Using Decimals to Prove $\zeta(n \geq 2)$ is Irrational

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Abstract

With a strange and ironic twist an open number theory problem, show $\zeta(n)$ is irrational for natural numbers greater than or equal to 2, is solved with the easiest of number theory concepts: the rules of representing fractions with decimals.

Introduction

If you are like me, someone who likes irrationality proofs, you probably delighted in experiencing how repeated divisions gives

$$\frac{1}{7} = .\overline{142857},$$

a repeating decimal. Also

$$\frac{1}{6} = .1\overline{6}$$

forms a mixed decimal. Hardy [2] reviews why numbers relatively prime to the base used give repeating decimals, like $1/7$, and fractions with denominators that share some but not all prime factors with the base used form mixed decimals. We see the latter with $1/6$. This shares a factor of 2 in its denominator with a 2 factor in the base used 10. Just to exhaust the possibilities, a fraction like $1/4$ shares all prime factors of its denominator with the base 10. It has a finite decimal representation, but there's more to the story.

We could write $1/4 = .25$ with $.24\overline{9}$ or $.(24)(99)$ in base 100 where the parentheses indicate a single symbol designating the number inside. With a little thought all fractions less than 1 can be written with a single decimal in some base and also as a single decimal with a trail of $(b-1)$, where b is the base. A natural name for this is the $.\overline{9}$ phenomenon.

This phenomenon introduces an annoying ambiguity when using decimal numbers. You can't really say that there is a unique decimal representation for finite decimal representations of a fraction. You can say there is a unique decimal representation for fractions that are repeating and mixed because their denominators are relatively prime to the base or, like $1/6$ base 10, share some but not all prime factors with the base. These are uniquely expressed with a decimal system. The other type of number, the irrationals, are also uniquely expressed or represented by a decimal base – they never repeat and require an infinite number of decimals.

We claim we can use this here-to-fore annoyance of a decimal system to good avail. We can use it to give a proof of the irrationality of $\zeta(n \geq 2)$. We will reduce these values by 1. This doesn't effect, of course, their status as irrational or rational numbers. Here's the notation we use:

$$z_n = \zeta(2) - 1 = \sum_{k=1}^{\infty} \frac{1}{k^n} - 1 \text{ and } s_k^n = \sum_{j=2}^{k} \frac{1}{j^n}.$$

There are proofs that z_2 and z_3 are irrational see [1]. These are currently the only two cases that are known to be irrational – at least without debate. We use a few non-debatable results from the debated proof given by [3] for the general result in the title of this article.

Assumption

We assume from a previous article the following.

Definition 1.

$$D_{j^n} = \{0, 1/j^n, \ldots, (j^n - 1)/j^n\} = \{0, .1, \ldots, .(j^n - 1)\} \textit{ base } j^n$$

Definition 2.

$$\bigcup_{j=2}^{k} D_{j^n} = \Xi_k^n$$

Corollary 1.

$$s_k^n \notin \Xi_k^n$$

The corollary says that partial sums are not expressible with one finite decimal when the bases are denominators of the terms used in the partial. We claim this immediately implies the irrationality of z_n.

Reasoning

If z_n was rational then for some k

$$z_n \in \Xi_k^n.$$

This forces z_n to be a repeating, mixed, or finite decimal in every base. But, when this rational is expressed in reduced form, it can only be a single decimal with the $.\overline{9}$ phenomenon in one base.

Upon reflection any infinite series that converges to a rational will never have any partials that are equal to its convergence point; when the partials are converted to the base given by the convergence point's denominator, the $.\overline{9}$ phenomenon will have to occur. The number will be of the form $.(a-1)\overline{(b-1)}$ where a/b is the convergence point.

As s_k^n are not equal to any single decimal from Ξ_k^n, when they are expressed in a base that is a denominator of a fraction in Ξ_k^n they will be mixed or repeating, but never finite. If they are never finite the $.\overline{9}$ phenomenon can't occur in the decimal's digits. Any base k^n will be reached in Ξ_k^n, so no base k^n allows for this form of expression of a rational. But base k^n allows for the expression of any rational as a finite decimal:

$$\frac{a}{b} = \frac{ab^{n-1}}{b^n}.$$

The number must be irrational.

Compare z_n's situation with that of any single decimal expressed with the $.\overline{9}$ phenomenon and you will see it! The partials must remain with the same prime factors as previously used to express previous partials: $.4\overline{9}$ gives an example. No matter the terms of the series, if it is converging to a rational, the partials when converted to the base given by the denominator (or a power of such a denominator) will have to converge to all 9's, – the $.\overline{9}$ phenomenon must occur.

References

[1] P. Eymard and J.-P. Lafon, *The Number π*, American Mathematical Society, Providence, RI, 2004.

[2] G. H. Hardy, E. M. Wright, R. Heath-Brown, J. Silverman, and A. Wiles, *An Introduction to the Theory of Numbers*, 6th ed., Oxford University Press, London, 2008.

[3] T. W. Jones, A Simple Proof of the Irrationality of $\zeta(n \geq 2)$, available at http://vixra.org/abs/1801.0140.