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# Division by Zero Fallacies using Transmathematics

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Three fallacies that illustrate why division by zero is frequently considered undefined operation are examined. The example fallacies consider the unique case of zero divided by zero. Two examples are fallacies of equality, and the other is an example of ambiguity in the solution for an equation. These fallacies are examined using the transmathematic number nullity $\Phi$. By utilizing nullity, division by zero is no longer an undefined or indeterminate operation, but a consistent, well-defined operation in arithmetic.

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# 1. Introduction

In mathematics, division by zero is formally an undefined or indeterminate operation. Three examples demonstrate why division by zero is an undefined operation. These examples are:

1. Ambiguity of value
2. Fallacy of equal numbers
3. Fallacy of equal proportions

## 1.1 Ambiguity of Value

The explanation for the ambiguity [Bit 1995] of division by zero is:

> "On the other hand, if we divide 0 by 0, we look for a number r such that 0 × r = 0. But 0 × r = 0 for any number r. Thus it appears 0 ÷ 0 could be any number we choose. Getting any answer we want when we divide 0 by 0 would be very confusing. Thus we agree that division by zero is undefined."

For the expression:

$$0 \times r = 0$$

Hence the calculation of *r* has the possibility of any number *n* such that:

$$-\infty < -n < -n + 1 < \ldots < -1 < 0 < +1 < \ldots < n - 1 < n < +\infty$$

Thus there are infinitely many solutions for the equation, and this ambiguity contradicts the certainty of one resulting number for a solution to satisfy the equation.

## 1.2 Fallacy of Equal Numbers

The fallacy of equal numbers [Nor 1944a] is explained with:

> Suppose that:
>
> $$a = b + c$$
>
> where *a*, *b*, and *c* are positive numbers. Then inasmuch as *a* is equal to *b* plus some number, *a* is greater than *b*. Multiply both sides by *a* - *b*. Then
>
> $$a^2 - ab = ab + ac - b^2 - bc.$$
>
> Subtract *ac* from both sides:

$$a^2 - ab - ac = ab + ac - b^2 - bc - ac.$$

Simplify the equation:

$$a^2 - ab - ac = ab - b^2 - bc.$$

Factor:

$$a(a - b - c) = b(a - b - c)$$

Divide both sides by *a - b - c*. Then

$$a = b$$

Thus *a*, which was originally assumed to be greater than *b*, has been show to be equal to *b*.

Using actual numbers, the fallacy is rewritten for a more concrete illustration.

Suppose that:

$$3 = 2 + 1$$

where *a = 3*, *b = 2*, and *c = 1* are positive numbers. Then inasmuch as *a* is equal to *b* plus some number, *3* is greater than *1*. Multiply both sides by *3 - 2*. Then

$$3(3 - 2) = (3 - 2)(2 + 1)$$

Expand the equation into multiplicative terms:

$$3 \cdot 3 - 3 \cdot 2 = 3 \cdot 2 + 3 \cdot 1 - 2 \cdot 2 - 1 \cdot 2.$$

Subtract *3·1* from both sides:

$$3 \cdot 3 - 3 \cdot 2 - 3 \cdot 1 = 3 \cdot 2 + 3 \cdot 1 - 2 \cdot 2 - 1 \cdot 2 - 3 \cdot 1.$$

Simplify the equations:

$$3 \cdot 3 - 3 \cdot 2 - 3 \cdot 1 = 3 \cdot 2 - 2 \cdot 2 - 1 \cdot 2.$$

Factor:

$$3(3 - 2 - 1) = 2(3 - 2 - 1)$$

Divide both sides by *3 - 2 - 1*. Then

$$3 = 2$$

Thus *3*, which was originally assumed to be greater than *2*, has been shown to be equal to *2*.

The result is a contradiction, where three is equal to two.

### 1.3 Fallacy of Equal Proportions

The fallacy of equal proportions [Nor 1944b] is explained with:

Given

$$\frac{a}{b} = \frac{a}{c},$$

Multiply both sides by *bc*. Then

$$ac = ab.$$

Divide both sides by *a*:

$$c = b.$$

But if $a = 0$, then this last step involves division by zero.

Using actual numbers, the fallacy is rewritten for a more concrete illustration, where a = 0, b = 1, and c = 2.

Given

$$\frac{0}{1} = \frac{0}{2}$$

Multiply both sides by *1·2*. Then

$$(1 \cdot 2) \cdot \frac{0}{1} = \frac{0}{2} \cdot (1 \cdot 2)$$

Simplifying the expression:

$$2 \cdot 0 = 1 \cdot 0$$

By the commutative property:

$$0 \cdot 2 = 0 \cdot 1$$

Divide both sides by *0*:

$$\frac{0}{0} \cdot 2 = \frac{0}{0} \cdot 1$$

Simplifying the expression:

$$2 = 1$$

The result is a contradiction where the number two is equal to one.

## 2. Application

Both examples of ambiguity and a contradiction for division by zero can be re-evaluated using the transmathematic number nullity represented by the Greek letter Φ phi.

### 2.1 Nullity

The standard, conventional thinking of mathematics is that nullity is undefined or indeterminate. Reid [Rei 2006] writes, "The expression 0/0 is neither mathematically meaningful nor meaningless. It is indeterminate." Nullity is formally defined with the expression:

$$\Phi = \frac{0}{0}$$

Nullity is a determinate number resulting from the specific case of zero divided by zero.

### 2.2 Ambiguity of Value

Utilizing nullity and re-evaluating the example:

$$0 \times r = 0$$

Divide by zero:

$$\frac{0}{0} \times r = \frac{0}{0}$$

Substitute the definition of nullity:

$$\Phi \times r = \Phi$$

Multiply by nullity:

$$\Phi = \Phi$$

The result is a logically true expression.

### 2.3 Fallacy of Equal Numbers

Utilizing nullity and re-evaluating the example:

Suppose that:

$$a = b + c$$

where *a*, *b*, and *c* are positive numbers. Then inasmuch as *a* is equal to *b* plus some number, *a* is greater than *b*.

Multiply both sides by *a* - *b*. Then

$$a^2 - ab = ab + ac - b^2 - bc.$$

Subtract *ac* from both sides:

$$a^2 - ab - ac = ab - b^2 - bc.$$

Factor:

$$a(a - b - c) = b(a - b - c)$$

Divide both sides by *a* - *b* - *c*. Then

$$a \cdot \frac{(a - b - c)}{(a - b - c)} = b \cdot \frac{(a - b - c)}{(a - b - c)}$$

Substituting for the expression *a* - *b* - *c* as zero:

$$a \cdot \frac{0}{0} = b \cdot \frac{0}{0}$$

Substituting into the expression with nullity:

$$a \cdot \Phi = b \cdot \Phi$$

Multiply by nullity:

$$\Phi = \Phi$$

The result is a logically true expression.

### 2.4 Fallacy of Equal Proportions

Utilizing nullity and re-evaluating the example:

Given

$$\frac{a}{b} = \frac{a}{c},$$

Multiply both sides by *bc*. Then

$$ac = ab.$$

Divide both sides by *a*:

$$\frac{a}{a} \cdot c = \frac{a}{a} \cdot b$$

Substitute $a = 0$ into the expression:

$$\frac{0}{0} \cdot c = \frac{0}{0} \cdot b$$

Substitute nullity:

$$\Phi \cdot c = \Phi \cdot b$$

Multiply by nullity:

$$\Phi = \Phi$$

The result is a logically true expression.

## 3. Discussion

The standard definitions of division by zero as an undefined operation are by fiat. The definition of division by zero is inconsistent with division by other numbers, and the solution is that division by zero is an undefined operation. Yet the overall problem of contradiction and ambiguity remains unresolved.

Yet the definition of division by zero in transmathematics has consistent results in both examples of division by zero. For each example, the multiplication of nullity with another term is nullity.

In transmathematics, this is formally the axiom of multiplicative nullity. This axiom states multiplication of a real or complex number *a* with nullity [And 2009a] is:

$$a \times \Phi = \Phi$$

The axiom of multiplicative nullity avoids both ambiguity and the contradiction in the operation of division by zero. This eliminates the special case where division by zero is undefined, thus division is defined for all possible numbers.

The ambiguity and contradiction in the operation of division by zero produces confusion in both students and teachers of mathematics. Crespo and Nicol [Cre 2006a] explain, "...teachers would be challenged to think of ways of explaining to young students why the answer to dividing by 0 is not 0 using mathematical arguments rather than appeal to authority."

The conclusion [Cres 2006b] is "...understanding of 0 and division by 0 were founded more on rule-based and flawed reasoning than on well-reasoned mathematical explanations and that they [teachers] lacked the experience and inclination to understand or appreciate different ideas and approaches to this topic."

Mathematics is logical, yet in teaching and understanding division by zero, there is flawed thinking of appeal to authority without any mathematical explanation. Division by zero in mathematics is simply forbidden.

## 4. Conclusion

The formalism of conventional mathematics is that division by zero is undefined or indeterminate. The examples using the definition of an undefined operation for division by zero result in ambiguity and contradiction. Reid [Rei 2006b] explains division by zero, "Mathematicians say, more technically, that it is indeterminate, and it took them centuries to realize that it is. Only then had they finally mastered zero the number." Yet with the transmathematic number nullity $\Phi$ produces a consistent mathematical result without simply defining the operation as indeterminate.

The transmathematical number nullity is used in the examples that consider the special and unique case of division of zero by zero. Yet there are two other transmathematical numbers, positive infinite $+\infty$ and negative infinity $-\infty$ that are the result of any positive or negative number divided by zero, respectively. All three transmathematical numbers are necessary in general for a consistent, uniform arithmetic that avoids contradictions.

## 5. References

[And 2009a] Anderson, James Transmathematics, `www.bookofparagon.com/Books/Transmathematics/Transmathematics.pdf`, 2009, p. 15.

[And 2009b] ibid, p. 25.

[Bit 1995] Bittinger and Keedy. Basic Mathematics, 7th edition, Addison-Wesley Publishing Company, Reading, Massachusetts, 1995, p. 533.

[Cre 2006a] Crespo, Sandra and Nicol, Cynthia. "Challenging Preservice Teachers' Mathematical Understanding: The Case of Division by Zero," School, Science, and Mathematics, Volume 106, Number 2, 2006, p. 84.

[Cre 2006b] ibid, p. 94.

[Nor 1944a] Northrup, Eugene P. Riddles in Mathematics a Book of Paradoxes, D. Van Nostrand Company, Inc., New York, New York, 1944, p. 84.

[Nor 1944b] ibid, p. 86.

[Rei 2006a] Reid, Constance. From Zero to Infinity What Makes Numbers Interesting, A. K. Peters, Ltd. Wellesley, Massachusetts, 2006, p. 8.

[Rei 2006b] ibid, p. 9.