Decomposition of the Riemann and Riccci curvatures

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Abstract

We propose a decomposition of the Riemann curvature due to the Bianchi identities. We deduce a decomposition of the Ricci curvature.

1 The Riemann and Ricci curvatures

In the case of a riemannian manifold (M, g), the Riemann curvature R can be defined with help of the Levi-Civita connection. The Ricci curvature Ric is then computed as a contraction of the Riemann curvature with respect of the metric. The Einstein equations are [Be]:

$$Ric = \lambda g$$

2 Decomposition of the Riemann curvature

The Riemann curvature R possesses symmetries like the Bianchi identities. The first Bianchi identity seems to imply that:

$$g(R(x,y)z,t) = \sum_i [g(S_i(x),z)g(S_i'(y),t) + g(S_i(y),t)g(S_i'(x),z) -$$

$$-g(S_i(x),t)g(S_i'(y),z) - g(S_i(y),z)g(S_i'(x),t)]$$

with S_i, S_i' symmetric endomorphisms of the tangent bundle $(S_i)^* = S_i, (S_i')^* = S_i'$. The second identity of Bianchi is implied if the endomorphisms S_i, S_i' are parallel with respect of the Levi-Civita connection $\nabla S_i = \nabla S_i' = 0$.

3 Decomposition of the Ricci curvature

The Ricci curvature is decomposed following the decomposition of the Riemann curvature:

$$Ric = \sum_i [tr(S_i')S_i + tr(S_i)S_i' - S_iS_i' - S_i'S_i]$$

In the case of an Einstein manifold, it can be supposed that $\nabla S_i = \nabla S'_i = 0$.

References

[Be] A.Besse, "Einstein Manifolds", Springer Verlag, Berlin, 1987.