QUESTIONS ON COLORING

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Abstract
We present some thoughts about a coloring of an arbitrary map

1 Introduction

Since the paper [1] of two US Americans Kenneth Appel and Wolfgang Haken it is well-known that the 'Four Color Theorem' is true. It was proven 1976 by the aid of computers, which have to consider nearly 2000 subcases. The four mathematicians Neil Robertson, Daniel P. Sanders, Paul Seymour and Robin Thomas in the year 1996 presented a new proof [2], where they reduced the number of subcases to about 600, but still it lacks an elementary proof without the aid of a computer. The following paper arose by the futile attempts to prove the Four Color Theorem.

Let Map be a map of N disjunct contries, $\mathsf{Map} = \{S_1, S_2, S_3, \ldots, S_{N-1}, S_N\}$. We assume that each country S_k is homeomorphic to the open unit circle $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$, and that the *border* of a country is homeomorphic to the unit circle $\{x^2 + y^2 = 1\}$ for $x, y \in \mathbb{R}$. Two countries are *neighboring* if and only if they have some common border homeomorphic to $\{x \in \mathbb{R} \mid 0 < x < 1\}$. If A and B are neighboring countries, then A is called a *neighbor* of B and B is a neighbor of A. We call (A, B) a *neighboring pair*. Two neighboring countries are also topological neighboring. A *coloring* of the map means that neighboring countries get different colors. In this case we call the map *colorable*. Note that two countries which meet only in a finite set of points are not neighboring.

2 Propositions

The following proposition is trivial, but we think it is important, and we read it nowhere.

Proposition 1. *Let* Map *be any map. This map is colorable with four colors if and only if* Map *is the union of* Map_1 *and* Map_2*, i.e.* $\mathsf{Map} = \mathsf{Map}_1 \cup \mathsf{Map}_2$*, and both* Map_1 *and* Map_2 *are colorable with just two colors.*

Remark 1. *We can choose disjunct sets* Map_1 *and* Map_2*.*

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We call a *half-plane* every subset of $\mathbb{R}^2$ which is homeomorphic to the upper half plane $\{(x,y) \mid x,y \in \mathbb{R},\ y > 0\}$, which is homeomorphic to $\mathbb{R}^2$. A subset of $\mathbb{R}^2$ which is homeomorphic to the vertical axis $\{(0,y) \mid y \in \mathbb{R}\}$ and which splits $\mathbb{R}^2$ into two disjunct half-planes is called a *line*. The set of all lines is called Lines.

A subset of $\mathbb{R}^2$ which is homeomorphic to the unit circle $\{(x,y) \mid x^2+y^2 = 1\}$ for $x,y \in \mathbb{R}$ is called a *circle*. The set of all circles is called $\mathsf{Circles}$. A circle seperates $\mathbb{R}^2$ into two disjunct parts, due to the Jordan curve theorem.
The following proposition is well-known. Please see [3], p. 166.

Proposition 2. *Assume a map where the borders of all countries are generated by a finite number of straight lines. Then two colors suffice for a coloring.*

Proposition 3. *Assume a map where the borders of all countries are generated by a finite number of elements from* Lines. *Additionally we assume that two arbitrary elements of the generating lines have only a finite set of common points. Two colors suffice for a coloring.*

Proof. Assumption of the induction: Let $\{L_1, L_2, \ldots, L_{R-1}, L_R\}$ be a set of lines. The generated map can be colored with two colors.

Beginning of the induction: We start with one color for the entire space $\mathbb{R}^2$. L_1 splits $\mathbb{R}^2$ in two subsets. We color one part with the first color, the other part with the second color. On one side of L_2 we change the colors, on the other side we keep them. As a result we get a coloring of the whole $\mathbb{R}^2$.

The induction step of k to $k+1$: Let $\{L_1, L_2, \ldots, L_{k-1}, L_k\}$ already be drawn. The $\mathbb{R}^2$ is parted in regions and it is colored with two colors. The line L_{k+1} splits the $\mathbb{R}^2$ in two half-planes. On an arbitrary side we leave the colors, on the other side we change all colors. As a result we get a coloring of the space $\mathbb{R}^2$. Note that we have only finite many intersection points between L_i and L_{k+1}, $1 \le i \le k$. Hence two regions with a same color only meet in a finite set of points. $\square$

Proposition 4. *Assume a map where the borders of all countries are generated by a finite number of elements from* Lines *and from* $\mathsf{Circles}$. *We assume that two elements of the generating curves have only a finite set of common points. It follows that two colors suffice.*

Proof. We use a corresponding proof as before in Proposition 3. $\square$

We extend the above considerations to the usual space with three dimensions. We consider the Euclidean space $\mathbb{R}^3$. Recall that for two points $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$ the distance is

$$dist(\vec{a}, \vec{b}) := \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}.$$

We take N subsets of $\mathbb{R}^3$. $N \in \mathbb{N}$, $\mathsf{Map} := \{S_1, S_2, S_3, \ldots, S_{N-1}, S_N\}$. An element $S_k \in \mathsf{Map}$ is called a *country*. We assume that the countries are disjunct, and we assume that each country $S_k, 1 \le k \le N$, is homeomorphic to the open unit ball of $\mathbb{R}^3$, i.e. to

$$\{(a,b,c) \in \mathbb{R}^3 \mid a^2 + b^2 + c^2 < 1\}.$$

We define the *border* of a country is the closure of that country, i.e. it is homeomorphic to the unit sphere, i.e. to

$$\{(a,b,c) \in \mathbb{R}^3 \mid a^2 + b^2 + c^2 = 1\}.$$

We define the space *disk*. A disk is any space homeomorphic to $\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le \gamma\}$ for a $\gamma > 0$.

We say that two countries $S \ne T$ in the $\mathbb{R}^3$ are *neighboring* if and only if the border of S and the border of T contain a common disk.

It is well-known that there is no corresponding theorem to the 'Four Color Theorem' in three dimensions. The number would be infinite. This is proven by the example of a number of lengthy cuboids on the bottom, and on the top the same number of cuboids shifted by 90 degrees. Every cuboid on the bottom touches every cuboid on the top.

It lacks the description of those maps which need only two or three or four or 'many' or a finite number of colors. Related are the questions if we restrict the shapes of the contries.

We call a *half-space* every subset of $\mathbb{R}^3$ which is homeomorphic to the upper half-space $\{(x, y, z) \mid \ x, y, z \in \mathbb{R}, z > 0\}$.

A subset of $\mathbb{R}^3$ which is homeomorphic to the horizontal plane $\{(x, y, 0) \mid x, y \in \mathbb{R}\}$ and which splits $\mathbb{R}^3$ into two disjunct half-spaces is called a *plane*. The set of all planes is called Planes.

A subset of $\mathbb{R}^3$ which is homeomorphic to the unit sphere $\{(x, y, z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$ for $x, y, z \in \mathbb{R}$ is called a *sphere*. The set of all spheres is called $\mathsf{Spheres}$. A sphere seperates $\mathbb{R}^3$ into two disjunct parts, due to the Jordan-Brouwer seperation theorem.

Proposition 5. *Assume a map in $\mathbb{R}^3$ where the borders of all countries are generated by a finite number of elements from* Planes *and from* $\mathsf{Spheres}$. *We assume that two of the generating elements contain no common disk. (As a consequence we get that all generating elements are pairwise different.)*
Then two colors suffice for a coloring.

Proof. Essentially it is the same proof as before in Proposition 3. □

Now we regard the contrary of the above proposition.

Remark 2. *Assume a map where the borders of all countries are generated by some elements $\{L_1, L_2, \ldots, L_{R-1}, L_R\}$ in $\mathbb{R}^3$. We assume that two elements* A *and* B *are elements of $\{L_1, L_2, \ldots, L_{R-1}, L_R\}$, and* A *and* B *contain a common disk. Then the above proof of Proposition 5 does not work. As a consequence the map may not be colorable with only two colors.*

We add a trivial proposition.

Proposition 6. *Take an arbitrary map. Assume that every country has at most n neighbors, $n \in \{0\} \cup \mathbb{N}$. Then $n + 1$ colors suffice for a coloring.*

By the work of Appel and Haken four colors are always sufficient to color an arbitrary map in $\mathbb{R}^2$. This produces further questions. It lacks a characterization out of graph theory of those maps which are colorable by only three colors. Please see [3], p. 167 and Proposition 7. Related are the questions if we restrict the shapes of the countries. For instance we can constrict to squares or triangles or squares and triangles. This opens a lot of possibilities.

3 Questions

Now we mention other possibilities to color a map.

Recall that for two points $\vec{a} = (a_1, a_2, a_3, \ldots, a_{n-1}, a_n)$, $\vec{b} = (b_1, b_2, b_3, \ldots, b_{n-1}, b_n) \in \mathbb{R}^n$ the usual distance is

$$dist(\vec{a}, \vec{b}) := \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2 + \ldots + (a_n - b_n)^2}. \tag{1}$$

Let *subset* be any subset of a metric space (X, d). A function $f : subset \to \mathbb{R}^n$ is called a *distance preserving function* if and only if it holds $d(s, t) = dist\,(f(s), f(t))$ for all $s, t \in subset$. We define finite subsets of natural numbers.

Definition 1. *Let for $n \in \mathbb{N}$ the set $\mathbb{N}_n := \{1, 2, 3, \ldots, n-2, n-1, n\} \subset \mathbb{N}$ be the set of the first n natural numbers.*

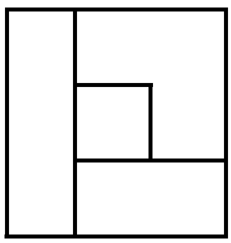
Let n be an arbitrary natural number. We determine that if two elements of $f : \mathbb{N}_n \to \mathsf{Map}$ from the image are in different countries, these countries are colored with different colors. We ask how many colors we need to color all maps, where f runs through the set of all distance preserving functions. We call this number $\heartsuit_n$.

Lemma 1. *For $n = 2$ it includes the Four Color Theorem.*

Proof. Let be $n = 2$. Let Map be an arbitrary map. The map Map has a finite number of countries. We enlarge every country A of Map into a country which we call $\overline{\mathsf{A}}$ and we get a new map called MAP such that the pair (A, B) is a neighboring pair of countries in Map if and only if the pair $(\overline{\mathsf{A}}, \overline{\mathsf{B}})$ is a neighboring pair in MAP. Further we make the enlargements such that for three countries A, B, C of Map with neighboring pairs (A, B) and (B, C) but A and C are not neighboring, then it holds for all $a \in \overline{\mathsf{A}}$ and $c \in \overline{\mathsf{C}}$ that $dist(a, c) > 1$. Further we make the enlargements such that for a neighboring pair (X, Y) in Map there is an element $x \in \overline{\mathsf{X}}$ and $y \in \overline{\mathsf{Y}}$ such that $dist(x, y) = 1$. Now the problem to color this map MAP by the set $\mathbb{N}_2$ is the same as the ordinary coloring problem, and MAP can be colored if and only if Map can be colored with the same number of colors. □

As a conclusion we get $\heartsuit_2 \geq 4$.

See the famous example of a map with only four countries. Four colors are necessary.



On the left hand side we show the example

of a map with four countries.

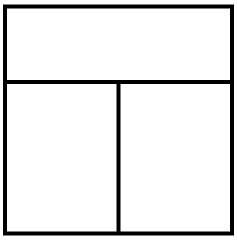
It requires four colors.

We call a *triple* three countries such that each country is a neighbor of the two others. We call a *way* a finite ordered set $(L_1, L_2, L_3, \ldots, L_{K-1}, L_K)$ of countries such that L_{i-1} and L_i are neighboring for $2 \leq i \leq K$. The *beginning* of the way is L_1, the *end* is L_K. The number K is called the *length* of the way. We name an *odd circle* a way $(L_1, L_2, L_3, \ldots, L_{K-1}, L_K)$ of countries such that additionally (L_1, L_K) is also a neighboring pair. Furthermore we demand that the set $\{L_1, L_2, L_3, \ldots, L_{K-1}, L_K\}$ has an odd number larger than four of elements.

It follows an important proposition.

Proposition 7. *Any map is colorable with two colors if and only if it contains neither a triple nor an odd circle.*

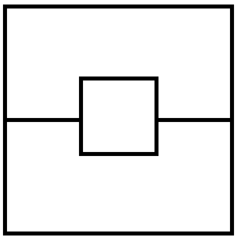
Proof. This proposition is known from graph theory. A proof beyond graph theory is yielded by [4] in the internet. $\square$



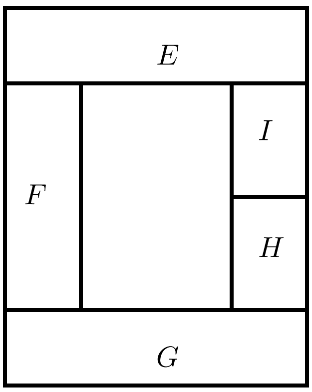
We show an example of a triple.

A triple is not colorable with two colors.

We need three colors.



We show another example of a triple.



On the left hand side is an odd circle

with countries $\{E, F, G, H, I\}$.

Note that it is not colorable with two colors.

From Map we make a metric space. Let A, B be different contries in Map. The distance of A and B is defined by the natural number $J-1$, where J is the minimal length of a way from A to B, i.e. there is a way that connects A and B, i.e. there is a set $\{Q_1, Q_2, \ldots, Q_{J-1}, Q_J\}$ with the beginning $A = Q_1$ and the end $B = Q_J$ and (Q_i, Q_{i+1}) is a neighboring pair for $1 \leq i \leq J-1$, and there is no shorter way from A to B. We define distance$(X, X) := 0$ for $X \in$ Map, and we have distance$(A, B) = 1$ if and only if (A, B) is a neighboring pair.

We introduce an infinite set of coloring constants. We generalize the ordinary neighborhood relation.
Let S be any subset of the natural numbers $= \{1, 2, 3, \ldots\}$, and let Map be any map. We define that two countries $A, B \in$ Map are *neighboring related to* S if and only if both countries have a distance in S. Note that for S $= \{1\}$ we get the ordinary neighborhood relation. We ask how many colors we need to color each map, where two countries which are neighboring related to S get different colors. We will call this number $♠_{\mathsf{S}}$. Note that for S $= \{1\}$ this is the question of the Four Color Theorem. By this theorem we get $♠_{\{1\}} = 4$.

We get corresponding questions if we restrict the shapes of the countries, for instance we take rectangles or triangles or something else.

We extend the above concept into higher dimensions. We consider the spaces $\mathbb{R}^n$, which we will call n-dimensional space, for $n > 1$. We take N subsets of $\mathbb{R}^n$. $N \in \mathbb{N}$, Map $:= \{S_1, S_2, S_3, \ldots, S_{N-1}, S_N\}$. An element $S_k \in$ Map is called a *country*. We assume that the countries are disjunct, and we assume that each country is homeomorphic to the open unit ball of $\mathbb{R}^n$, i.e. to

$$\{(a_1, a_2, a_3, \ldots, a_{n-1}, a_n) \in \mathbb{R}^n \mid a_1^2 + a_2^2 + a_3^2 + \ldots + a_{n-1}^2 + a_n^2 < 1\}.$$

We define that the border of each country is the closure of the country. It follows that the border is homeomorphic to

$$\{(a_1, a_2, a_3, \ldots, a_{n-1}, a_n) \in \mathbb{R}^n \mid a_1^2 + a_2^2 + a_3^2 + \ldots + a_{n-1}^2 + a_n^2 = 1\}.$$

Definition 2. *Any finite subset of $\mathbb{R}^n$ is called a* set of vertices.

Remark 3. *Note that the* set of vertices *is a generalization of the sets $\mathbb{N}_n$, since $\mathbb{N}_n \subset \mathbb{N} \subset \mathbb{R} \subset \mathbb{R}^k$.*

Let f be a function with any set of vertices of the $\mathbb{R}^n$ as the domain and the codomain $\mathbb{R}^n$. We define that if two elements of the image are in different countries these countries are colored with different colors. We ask how many colors we need to color all maps in the $\mathbb{R}^n$, where f runs through all distance preserving functions.

Proposition 8. *Let the set of vertices be consist of K points. Then the number of the needed colors is at least K. It may be infinite.*

Proof. We take a map such that each element of the image points of the set of vertices is in a different country. □

Proposition 9. *Let the set of vertices be consist of more than one point. Let be $n \geq 2$. The above question is an extension of the 'Four Color Theorem'. The sought number is at least four.*

Proof. If it is necessary we make an enlargement of a country as we described it in the proof of Lemma 1. □

Remark 4. *In the definition Definition 2 we have in our minds some geometrical shape. The set of vertices could be the vertices of a polygon, or a star, or something else. Only the image points of the set of vertices are important.*

We can change also the maps. Recall the town Kaliningrad (the former Königsberg in Prussia). It lies in an exclave of Russia, this means that one has to transit a foreign territory to reach the town. Russia consists of two parts. We formulate a question. We ask how many colors we need to color an arbitrary map, where one of the countries consists of two parts. Each part is homeomorphic to a square. This concept can be generalized. Let Map be a map of K_1 countries. Up to K_2 countries consist of two parts, up to K_3 countries consist of three parts ... up to K_L countries consist of L parts. Of course it holds $K_1 \geq K_2 \geq K_3 \geq \ldots \geq K_L$. We can repeat the question. What is the number of colors to color all such maps? We also can restrict the shape of the countries.

We mention also the question how many colors we need to color the $\mathbb{R}^n$, where we use a set of vertices of the Definition 2. We ask how many colors we need to color the entire Euclidean space $\mathbb{R}^n$, where different image points of $f : \mathsf{set\ of\ vertices} \to \mathbb{R}^n$ have different colors, where f runs through all distance preserving functions. Note that this does not include Erdös's Open Problem 4.4, which is mentioned in [3], p.38. Further note that for a set of vertices of two points it is the well-known problem of Hadwiger-Nelson. See [3], where if $n = 2$ it is called the 'Chromatic Number of the Plane', its symbol is χ. Recently it is discovered that this number is at least 5. See [5]. Another problem is named the 'Polychromatic Number of the Plane' with the symbol χ_P, which asks for the smallest number of colors which is needed for a coloring of the plane, where no color realizes all distances. Please see [3], p. 32. We create our own problems with symbols χ_{1A} , χ_{1B} , χ_M and χ_T. For χ_{1A} we ask for the smallest number of colors needed for coloring the plane in such way that there is at least one color which has exactly one distance that is not realized. For χ_{1B} we ask also for the smallest number of colors needed for coloring the plane in such way that all covering colors have exactly one distance that is not realized. For χ_M we ask for the smallest number of colors to color the entire plane, and it holds that for all covering colors there is exactly one distance which is not realized. This distance is the same for all colors. For χ_T we ask for the smallest number of colors needed for coloring the plane in such way that all colors have exactly the same nonempty set of distances which are not realized. For χ_M we suggest the name 'Monochromatic Number of the Plane', and for χ_T we suggest 'Thuerey Number of the Plane'. It holds

$$5 \leq \chi \leq 7 \qquad \text{and} \qquad 4 \leq \chi_P \leq 6 \qquad \text{and} \qquad \chi_P \leq \chi \ .$$

In the case that there is a coloring of the plane which fulfills the conditions for χ_M also the conditions for χ_T, χ_{1A} and χ_{1B} are satisfied, and there are inequalities

$$\chi_{1A} \leq \chi_{1B} \leq \chi_M \qquad \text{and} \qquad 4 \ \leq \chi_P \leq \chi_T \leq \chi_M \qquad \text{and} \qquad \chi \leq \chi_M \ .$$

We pose the same questions, where we replace the entire $\mathbb{R}^n$ by a subset.

Proposition 10. *Let the set of vertices be consist of two points. If $n = 1$ two colors suffice.*

Proof. The two points have a distance of α, i.e. $\alpha > 0$. We take half-open intervals. We use the intervals $[\alpha \cdot k, \alpha \cdot (k+1)[, k \in \mathbb{Z}$. We color these intervals alternating with two colors. □

Instead of the usual distance from line (1) finally we got the idea to change also the distance. For two vectors $\vec{a} = (a_1, a_2, \ldots, a_n)$ and $\vec{b} = (b_1, b_2, \ldots, b_n)$ from $\mathbb{R}^n$ we can take the 'taxicab metric'

$$\mathrm{dist}(\vec{a}, \vec{b}) := |a_1 - b_1| + |a_2 - b_2| + \ldots + |a_{n-1} - b_{n-1}| + |a_n - b_n|$$

or the 'maximum metric'

$$\mathrm{dist}(\vec{a}, \vec{b}) := \max\{|a_1 - b_1|, |a_2 - b_2|, \ldots, |a_{n-1} - b_{n-1}|, |a_n - b_n|\}.$$

With this other distances in the $\mathbb{R}^n$ we can consider the questiones of χ, χ_P, χ_{1A}, χ_{1B}, χ_M and

χ_T which we mentioned above.

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