and by the parity criterion either $p = 2$ or $q = 2$ which is settled by direct inspection.

3.45. **83520.** $n = 83520 = 2^6 \cdot 3^2 \cdot 5 \cdot 29$, $\tau(n) = 84$, $n + 1 = 83521 = 17^4$, $\tau(n + 1) = 5$ [6, A030514] .

Contribution from Jon Schoenfield [11]: Since $\tau(n + 1) = 5$, $n + 1 = p^4$ for some prime p, so $n = p^4 - 1$, which does not have 84 divisors for any $p \leq 5$, so $p > 5$; for all such primes p, $2^4 \cdot 3 \cdot 5$ divides $p^4 - 1$ (Theorem 5). So n has at least 3 distinct prime factors; the only prime signatures yielding 84 divisors and having at least 3 distinct prime factors are $q^{20}rs$, $q^{13}r^2s$, q^6r^5s, $q^6r^3s^2$, and q^6r^2st.

If n has exactly 3 distinct prime factors, then since $2^4 \cdot 3 \cdot 5 \mid n$, the only possibilities are $2^{20} \cdot 3 \cdot 5$, $2^{13} \cdot 3^2 \cdot 5$, $2^{13} \cdot 3 \cdot 5^2$, $2^6 \cdot 3^5 \cdot 5$, $2^6 \cdot 3 \cdot 5^5$, $2^5 \cdot 3^6 \cdot 5$, $2^5 \cdot 3 \cdot 5^6$, $2^6 \cdot 3^3 \cdot 5^2$, and $2^6 \cdot 3^2 \cdot 5^3$, but none of those values satisfy $\tau(n + 1) = 5$, so n must have more than 3 distinct prime divisors; its prime signature must then be q^6r^2st, so $n = 2^6r^2st$, where r, s, and t are distinct odd primes, two of which are 3 and 5. Note that n has exactly one prime divisor > 5 and is not divisible by 5^3.

For odd primes p, $p^4 - 1 = (p - 1)(p + 1)(p^2 + 1)$ and these three factors are all even and share no prime divisors other than 2. Either $(p - 1)(p + 1)$ or $p^2 + 1$ must be 5-smooth [6, A051037] [otherwise, $n = (p - 1)(p + 1)(p^2 + 1)$ would have at least two distinct prime divisors > 5]. Since p is a prime > 5, $p^2 + 1 \equiv 2 \pmod{2^3 \cdot 3}$ by Eqs. (9) and (25). So $p^2 + 1$ is divisible by 2 but not by 3 or 4, and $2 \mid p^2 + 1$ with $2^5 \mid p^2 - 1$; thus, if $p^2 + 1$ were 5-smooth, it would be of the form $2^1 \cdot 3^0 \cdot 5^L$

where $L < 3$ (since $5^3 \nmid n$), but $L = 0, 1$, or 2 would correspond to $p^2 + 1 = 2, 10$, or 50, i.e., $p = 1, 3$, or 7, none of which would satisfy $\tau(p^4 - 1) = 84$. So $p^2 + 1$ is not 5-smooth.

So $p^2 - 1$ is 5-smooth. The coprime factorizations of $(p^2 + 1)(p^2 - 1) = 2^6 r^2 st$ must be one of

- $2r^2 st = p^2 + 1 \wedge 2^5 = p^2 - 1$. Impossible since $2^5 + 1 \nsim p^2$.
- $2st = p^2 + 1 \wedge 2^5 r^2 = p^2 - 1$. Because $p^2 - 1$ is 5-smooth, either $r = 3$ (which gives the known solution $p = 17$) or $r = 5$ (but $2^5 \cdot 5^2 + 1 \nsim p^2$).
- $2r^2 = p^2 + 1 \wedge 2^5 st = p^2 - 1$. Because $p^2 - 1$ is 5-smooth, $st = 3 \cdot 5$, but $2^5 \cdot 3 \cdot 5 + 1 \nsim p^2$.
- $2 = p^2 + 1 \wedge 2^5 r^2 st = p^2 - 1$. Impossible because $2 - 1 \nsim p^2$.
- $2r^2 s = p^2 + 1 \wedge 2^5 t = p^2 - 1$. Because $p^2 - 1$ is 5-smooth, either $t = 3$ (but $2^5 \cdot 3 + 1 \nsim p^2$) or $t = 5$ (but $2^5 \cdot 5 + 1 \nsim p^2$).
- $2s = p^2 + 1 \wedge 2^5 r^2 t = p^2 - 1$. Because $p^2 - 1$ is 5-smooth, either $r = 3$ and $t = 5$ (but $2^5 \cdot 3^2 \cdot 5 + 1 \nsim p^2$) or $r = 5$ and $t = 3$ (but $2^5 \cdot 5^2 \cdot 3 + 1 = 7^2$ and $7^2 + 1 \nsim 2s$).

3.46. **117648.** $n = 117648 = 2^4 \cdot 3^2 \cdot 19 \cdot 43$, $\tau(n) = 60$, $n + 1 = 117649 = 7^6$, $\tau(n+1) = 7$. $\tau = 60$ requires $n = p^{59}$, pq^{29}, p^2q^{19}, p^3q^{14}, p^4q^{11}, p^5q^9, pqr^{14}, pq^2r^9, $p^2q^3r^4$, pq^4r^5, or pqr^2s^4. $\tau = 7$ requires $n + 1 = t^6$.

The case $t = 2$ is discarded by direction inspection, because $\tau(2^6 - 1) \neq 60$, so t is odd and n is even. The cases $t = 3$ and 5 are discarded also by direct evaluation, and $t = 7$ leads to the known solution. For primes $t > 7$, Theorem 6 requires $2^3 \cdot 3^2 \cdot 7 \mid n$, so only the prime signatures with a minimum of three distinct primes remain.

- $pqr^{14} = t^6 - 1$. Does not support at least two primes with exponents ≥ 2 required by the theorem.
- $pq^2r^9 = t^6 - 1$. The theorem is matched only with $r = 2$, $q = 3$, $p = 7$, but $7 \cdot 3^2 \cdot 2^9 + 1 \nsim t^6$.
- $p^2q^3r^4 = t^6 - 1$. The theorem is matched by
  - $p = 3$, $q = 2$, $r = 7$, but $3^2 \cdot 2^3 \cdot 7^4 + 1 \nsim t^6$.
  - $p = 3$, $q = 7$, $r = 2$, but $3^2 \cdot 7^3 \cdot 2^4 + 1 \nsim t^6$.
  - $p = 7$, $q = 2$, $r = 3$, but $7^2 \cdot 2^3 \cdot 3^4 + 1 \nsim t^6$.
  - $p = 7$, $q = 3$, $r = 2$, but $7^2 \cdot 3^3 \cdot 2^4 + 1 \nsim t^6$.
- $pq^4r^5 = t^6 - 1$. The theorem is matched by
  - $p = 7$, $q = 2$, $r = 3$, but $7 \cdot 2^4 \cdot 3^5 + 1 \nsim t^6$.
  - $p = 7$, $q = 3$, $r = 2$, but $7 \cdot 3^4 \cdot 2^5 + 1 \nsim t^6$.
- $pqr^2s^4 = t^6 - 1$. The theorem is only matched if $s = 2$, $r = 3$, so $2^4 \cdot 3^2 \cdot 7q = t^6 - 1 = (t + 1)(t - 1)(t^2 + t + 1)(t^2 - t + 1)$. The two even factors have greatest common divisor $(t + 1, t - 1) = 2$.

  So one of $t \pm 1$ is an odd multiple of 2^3, the other an odd multiple of 2. The greatest common divisor of $t^2 - 1$ and $t^2 + t + 1$ is 3 if $t \equiv 1 \pmod 3$ and 1 if $t \equiv \{0, 2\} \pmod 3$. The greatest common divisor of $t^2 - 1$ and $t^2 - t + 1$ is 3 if $t \equiv 2 \pmod 3$ and 1 if $t \equiv \{0, 1\} \pmod 3$. (The case $t \equiv 0 \pmod 3$ can be ignored because $t = 3$ does not yield a solution.) $t^2 - t + 1$ and $t^2 + t + 1$ are coprime. How do the three factors 3^2, 7 and q emerge from $(t^2 - 1)/16$, $t^2 + t + 1$ and $t^2 - t + 1$ (all three of which are > 1 because $t > 7$)?

Because $(t^2-1)/16$ shares a factor of 3 with either t^2+t+1 ($t \equiv 1 \pmod 3$) or t^2-t+1 ($t \equiv 2 \pmod 3$), and because 3^2 appears in pqr^2s^4, we have

- if $t \equiv 1 \pmod 3$, $3 \mid (t^2-1)/16$ and $3 \mid t^2+t+1$. The factors 7 and q must be distributed over the factors $t^2 \pm t+1$, because otherwise one of them remains fixed at 1:
  * $3 = (t^2-1)/16 \wedge 3q = t^2+t+1 \wedge 7 = t^2-t+1$. Obviously no solution.
  * $3 = (t^2-1)/16 \wedge 3 \cdot 7 = t^2+t+1 \wedge q = t^2-t+1$. Obviously no solution.
- if $t \equiv 2 \pmod 3$, $3 \mid (t^2-1)/16$ and $3 \mid t^2-t+1$. The factors 7 and q must be distributed over the factors $t^2 \pm t+1$, because otherwise one of them remains fixed at 1:
  * $3 = (t^2-1)/16 \wedge 3q = t^2-t+1 \wedge 7 = t^2+t+1$. Obviously no solution.
  * $3 = (t^2-1)/16 \wedge 3 \cdot 7 = t^2-t+1 \wedge q = t^2+t+1$. Obviously no solution.

4. Conjectured Unique

4.1. **50175.** $n = 50175 = 3^2 \cdot 5^2 \cdot 223$, $\tau(n) = 18$ [6, A030636] , $n+1 = 50176 = 2^{10} \cdot 7^2$, $\tau(n+1) = 33$ [6, A175743] . $\tau = 18$ requires $n = p^{17}$ or pq^8 or p^2q^5 or pq^2r^2. $\tau = 33$ requires $n+1 = s^{32}$ or s^2t^{10}. The pairs of signatures are

- $p^{17} = s^{32}-1$. There are no solutions as outlined in App. B.1.
- $p^{17} = s^2t^{10}-1$. There are no solutions as outlined in App. B.1.
- $pq^8 = s^{32}-1$. $s \neq 2$ because $2^{32}-1 \nsim pq^8$. s is an odd number, so $2^7|s^{32}-1$ by Theorem 4, which matches the RHS only if $q = 2$, so $2^8p = s^{32}-1$. Further by the Mod-3 criterion either $p = 3$ or $s = 3$, but $2^8 \cdot 3+1 \nsim s^{32}$ and $3^{32}-1 \nsim 2^8p$.
- $pq^8 = s^2t^{10}-1 = (st^5+1)(st^5-1)$.
  - If st is even, the RHS are two coprime odd numbers such that
    * $p = st^5+1 \wedge q^8 = st^5-1$. The Mod-3 criterion requires either $p = 3$ (impossible since $3-1 \nsim st^5$) or $q = 3$ (impossible since $3^8+1 \nsim st^5$) or $s = 3$ or $t = 3$ (impossible because then $st^5 \equiv 0 \pmod 3$ and $q^8 \equiv -1 \pmod 3$, which violates Theorem 7).
    * or $q^8 = st^5+1 \wedge p = st^5-1$. The Mod-3 criterion requires either $q = 3$ (impossible since 3^8-1 does not have the form st^5) or $p = 3$ (impossible since $3+1 \nsim st^5$ or (remember st is even) either $s = 2 \wedge t = 3$ (but $2 \cdot 3^5+1 \nsim q^8$) or $s = 3 \wedge t = 2$ (but $3 \cdot 2^5+1 \nsim q^8$).
  - If st is odd, $s^2t^{10}-1$ is divisible by 2^3 and a product of two even factors $st^5 \pm 1$ with a greatest common divisor of 2. $q = 2$ on the LHS. The possible factorizations of 2^8p (with the constraint that $p > 2$) are
    * $2^7p = st^5+1 \wedge 2 = st^5-1$. Fails because $2+1 \nsim st^5$.
    * $2p = st^5+1 \wedge 2^7 = st^5-1$. Requires by subtraction $2p-2^7 = 2$, but that $p = 65$ is not prime.
    * $2^7 = st^5+1 \wedge 2p = st^5-1$. Requires by subtraction $2^7-2p = 2$, but that $p = 63$ is not prime.

- $p^2q^5 = s^{32} - 1$. s is not 2 because $2^{32} - 1 \nsim p^2q^5$. Then for odd s Eq. (21) requires a factor 2^7 on the LHS but this is not supported by p^2q^5.
- $p^2q^5 = s^2t^{10} - 1$.
  - If st is even, the RHS is the product of two coprime odd numbers such that
    - $p^2 = st^5 + 1 \wedge q^5 = st^5 - 1$. The Mod-3 criterion requires either $p = 3$ or $q = 3$ (both fail to match the $st^5 \pm 1$ formats) or $s = 3$ or $t = 3$. Because st is even, either $st^5 = 2 \cdot 3^5$ or $3 \cdot 2^5$, but in both cases $st^5 + 1$ do not match the prime signature p^2.
    - or $q^5 = st^5 + 1 \wedge p^2 = st^5 - 1$. The Mod-3 criterion requires either $p = 3$ or $q = 3$ (both fail to match their $st^5 \pm 1$ formats) or $s = 3$ or $t = 3$. Because st is even, either $st^5 = 2 \cdot 3^5$ (but $2 \cdot 3^5 + 1 \nsim q^5$) or $3 \cdot 2^5$ (but $3 \cdot 2^5 + 1 \nsim q^5$).
  - If st is odd, $(st^5 + 1)(st^5 - 1)$ is a product of two adjacent positive numbers with greatest common divisor 2 and $q = 2$ on the other side from Eq. 9. The factorizations of $2^5p^2 = (st^5 + 1)(st^5 - 1)$ are one of
    - $2p^2 = st^5 + 1 \wedge 2^4 = st^5 - 1$. Impossible because $2^4 + 1 \nsim st^5$.
    - $2^4p^2 = st^5 + 1 \wedge 2 = st^5 - 1$. Impossible because $2 + 1 \nsim st^5$.
    - $2^4 = st^5 + 1 \wedge 2p^2 = st^5 - 1$. Impossible because $2^4 - 1 \nsim st^5$.
- $pq^2r^2 = s^{32} - 1$. s is not 2 because $2^{32} - 1 \nsim pq^2r^2$. Then for odd s Eq. (21) requires a factor 2^7 on the LHS, but this is not supported by pq^2r^2.
- $pq^2r^2 = s^2t^{10} - 1$. The branch with the known solution.
  - If st is odd, $s^2t^{10} - 1$ is divisible by 2^3 by Eq. (9) but the exponents of pq^2r^2 are too small to support solutions.
  - If st is even, $(st^5+1)(st^5-1)$ is a product of two coprime odd numbers. The possible splitting of the three factors p, q^2 and r^2 over these two are (note that q^2 and r^2 are equivalent here)
    - $pq^2r^2 = st^5 + 1 \wedge 1 = st^5 - 1$. Impossible because $1 + 1 \nsim st^5$.
    - $q^2r^2 = st^5 + 1 \wedge p = st^5 - 1$. Because qr is odd, $8|(qr)^2 - 1$, so $t = 2$, so $(qr)^2 = 2^5s + 1 \wedge p = 2^5s - 1$. $(qr + 1)(qr - 1)$ is a product of two adjacent even numbers, so the splitting of the factors 2^5s is one of
      - $2s = qr+1 \wedge 2^4 = qr-1 \wedge p = 2^5s-1$. Subtraction requires $2s - 2^4 = 2$, but $s = 9$ is not prime.
      - $2^4s = qr+1 \wedge 2 = qr-1 \wedge p = 2^5s-1$. Subtraction requires $2^4s - 2 = 2$, but such prime s does not exist.
      - $2 = qr+1 \wedge 2^4s = qr-1 \wedge p = 2^5s-1$. Impossible because $2 < 2^4s$
      - $2^4 = qr+1 \wedge 2s = qr-1 \wedge p = 2^5s-1$. Subtraction requires $2^4 - 2s = 2$, so $s = 7$, $p = 223$, $qr = 3 \times 5$, which is the known solution.
    - $pr^2 = st^5 + 1 \wedge q^2 = st^5 - 1$. Since st is even, q is odd, so $q^2 \equiv 1 \pmod 4$, which needs to be balanced by $st^5 - 1 \equiv 1 \pmod 4$. Checking all 4×4 cases of s and t, this requires $s \equiv 2 \pmod 4$ and $t \equiv \{1, 3\} \pmod 4$. Since st is even, $s = 2$ and t is odd, so $pr^2 = 2t^5 + 1 \wedge q^2 = 2t^5 - 1$. No solutions because $q^2 = 2t^5 - 1$ has no solutions in the primes [5].

* $r^2 = st^5 + 1 \wedge pq^2 = st^5 - 1$. Here $r^2 - 1 = st^5$ with odd r such that $2^3 | r^2 - 1$, so $t = 2$ and $(r+1)(r-1) = 2^5 s \wedge pq^2 = 2^5 s - 1$. The Mod-3 criterion needs either $r = 3$ (impossible since $(3+1)(3-1) \not\approx 2^5 s$ or $s = 3$ (impossible since $2^5 \cdot 3 + 1 \not\approx r^2$) or $p = 3$ or $q = 3$. The two latter choices are not possible because then the two sides of $r^2 - 1 = 2^5 s$ would differ modulo 3.
* $p = st^5 + 1 \wedge q^2 r^2 = st^5 - 1$. The Mod-3 criterion requires that either $p = 3$ (but $3 - 1 \not\approx st^5$) or $q = 3$ (equivalent to $r = 3$) or $s = 3$ or $t = 3$.
  · $p = st^5 + 1 \wedge 3^2 r^2 = st^5 - 1$. t cannot be 2, because that would require $3^2 r^2 = 2^5 s - 1$. By Eq. (9) odd squares are $(3r)^2 + 1 \equiv 2 \pmod{2^3}$, but the equation would require odd squares $(3r)^2 + 1 \equiv 0 \pmod{2^5}$, which is incompatible. So $s = 2$, and the focus is on the existence of solutions to the coupled Diophantine equations $p = 2t^5 + 1 \wedge 3^2 r^2 = 2t^5 - 1$ for odd distinct primes p, t and r larger than 3. This manuscript does not offer a proof that no such triples exist, so the question whether 50175 can be moved into Section 3 remains unanswered here.
  · $p = 3t^5 + 1 \wedge q^2 r^2 = 3t^5 - 1$. Impossible since $3t^5 \equiv 0 \pmod 3$ and $q^2 r^2 \equiv 1 \pmod 3$ since $q \cdot r > 3$.
  · $p = 3^5 s + 1 \wedge q^2 r^2 = 3^5 s - 1$. Impossible since $3^5 s \equiv 0 \pmod 3$ and $q^2 r^2 \equiv 1 \pmod 3$ since $q \cdot r > 3$.
* $1 = st^5 + 1 \wedge pq^2 r^2 = st^5 - 1$. Impossible because $st^5 + 1 > st^5 - 1$.

4.2. **59049.** $n = 59049 = 3^{10}$, $\tau(n) = 11$ [6, A030629] , $n+1 = 59050 = 2 \cdot 5^2 \cdot 1181$, $\tau(n+1) = 12$ [6, A030630] . $\tau = 11$ requires $n = p^{10}$. $\tau = 12$ requires $n+1 = q^{11}$, $q^5 r$, $q^3 r^2$, or $q^2 rs$.

A proof that $n = 59049$ is a term *unless* $n > 10^{40000}$ [11]:

$n + 1 = p^{10} + 1 = xy$ where $x = p^2 + 1$ and $y = p^8 - p^6 + p^4 - p^2 + 1$.

$p \neq 2$ since $\tau(2^{10}+1) \neq 12$. So p is an odd prime, so $p^2 \equiv 1 \pmod 8$; $x = p^2+1 \equiv 2 \pmod 8$ (which is an even composite number), and y is odd, so $n+1 = xy = 2q^5$ or $2q^2 s$.

$y = x(p^6 - 2p^4 + 3p^2 - 4) + 5$, so $(y, x) = 1$ or 5.

Note that y is never a square; for every odd prime p (and for all other odd numbers $p \geq 3$), the fractional part of $\sqrt{y}$ is in the interval $[0.843997..., 0.875]$, converging to $0.875 = 7/8$.

If $n + 1 = 2q^5$, then since x is an even number > 2, and since y is not a square, we must have $x = 2q^2$ and $y = q^3$ or $x = 2q^4$ and $y = q$, but the latter would mean $y > x$ (impossible). So $x = 2q^2$ and $y = q^3$.

But $y > x^3$ for all primes p, so $x = 2q^2 \wedge y = q^3$ is also impossible.

So $n + 1 = xy = 2q^2 s$, and $2 \mid x$, and $x > 2$, and $y \neq q^2$.

Thus if $(x, y) = 1$, then $x = 2q^2$, $y = s$ (addressed below).

The only other way to split up $n + 1 = 2q^2 s$ would be $x = 2q$, $y = qs$, but this would require $(x, y) = 5$, and this would mean $x = 2 \cdot 5 = 10$, $y = 5s$; $x = p^2 + 1$, so $x = 10$ means $p = 3$, $y = 3^8 - 3^6 + 3^4 - 3^2 + 1 = 5905 = 5 \cdot 1181$, so $n + 1 = xy = 10 \cdot 5905 = 59050$; this is the known solution $n = 59049 = 3^{10}$.

So if there is any solution other than $n = 59049$, it must satisfy $n+1 = p^{10}+1 = xy$ where $x = p^2 + 1$ and $y = p^8 - p^6 + p^4 - p^2 + 1$, with $x = 2q^2$ and $y = s$.

So p must be a prime such that $p^2+1=2q^2$ and $p^8-p^6+p^4-p^2+1=s$ is prime.

There *do* exist primes p such that $p^2+1=2q^2$. In general, pairs of positive integers (y,z) such that $y^2+1=2z^2$ are (1, 1), (7, 5), (41, 29), (239, 169), (1393, 985), (8119, 5741), (47321, 33461), (275807, 195025), ... (the y values are [6, A002315] the z values are [6, A001653])

Thus if there is a solution other than $n=59049$, it is the 10th power of a prime p that meets three requirements:

(1) p is a term in [6, A002315],
(2) the accompanying term in [6, A001653], i.e., the number q that satisfies $p^2+1=2q^2$, is also a prime, and
(3) $p^8-p^6+p^4-p^2+1=1+(p^4+1)p^2(p+1)(p-1)$ is also a prime.

Ordered pairs that satisfy the first two requirements are rare:

p	q
7	5
41	29
63018038201	44560482149
19175002942688032928599	13558774610046711780701

[6, A086397] lists (after its initial term, 3) the numbers p in such pairs. [6, A118612] lists (after its initial term, 2) the numbers q in such pairs, and each of those two sequences has comment that says *Next term, if it exists, is bigger than 489 digits (the 1279th convergent to sqrt(2)).* (I checked farther; the next term, if it exists, exceeds 10^{5000}.)

For each of the above pairs, the corresponding value of $p^8-p^6+p^4-p^2+1$ is *not* a prime.

So if there's a number other than 59049 such that $\tau(n)=11$ and $\tau(n+1)=12$, it's at least $p^8-p^6+p^4-p^2+1$ where $p\geq 10^{5000}$, so $n\geq 10^{40000}$.

But primes aren't terribly rare among values of $y=p^8-p^6+p^4-p^2+1$ for primes p; y is prime for each of the following primes: 11, 19, 101, 109, 139, 149, 181, 229, 311, 479, 601, 619, 631, 659, 719, 1201, 1229, 1511, 1531, 1571, 1699, 1721, 1879, 1931, 1999, 2221, 2239, 2269, 2549, 2879, 2939, 2969, 3221, 3391, 3511, 3539, 3541, 3719, 3761, 4211, 4241, 4421, 4441, 4621, 4649, 5059, 5471, 5519, 5569, 5869, 5939, 6151, 6361, 6571, 6659, 6781, 7069, 7151, 7159, 7489, 7499, 7759, 8419, 8951, 9419, 9649, 9929, ...

Is there some way to prove that if p is prime and the corresponding q such that $p^2+1=2q^2$ then $p^8-p^6+p^4-p^2+1$ cannot be prime?

4.3. **130320.** $n=130320=2^4\cdot 3^2\cdot 5\cdot 181$, $\tau(n)=60$, $n+1=130321=19^4$, $\tau(n+1)=5$.

4.4. **146688.** $n=146688=2^8\cdot 3\cdot 191$, $\tau(n)=36$, $n+1=146689=383^2$, $\tau(n+1)=3$.

4.5. **262143.** $n=262143=3^3\cdot 7\cdot 19\cdot 73$, $\tau(n)=32$, $n+1=262144=2^{18}$, $\tau(n+1)=19$.

4.6. **263168.** $n=263168=2^{10}\cdot 257$, $\tau(n)=22$, $n+1=263169=3^6\cdot 19^2$, $\tau(n+1)=21$.

4.7. **279840.** $n = 279840 = 2^5 \cdot 3 \cdot 5 \cdot 11 \cdot 53$, $\tau(n) = 96$, $n+1 = 279841 = 23^4$, $\tau(n+1) = 5$.

4.8. **331775.** $n = 331775 = 5^2 \cdot 23 \cdot 577$, $\tau(n) = 12$, $n+1 = 331776 = 2^{12} \cdot 3^4$, $\tau(n+1) = 65$.

4.9. **529983.** $n = 529983 = 3^6 \cdot 727$, $\tau(n) = 14$, $n+1 = 529984 = 2^6 \cdot 7^2 \cdot 13^2$, $\tau(n+1) = 63$.

4.10. **531440.** $n = 531440 = 2^4 \cdot 5 \cdot 7 \cdot 13 \cdot 73$, $\tau(n) = 80$, $n+1 = 531441 = 3^{12}$, $\tau(n+1) = 13$,

4.11. **746495.** $n = 746495 = 5 \cdot 173 \cdot 863$, $\tau(n) = 8$, $n+1 = 746496 = 2^{10} \cdot 3^6$, $\tau(n+1) = 77$.

4.12. **923520.** $n = 923520 = 2^7 \cdot 3 \cdot 5 \cdot 13 \cdot 37$, $\tau(n) = 128$, $n+1 = 923521 = 31^4$, $\tau(n+1) = 5$.

5. Proven Non-Unique

In chapter 5 we list those n which are absent from the sequence because one can find an m which fulfills the requirement of (3).

5.1. Individual Entries Proven Non-unique.

- $n = 23103 = 3^2 \cdot 17 \cdot 151$, $\tau(n) = 12$, $\tau(n+1) = 21$. matched by $m = 4631103 = 3^2 \cdot 239 \cdot 2153$ for example.
- $n = 25920 = 2^6 \cdot 3^4 \cdot 5$, $\tau(n) = 70$, $\tau(n+1) = 9$. matched by $m = 34023888 = 2^4 \cdot 3^6 \cdot 2917$ for example.
- $n = 33124 = 2^2 \cdot 7^2 \cdot 13^2$, $\tau(n) = 27$, $\tau(n+1) = 10$. matched by $m = 5373124 = 2^2 \cdot 19^2 \cdot 61^2$ for example.
- $n = 39600 = 2^4 \cdot 3^2 \cdot 5^2 \cdot 11$, $\tau(n) = 90$, $\tau(n+1) = 3$. matched by $m = 75916368 = 2^4 \cdot 3^2 \cdot 11^2 \cdot 4357$ for example.
- $n = 46656 = 2^6 \cdot 3^6$, $\tau(n) = 49$, $\tau(n+1) = 8$. matched by $m = 7529536 = 2^6 \cdot 7^6$ for example.
- $n = 28224 = 2^6 \cdot 3^2 \cdot 7^2$, $\tau(n) = 63$, $\tau(n+1) = 6$. matched by $m = 5271616 = 2^6 \cdot 7^2 \cdot 41^2$ for example.
- $n = 71824 = 2^4 \cdot 67^2$ matched by $m = 868834576 = 2^4 \cdot 7369^2$ for example.
- $n = 82944 = 2^{10} \cdot 3^4$ matched by $m = 1686498489 = 3^{10} \cdot 13^4$ for example.
- $n = 250000 = 2^4 \cdot 5^6$ matched by $m = 2368574224 = 2^4 \cdot 23^6$ for example.
- $n = 262144 = 2^{18}$ matched by $m = 387420489 = 3^{18}$ for example.
- $n = 421200 = 2^4 \cdot 3^4 \cdot 5^2 \cdot 13$ matched by $m = 262472400 = 2^4 \cdot 3^4 \cdot 5^2 \cdot 8101$ or $1008253008 = 2^4 \cdot 3^4 \cdot 7^2 \cdot 15877$ for example.
- $n = 589824 = 2^{16} \cdot 3^2$ matched by $m = 59520385024 = 2^{16} \cdot 953^2$ or $261358157824 = 2^{16} \cdot 1997^2$ for example.
- $n = 640000 = 2^{10} \cdot 5^4$ matched by $m = 718240924712219661023456897050078235203037193616 = 2^4 \cdot 46261^{10}$ for example.
- $n = 641600 = 2^6 \cdot 5^2 \cdot 401$ matched by $m = 12408400448 = 2^6 \cdot 59^2 \cdot 55697$ for example.
- $n = 651249 = 3^2 \cdot 269^2$ matched by $m = 290941249 = 3^2 \cdot 461^2$ or $321951249 = 3^2 \cdot 5981^2$ for example.
- $n = 746496 = 2^{10} \cdot 3^6$ matched by $m = 4942652416 = 2^{10} \cdot 13^6$ or $151588750336 = 2^{10} \cdot 23^6$ for example.

- $n = 777924 = 2^2 \cdot 3^4 \cdot 7^4$ matched by $m = 410791824 = 2^4 \cdot 3^4 \cdot 563^2$ or $1983099024 = 2^4 \cdot 3^4 \cdot 1237^2$ for example.
- $n = 860624$ matched by $m = 150309999$
- $n = 861183 = 3^2 \cdot 103 \cdot 929$ matched by $m = 369869823 = 3^2 \cdot 2137 \cdot 19231$ or $1063281663 = 3^2 \cdot 3623 \cdot 32609$ for example.
- $n = 937024 = 2^6 \cdot 11^4$ matched by $m = 886133824 = 2^6 \cdot 61^4$ or $64537845849 = 3^6 \cdot 97^4$ for example.
- $n = 1000000$ matched by $m = 689869781056$ for example [11].

5.2. **Broad band search (small m).** A numerical search for unique pairs started by Jack Brennen in Jun 11, 2009 in the seqfan list was extended by Jon Schoenfield in August 2019. All numbers in the range $n \le 1000000$ except the individuals listed in Chapters 3 and 4 have non-unique τ-pairs.

6. Summary

We filtered numerically a maximum set of 58 pairs $(\tau(n),\tau(n+1))$ in the range $n < 1,000,000$ which seem to be unique in the infinite sequence of such pairs, and showed for 46 of them that they are indeed unique by computing all solutions for their set of associated Diophantine equations that emerge from the prime signatures of n and $n+1$.

Appendix A. Squares Near-Misses

In the context of this manuscript, Squares Near-Misses are numbers of the form $n^2 - 1$. The associated subset of the τ-functions is

(6)
$$\tau(n^2-1) = 0, 2, 4, 4, 8, 4, 10, 6, 10, 6, 16, 4, 16, 8, 12, 8, 18, 4, 24, 8, 16, 8, 20, 6, \ldots (n \ge 1)$$

These are even numbers. [Proof: n^2-1 is not a square basically because the squares have a mutual distance larger than 1 — except the squares 0^2 and 1^2 which are not of interest here. So in the unique prime factorization of $n^2 - 1$ there is at least one prime factor with an odd exponent, and each odd exponent multiplies τ by an even number according to Eq. (2).] More explicit: Let $\nu_p(n)$ characterize the maximum power of p that divides n:

$$\nu_p(n) = \max_{\alpha \ge 0} \{p^\alpha \mid n\}. \tag{7}$$

Then the maximum power of 2 in $\tau(n^2 - 1)$, $\nu_2(\tau(n^2 - 1))$, is the number of primes with odd exponent in the prime factorization of $n^2 - 1$.

The Prime Squares Near-Misses are the subset where the s are primes, [6, A084920]

$$p^2 - 1 = 3, 8, 24, 48, 120, 168, 288, 360, 528, 840, 960, 1368, 1680, 1848, \ldots \tag{8}$$

With the exception of the initial 3 these are multiples of 4 because the factors $p+1$ and $p-1$ are individually multiples of 2.

For odd numbers of the form $a = 2k + 1$ we have $a^2 - 1 = 4k(k + 1)$, $a^3 - 1 = 2k(4k^2 + 6k + 3)$, $a^4 - 1 = 8k(k + 1)(2k^2 + 2k + 1)$, and so on, where $k(k + 1)$ is an even number:

Theorem 4. *For odd a*

$$2^3 \mid a^2 - 1 \tag{9}$$
$$2^4 \mid a^4 - 1 \tag{10}$$
$$2^3 \mid a^6 - 1 \tag{11}$$
$$2^5 \mid a^8 - 1 \tag{12}$$
$$2^3 \mid a^{10} - 1 \tag{13}$$
$$2^4 \mid a^{12} - 1 \tag{14}$$
$$2^3 \mid a^{14} - 1 \tag{15}$$
$$2^6 \mid a^{16} - 1 \tag{16}$$
$$2^3 \mid a^{18} - 1 \tag{17}$$
$$2^4 \mid a^{20} - 1 \tag{18}$$
$$2^5 \mid a^{24} - 1 \tag{19}$$
$$2^3 \mid a^{26} - 1 \tag{20}$$
$$2^7 \mid a^{32} - 1 \tag{21}$$
$$2^6 \mid a^{48} - 1 \tag{22}$$
$$2^3 \mid a^{62} - 1 \tag{23}$$
$$2^8 \mid a^{64} - 1 \tag{24}$$

With the exception of the initial 3 and 8 they are multiples of 3:

$$3 \mid p^2 - 1 \quad (\text{primes}\, p > 3) \tag{25}$$

Proof. For primes of the form $p = 3k + 1$ we have $p^2 - 1 = 3k(3k + 1)$ and for primes of the form $p = 3k + 2$ we have $p^2 - 1 = 3(k + 1)(3k + 1)$. □

For odd primes Eq. (10) may be extended as follows [11]:

Theorem 5. $2^4 \cdot 3 \cdot 5 \mid p^4 - 1 \quad (\text{prime}\, p > 5)$.

Proof. The factor 5 is shown by using the prime residue classes $p \equiv \{1, 2, 3, 4\} \pmod 5$ such that $p^2 \equiv \{1, 4\} \pmod 5$ and $p^4 \equiv 1 \pmod 5$. The factor 3 is inherited from Eq. (25) because $p^2 - 1 \mid p^4 - 1$. □

Most of this section summarizes standard divisibility properties of Cyclotomic Polynomials [3, 8, 2].

$$p^2 - 1 = \Phi_1(p)\Phi_2(p), \tag{26}$$
$$p^4 - 1 = \Phi_1(p)\Phi_2(p)\Phi_4(p), \tag{27}$$
$$p^6 - 1 = \Phi_1(p)\Phi_2(p)\Phi_3(p)\Phi_6(p), \tag{28}$$
$$p^8 - 1 = \Phi_1(p)\Phi_2(p)\Phi_4(p)\Phi_8(p), \tag{29}$$
$$p^{10} - 1 = \Phi_1(p)\Phi_2(p)\Phi_5(p)\Phi_{10}(p), \tag{30}$$
$$p^{12} - 1 = \Phi_1(p)\Phi_2(p)\Phi_3(p)\Phi_6(p)\Phi_4(p)\Phi_{12}(p). \tag{31}$$

For odd primes Eq. (11) may be extended as follows:

Theorem 6. $2^3 \cdot 3^2 \cdot 7 \mid p^6 - 1 \quad (\text{prime}\, p > 7)$.

Proof. The factor 7 is shown by using the prime residue classes $p \equiv \{1,2,3,4,5,6\} \pmod 7$ such that $p^2 \equiv \{1,2,4\} \pmod 7$, $p^3 \equiv \{1,6\} \pmod 7$ and finally $p^6 \equiv 1 \pmod 7$. The factor 3^2 is shown by using the prime residue classes $p \equiv \{1,2,4,5,7,8\} \pmod 9$ such that $p^2 \equiv \{1,4,7\} \pmod 9$, $p^3 \equiv \{1,8\} \pmod 9$, $p^4 \equiv \{1,4,7\} \pmod 9$, and $p^6 \equiv 1 \pmod 9$. □

For odd primes Eq. (12) may be extended as follows:

Theorem 7. $2^5 \cdot 3 \cdot 5 \mid p^8 - 1 \quad (\text{prime}\, p > 5)$.

Proof. The factor 2^5 is inherited from (12) and the factor $3 \cdot 5$ is inherited from Theorem 5, since $p^4 - 1 | p^8 - 1$. □

For odd primes Eq. (13) may be extended as follows:

Theorem 8. $2^3 \cdot 3 \mid p^{10} - 1 \quad (\text{prime}\, p > 3)$.

Proof. The factor 2^3 is inherited from (13) and the factor 3 is inherited from (25), since $p^2 - 1 | p^{10} - 1$. □

For odd primes Eq. (16) may be extended as follows:

Theorem 9. $2^6 \cdot 3 \cdot 5 \mid p^{16} - 1 \quad (\text{prime}\, p > 5)$.

Proof. The factor 2^6 is inherited from (16) and the factor $3 \cdot 5$ is inherited from Theorem 7, since $p^8 - 1 | p^{16} - 1$. □

For odd primes Eq. (19) may be specialized as follows:

Theorem 10. $2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13 \mid p^{24} - 1 \quad (\text{prime}\, p > 13)$.

Proof. The factor 13 is shown by using the prime residue classes $p \equiv \{1,2,3,\ldots,12\} \pmod{13}$ such that $p^3 \equiv \{1,5,8,12\} \pmod{13}$, $p^6 \equiv \{1,12\} \pmod{13}$, $p^{12} \equiv 1 \pmod{13}$.

The factor $3^2 \cdot 7$ is inherited from Theorem 6 because $p^6 - 1 \mid p^{24} - 1$. The factor 5 is inherited from Theorem 5 because $p^4 - 1 \mid p^{24} - 1$. □

Appendix B. Generic Prime Signatures

B.1. $a^2 - 1 = p^j$**.** Can a number of the form $a^2 - 1 = (a+1)(a-1)$ equal a power of a prime, p^j? The answer is obviously *no* if $j = 1$, because a prime cannot have two distinct factors $a \pm 1$. The severe constraint is that the two divisors $a+1$ and $a-1$ are near neighbors. Solutions require that p^α is split into one of the forms $p \cdot p^{j-1}$, $p^2 \cdot p^{j-2}$,$\ldots p^{\lfloor (j-1)/2} \cdot p^{\lceil (j+1)/2 \rceil}$, where the smaller factor $p^i = a - 1$ and the larger factor $p^{j-i} = a + 1$, $2i < j$. Subtraction of these two equations requires $2 = p^{j-i} - p^i = p^i(p^{j-2i} - 1)$, which requires either $2 = p^i$, $1 = p^{j-2i} - 1$ or $p^i = 1$ (impossible) and $p^{j-2i} - 1 = 2$. This winds down to $p = 2$, $i = 1$, therefore $p^{j-2} = 2$ and finally $j = 3$. So the only solution is $3^2 - 1 = 2^3$.

B.2. $a^2 - 1 = pq^j$**.** Can a number of the form $a^2 - 1 = (a+1)(a-1)$ equal a prime times another prime's power, pq^j?

B.2.1. $j = 1$. There are numerous solutions for $j = 1$ with $a = 4, 6, 12, 18, 30, \ldots$. [6, A014574]. There are no solutions where a is a prime.

Proof. The candidates in the prime residue classes $a \equiv \{1,5\} \pmod 6$ have $a^2 - 1 \equiv 0 \pmod 6$ which requires $6 \mid pq$ and $p = 2$, $q = 3$, which does not solve the equation (by direct inspection). □

B.2.2. $j = 2$. The candidates of the factorization of pq^2 are $p \cdot q^2$ and $(pq) \cdot q$. (The $1 \cdot pq^2$ and $a-1 = 1$, $a+1 = pq^2$ does not generate solutions.) Their possible orderings are $p < q^2$, $p > q^2$ and $q < pq$, so assignment of $a \pm 1$ to the two factors classifies as:

- $a-1 = p$, $a+1 = q^2$: There are numerous solutions with $a = 8, 24, 48, 168, 360, \ldots$. If we consider the prime classes $q \equiv \{1, 3, 5, 7\} \pmod 8$ we conclude that $8|a$.
- $a-1 = q^2$, $a+1 = p$: There is only the solution with $a = 10$, $q = 3$ and $p = 11$. There are no others because this requires $p = q^2 + 2$, and with the classifications $q \equiv \{1, 2\} \pmod 3$ with find $q^2 \equiv 1 \pmod 3$ such that $q^2 + 2$ is divisible by 3 and cannot equal a prime p.
- $a-1 = q$, $a+1 = pq$: The ratio $(pq)/q = (a+1)/(a-1) = 1 + 2/(a-1)$ needs to be integer, which requires $a = 2$ or $a = 3$. By direct inspection none of these has the proper format.

B.2.3. $j = 3$. The candidates of the factorization of $pq^3 = (a+1)(a-1)$ in two factors are $p \cdot q^3$, $(pq) \cdot q^2$, and $(pq^2) \cdot q$. (The $1 \cdot pq^3$ is left aside as reasoned in App. B.2.2.) Their possible orderings are $p < q^3$, $p > q^3$, $pq > q^2$, $pq < q^2$, $q < pq^2$, so assignments of $a \pm 1$ are:

- $a-1 = p$, $a+1 = q^3$: There are numerous solutions at $a = 6858, 29790, 50652, 300762, 1295028, \ldots$.
- $a-1 = q^3$, $a+1 = p$: There are numerous solutions at $a = 2^2 \cdot 7$, $2 \cdot 3^2 \cdot 7$, $2 \cdot 3^2 \cdot 5 \cdot 271$, $2^3 \cdot 3^3 \cdot 1667, \ldots$
- $a-1 = q^2$, $a+1 = pq$: Subtraction of these two equations yields $2 = q(p-q)$, so $q = 2$, $p = 3$, $a = 25$ is the only solution.
- $a-1 = pq$, $a+1 = q^2$: Subtraction of these two equations yields $2 = q(q-p)$, so there is no solution.
- $a-1 = q$, $a+1 = pq^2$: The ratio $(pq^2)/q = (a+1)/(a-1) = 1 + 2/(a-1)$ needs to be integer, which requires $a = 2$ or $a = 3$. By direct inspection none of these has the proper format.

B.2.4. $j = 4$. The candidates of the factorization of pq^4 in two factors are $p \cdot q^4$, $(pq) \cdot q^3$, $(pq^2) \cdot q^2$ and $(pq^3) \cdot q$. (The $1 \cdot pq^4$ is left aside as reasoned in App. B.2.2.) Their possible orderings with $p \lesseqgtr q$ are $p < q^4$, $p > q^4$, $pq > q^3$, $pq < q^3$, $q^2 < pq^2$, $q < pq^3$, so assignments of $a \pm 1$ are:

- $a-1 = p$, $a+1 = q^4$: Numerous solutions via $a = 2^4 \cdot 5$, $2^5 \cdot 3 \cdot 5^2$, $2^4 \cdot 3 \cdot 5 \cdot 61$, $2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 17, \ldots$ exist. With the exception of the smallest, $a = 80$, all are multiples of $2^4 \cdot 3 \cdot 5 = 240$, Theorem 5.
- $a-1 = q^4$, $a+1 = p$: There is only one solution with $a = 82 = 2 \cdot 41$, $q = 3$, $p = 83$. There are no others because this requires $p = q^4 + 2$, and with the classifications $q \equiv \{1, 2\} \pmod 3$ with find $q^4 \equiv 1 \pmod 3$ such that $q^4 + 2$ is divisible by 3 and cannot equal a prime p.
- $a-1 = q^3$, $a+1 = pq$: Subtraction requires $2 = q(p - q^2)$, so there is only the solution with $a = 9$, $q = 2$, $p = 5$.
- $a-1 = pq$, $a+1 = q^3$: Subtraction requires $2 = q(q^2 - p)$, so there is only the solution with $q = 2$, $p = 3$ and $a = 7$.
- $a-1 = q^2$, $a+1 = pq^2$: The ratio $(pq^2)/q^2 = (a+1)/(a-1) = 1 + 2/(a-1)$ needs to be integer, which requires $a = 2$ or $a = 3$. By direct inspection none of these has the proper format.

- $a-1=q$, $a+1=pq^3$: The ratio $(pq^3)/q=(a+1)/(a-1)=1+2/(a-1)$ needs to be integer, which requires $a=2$ or $a=3$. By direct inspection none of these has the proper format.

B.3. $a^4-1=pq^3$**.** Can a number of the format $a^4-1=(a^2+1)(a+1)(a-1)$ equal a prime times a different prime cubed, pq^3? For $0\le a\le 1$ the answer is negative because the LHS of the equation is not positive. For $a\ge 2$, the RHS has three distinct, well-ordered factors $a^2+1>a+1>a-1$. There is only one factorization of pq^3 which supports three distinct factors: $p\cdot q\cdot q^2$. (At this place factorizations like $(pq^2)\cdot q\cdot 1$ with smallest part equal 1 are discarded because that implies $a=2$ and $2^4-1=3\cdot 5\ne pq^3$.) There are three different possible orders of p relative to q and q^2:

- $p<q<q^2$: Here $q=a+1$, which implies $q^2=a^2+2a+1$, and $q^2=a^2+1$, which requires $a=0$ and does not provide solutions.
- $q<p<q^2$: Here $q=a-1$, which implies $q^2=a^2-2a+1$, and $q^2=a^2+1$, which requires $a=0$ and does not provide solutions.
- $q<q^2<p$: Here $q=a-1$ and $q^2=a+1$, which subtracted require $2=q^2-q=q(q-1)$, so $q=2$, therefore $a=3$. [In short: 2 and 4 is the only twin prime pair of the form (q,q^2).] In addition we need $p=a^2+1$ which becomes $p=10$ since $a=3$, which is not prime, as required.

Summary: there are no solutions to $a^4-1=pq^3$.

Appendix C. Individual Pairs

C.1. **Pairs of 4's.** The number 4 appears in the sequence of τ when $n=p^3$ or $n=pq$, cubed primes or squarefree semiprimes.

C.1.1. *Pair of Semiprimes.* Squarefree semiprimes followed by squarefree semiprimes require $n=pq=rs-1$ with distinct primes p, q, r and s, where exactly one of them is 2, and that condition is often met, see [6, A263990].

Remark 1. *There are even triples of consecutive 4's in* (1)*, for example starting at* $n=33$*, or* 85 *or* 93*, see* [6, A039833].

C.1.2. *Cube plus Semiprime.* A prime cubed followed by a semiprime means $2^3=rs-1$ (which obviously has no solution because $2^3+1=3^2\ne rs$) or $p^3=2s-1$, which also has no solution.

Proof. The case where $p=3$ requires $s=14$ which is not prime. The cases in the residuum class $p=3k+1$, $k\ge 2$, lead to $p^3+1=(3k+2)(9k^2+3k+1)$ which do not have the format $2s$, and the cases $p=3k+2$, $k\ge 1$ lead to $p^3+1=9(k+1)(3k^2+3k+1)$ which also do not have the format $2s$. $\square$

C.1.3. *Semiprime plus Cube.* A squarefree semiprime followed by a prime cubed, $rs=p^3-1$, which has one solution: The case $p=2$ has obviously no solution, and $2s=p^3-1$ has one solution.

Proof. The case where $p=3$ furnishes a solution with $s=13$. The cases in the residuum class $p=3k+1$ ($k\ge 2$) lead to $p^3-1=9k(3k^2+3k+1)$ which does not have the format $2s$, and the cases $p=3k+2$ ($k\ge 1$) lead to $p^3-1=(3k+1)(9k^2+15k+7)$ which also does not have the format $2s$. $\square$

C.2. **The Pair (6,2).** Pairs of (6,2) in the τ-sequence are generated when a number of the form $n = p^5$ or $n = pq^2$ is followed by prime $n + 1 = s$. The case $p^5 = s - 1$ has neither a solution $2^5 = s - 1$ nor $p^5 = 2 - 1$. The case $pq^2 = s - 1$ has the formats

- $2q^2 = s - 1$: There is a solution with $q = 3$, $s = 19$. There are no further solutions where $q \equiv \{1, 2\} \pmod 3$ because then $q^2 \equiv 1 \pmod 3$, $2q^2 \equiv 2 \pmod 3$ such that $2q^2 + 1$ is divisible by 3 and not a prime s.
- $2^2p = s-1$: has many solutions, $p = 3, 7, 13, 37, 43, 67, 73, 79, 97, 127, 139, \ldots$, all of the form $p \not\equiv 2 \pmod 3$, [6, A023212].
- $pq^2 = 2 - 1$: Obviously not solvable.

C.3. **The Pair (2,4).** Pairs of (2,4) in the τ-sequence appear when a number of the form $n = p$ is followed by a number of the form $n + 1 = q^3$ or $n + 1 = rs$.

- $n = p = q^3 - 1$. Using the parity argument we only find $q = 2$, $p = 7$ here.
- $n = p = rs - 1$. Using the parity argument we find no solutions where $p = 2$ but many solutions where $p = 2s - 1$, [6, A005382].

C.4. **The Pair (3,6).** Pairs of (3,6) in the τ-sequence appear when a number of the form $n = p^2$ is followed by a number of the form $n+1 = s^5$ (the parity argument quickly reveals that there are none) or $n + 1 = s^2t$. The parity argument again shows that $p^2 = s^2t - 1$ have only solutions with $p^2 = 2s^2 - 1$. The candidates for s are [6, A001653], and here we are only interested in the sub-sequence where s is prime, [6, A056869], $s = 5, 29, 169, \ldots$ such that $p = 1, 7, 41, 8119, 47321 \ldots$ and reduce this further to cases where p is a prime larger than 3 (apparently [6, A086397]).

References

1. Shabnam Akhtari, *The diophantine equation $ax^4 - by^2 = 1$*, J. Reine Angew. Math. **2009** (2009), no. 630, 33–57. MR 2526785
2. Tom M. Apostol, *The resultant of the cyclotomic polynomials $f_m(ax)$ and $f_n(bx)$*, Math. Comp. **29** (1975), no. 129, 1–6. MR 0366801
3. Nathan Bliss, Ben Fulan, Stephen Lovett, and Jeff Sommars, *Strong divisibility, cyclotomic polynomials and iterated polynomials*, Am. Math. Monthly **120** (2013), no. 6, 519–536. MR 3063117
4. Jian Hua Chen and Paul Voutier, *Complete solution of the diophantine equation $x^2+1 = dy^4$ and a related family of quartic Thue equations*, J. Number Theory **62** (1997), no. 1, 71–99. MR 1430002
5. Peter Crescenzo, *A diophantine equation which arises in the theory of finite groups*, Adv. Math. **17** (1975), no. 1, 25–29. MR 0371812
6. O. E. I. S. Foundation Inc., *The On-Line Encyclopedia Of Integer Sequences*, (2019), https://oeis.org/. MR 3822822
7. Maximilian Hasler, *Pairs occurring only once among # of divisors*, SeqFan Mailing list, Jun 2009.
8. Wayne L. McDaniel, *On multiple prime divisors of cyclotomic polynomials*, Math. Comp. **28** (1974), no. 127, 847–850. MR 0387177
9. Tony D. Noe, priv. commun., 3Aug 2009.
10. ———, *Pairs occurring only once among # of divisors*, SeqFan Mailing list, Jun 2009.
11. Jon E. Schoenfield, *OeisWiki email*, priv. commun., Aug 2019.
12. Hagen von Eitzen, *Pairs occurring only once among # of divisors*, SeqFan Mailing list, Jun 2009.
13. Joshua Zelinsky, *Tau numbers: a partial proof of a conjecture and other results*, J. Integer Seq. **5** (2002), 02.2.8. MR 1951564

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