The Nature of The Φ(m) Function

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Abstract

In number theory, for the continuous product formula $\Pi(1-\frac{2}{p})$, the meaning is unclear.This paper gives the definition and nature of Φ(m) function, as well as the relationship between Φ(m) and Euler's totient function φ(m). In number theory, Euler function φ(m) is widely used, Φ(m) function if there are other applications, Some attempts are made in this paper.
Keywords: Φ(m) function;Euler function φ(m);co-prime
MSC: 11A41;11R04

Notation

P: a prime number.
m: a positive integer,
φ(m): Euler's totient function.
Φ(m): Φ function.
(a,b)=1: a and b are co-prime.
Π:the sign of the continued product.
～: denotes equivalence relation.

Definitions and Natures

Euler function φ(m)

In number theory, for the positive integer m, Euler function is the number of the positive integer q less than m and (q,m)=1.

$\varphi(m) = m\Pi(1-\frac{1}{p})$ (p is a prime factor of m)

Φ(m) function [1] [2] [3]

1. In number theory, for an even number m (m≥6), Function Φ(m) is the number of the positive odd q less than m, (q,m)=1 and (q-2k,m)=1 or (q+2k,m)=1 (k≥1).
Obviously, q < m, q-2k is not necessarily a positive integer or q+2k is not necessarily less than m.

If $k=2^n$, $\Phi(m)=\frac{m}{2}\Pi(1-\frac{2}{p})$ (p is the odd factor of m)

If the odd factor of k is different from that of m, $\Phi(m)=\frac{m}{2}\Pi(1-\frac{2}{p})$ (p is the odd factor of m)

If k shares an odd factor p with m, $\Phi(m)=\frac{m}{2}\Pi(1-\frac{2}{p})\Pi\frac{(p-1)}{(p-2)}$

($\Pi(1-\frac{2}{p})$,p is the odd factor of m,$\Pi\frac{(p-1)}{(p-2)}$,P is the common odd factor of k and m.)

If k is the same as the odd factor of m,Φ(m)=φ(m)
For different 2k, if the difference of 2k is a multiple of m, Φ(m) is equal, q is the same.

In particular,if $m=2^n$,Φ(m)= φ(m) =$\frac{m}{2}$

2. If m is odd (m≥3), Function Φ(m) is the number of the positive integer q less than m, (q,m)=1 and (q-k,m)=1 or (q+k,m)=1 (k≥1).

Obviously, q < m, q-k is not necessarily a positive integer or q+k is not necessarily less than m.

If $k=2^n$,$\Phi(m)=m\Pi(1-\frac{2}{p})$ (p is the odd factor of m)

If the odd factor of k is different from that of m,$\Phi(m)=m\Pi(1-\frac{2}{p})$ (p is the odd factor of m)

If k shares an odd factor p with m,$\Phi(m)=m\Pi(1-\frac{2}{p})\Pi\frac{(p-1)}{(p-2)}$

($\Pi(1-\frac{2}{p})$,p is the odd factor of m,$\Pi\frac{(p-1)}{(p-2)}$,p is the common odd factor of k and m)

If k is the same as the odd factor of m,Φ(m)=φ(m)
For different 2k, if the difference of 2k is a multiple of m, Φ(m) is equal, q is the same.
For an odd number m(m≥3),Φ(2m)=Φ(m)

3. In number theory, for an even number m (m≥6), Function Φ (m) is the number of the positive odd q less than m, (q,m)=1 and (m+2k-q,m)=1(k≥1).

If $k=2^n$,$\Phi(m)=\frac{m}{2}\Pi(1-\frac{2}{p})$ (p is the odd factor of m)

If the odd factor of k is different from that of m,$\Phi(m)=\frac{m}{2}\Pi(1-\frac{2}{p})$ (p is the odd factor of m)

If k shares an odd factor p with m,$\Phi(m)=\frac{m}{2}\Pi(1-\frac{2}{p})\Pi\frac{(p-1)}{(p-2)}$

($\Pi(1-\frac{2}{p})$,p is the odd factor of m,$\Pi\frac{(p-1)}{(p-2)}$,p is the common odd factor of k and m.)

If k is the same as the odd factor of m,Φ(m)=φ(m)
For different 2k, if the difference of 2k is a multiple of m, Φ(m) is equal, q is the same.

The proof of Φ(m) Function

The proof of Φ(m) function is similar to euler function Φ(m), slightly.

For example

1：m=30,2k=2 or 2k=4 or 2k=8 or 2k=16 or 2k=32 or $2k=2^n$,$\Phi(m)=\frac{m}{2}\Pi(1-\frac{2}{p})=3$,

q is not more than 30, (q,30)=1 and (q - 2k,30)=1(k≥1),

The number of odd pairs (q-2,q) is 3, (q-2,q):(-1 1),(11 13),(17 19).

The number of odd pairs (q-4,q) is 3, (q-4,q):(7 11),(13 17),(19 23).

The number of odd pairs (q-8,q) is 3, (q-8,q):(-7 1),(-1 7),(11 19).

The number of odd pairs (q-16,q) is 3, (q-16,q):(1 17),(7 23),(13 29).

The number of odd pairs (q-32,q) is 3, (q-32,q):(-31 1),(-19 13),(-13 19).

The number of odd pairs (q-64,q) is 3, (q-64,q):(53 11),(-47 17),(-41 23).

2k=2 and 2k=32,the difference between 2k is 30,q is the same,q：1,13,19.

2k=4 and 2k=64,the difference between 2k is 60,q is the same,q：11,17,23.

2：m=30,2k=6 or 2k=12 or 2k=24 or 2k=48 or $2k=3^N\times2^n$，$\Phi(m)=\frac{m}{2}\Pi(1-\frac{2}{p})\Pi\frac{(p-1)}{(p-2)}=6$,

q is not more than 30, (q,30)=1 and (q - 2k,30)=1(k≥1),

The number of odd pairs (q-6,q) is 6, (q-6,q):(1 7),(7 13),(13 19),(11 17),(17 23),(23 29).

The number of odd pairs (q-12,q) is 6, (q-12,q):(-11 1),(1 13),(7 19),(17 29),(-1 11),(11 23).

The number of odd pairs (q-24,q) is 6, (q-24,q):(-23 1),(-17 7),(-13 11),(-11 13),(-7 17),(-1 23).

The number of odd pairs (q-48,q) is 6, (q-48,q):(-47 1),(-41 7),(-37 11),(-31 17),(-29 19),(-19 29).

3：m=30,2k=30 or 2k=60 or 2k is a multiple of 30,$\Phi(m)=\frac{m}{2}\Pi(1-\frac{2}{p})\Pi\frac{(p-1)}{(p-2)}=\frac{m}{2}\Pi(1-\frac{1}{p})=\varphi(m)=8$,

q is not more than 30, (q,30)=1 and (q - 2k,30)=1(k≥1),

The number of odd pairs (q-30,q) is 8, (q-30,q):(-29 1),(-23 7),(-19 11),(-17 13),(-13 17),(-11 19),(-7 23),(-1 29).

The number of odd pairs (q-60,q) is 8, (q-60,q):(-59 1),(-53 7),(-49 11),(-47 13),(-43 17),(-41 19),(-37 23),(-31 29).

4：m=30,2k=14 or 2k=28 or 2k=56 or $2k=7^N\times2^n$,$\Phi(m)=\frac{m}{2}\Pi(1-\frac{2}{p})=3$,

q is not more than 30, (q,30)=1 and (q - 2k,30)=1(k≥1),

The number of odd pairs (q-14,q) is 3, (q-14,q): (-13 1),(-7 7),(-1 13).

The number of odd pairs (q-28,q) is 3, (q-28,q) :(1 29),(-11 17),(-17 11).

The number of odd pairs (q-56,q) is 3, (q-56,q): (-49 7),(-43 13),(-37 19).

5：m=30,2k=2 or 2k=4 or 2k=8 or 2k=16 or 2k=32 or $2k=2^n$,$\Phi(m)=\frac{m}{2}\Pi(1-\frac{2}{p})=3$,

q is not more than 30, (q,30)=1 and (30+2k-q,30)=1(k≥1),

(q,30)=1 and (30+2-q,30) =1, the number of q is 3,q: 1 ,13 ,19.

(q,30)=1 and (30+4-q,30) =1, the number of q is 3,q: 11 ,17 ,23.

(q,30)=1 and (30+8-q,30) =1, the number of q is 3,q: 1 , 7 ,19.

(q,30)=1 and (30+16-q,30) =1, the number of q is 3,q: 17 ,19 ,23.

(q,30)=1 and (30+32-q,30) =1, the number of q is 3,q: 1 ,13 ,19.

(q,30)=1 and (30+64-q,30) =1, the number of q is 3,q: 11 ,17 ,23.

2k=2 and 2k=32,the difference between 2k is 30,q is the same,q：1 ,13 ,19.

2k=4 and 2k=64 ,the difference between 2k is 60,q is the same,q：11 ,17 ,23.

6：m=30,2k=6 or 2k=12 or 2k=24 or 2k=48 or $2k=3^N\times2^n$,$\Phi(m)=\frac{m}{2}\Pi(1-\frac{2}{p})\Pi\frac{(p-1)}{(p-2)}=6$,

q is not more than 30, (q,30)=1 and (30+2k-q,30)=1(k≥1),

(q,30)=1 and (30+6-q,30) =1, the number of q is 6,q: 7 ,13 ,17 ,19 ,23 ,29.

(q,30)=1 and (30+12-q,30) =1, the number of q is 6,q: 1 ,11 ,13 ,19 ,23 ,29.

(q,30)=1 and (30+24-q,30) =1, the number of q is 6,q: 1 ,7 ,11 ,13 ,17 ,23.

(q,30)=1 and (30+48-q,30) =1, the number of q is 6,q: 1 ,7 ,11 ,17 ,19 ,29.

7：m=30,2k=30 or 2k=60 or 2k is a multiple of 30，Φ(m)= $\frac{m}{2}\Pi(1-\frac{2}{p})\Pi\frac{(p-1)}{(p-2)}=\frac{m}{2}\Pi(1-\frac{1}{p})$=φ(m)=8,

q is not more than 30, (q,30)=1 and (30+2k-q,30)=1(k≥1),

(q,30)=1 and (30+30-q,30) =1, the number of q is 8,q: 1 ,7 ,11 ,13 ,17 ,19 ,23 ,29.

(q,30)=1 and (30+60-q,30) =1, the number of q is 8,q: 1 ,7 ,11 ,13 ,17 ,19 ,23 ,29.

8：m=30,2k=14 or 2k=28 or 2k=56 or 2k=$7^N\times2^n$，Φ(m)= $\frac{m}{2}\Pi(1-\frac{2}{p})$ =3,

q is not more than 30, (q,30)=1 and (30+2k-q,30)=1(k≥1),

(q,30)=1 and (30+14-q,30) =1, the number of q is 3,q: 1 ,7 ,13.

(q,30)=1 and (30+28-q,30) =1, the number of q is 3,q: 11 ,17 ,29.

(q,30)=1 and (30+56-q,30) =1, the number of q is 3,q: 7 ,13 ,19.

Application of Φ(m) function: The generalized Goldbach's conjecture

The Goldbach's conjecture is that ∀n∈N*,∃ p, q∈P, such that 2n=p+q.

The generalized Goldbach's conjecture[4]:

Given m,a∈N ,(a,m)=1,for every $\frac{x}{2}$≡a (mod m) large enough (x is even integer), ∃ p,q∈P,such that x=p+q and p≡q≡a (mod m).

Let G(a, m, x) is the number of representatives a large even integer x as a sum of two primes p and q, Among them (a,m)=1, p≡q≡a (mod m).
Let G(x) is the number of representatives a large even integer x as a sum of two primes.
Φ(m) is the number of categories of G(x) for mod m.

1. If m=2^n, G(a, m, x) $\sim\frac{1}{\Phi(m)}$G(x) $\sim\frac{1}{\varphi(m)}$G(x)
2. If m is even, G(a, m, x) $\sim\frac{1}{\Phi(m)}$G(x), Φ(m)=$\frac{m}{2}\Pi(1-\frac{2}{p})$ (p is the odd factor of m)
3. If m is odd, G(a, m, x) $\sim\frac{1}{\Phi(m)}$G(x), Φ(m)=m$\Pi(1-\frac{2}{p})$ (p is the odd factor of m)

Where G(x) $\sim 2C\cdot\Pi\frac{(p-1)}{(p-2)}\cdot\frac{x}{\ln^2 x}$ (p is the odd factor of x. C =$\Pi(1-\frac{1}{(p-1)^2})$, p is an odd prime number.)
It can be seen,while m=2;m=3 or m=6 aiternative,

$G(x) = G(x, a, m) \sim G(x)$

this formula prompts the expression of Goldbach conjecture.

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